

Chirality and Helicity vs. Spin and Torsion Differential Topology^{or} and Electromagnetism

R. M. Kiehn

University of Houston

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Chamonix, France

rkiehn2352@aol.com

Abstract: In this article it is recognized that Chirality is to be associated with enantiomorphic pairs which induce Optical Activity, while Helicity is to be associated enantiomorphic pairs which induce a Faraday effect. Experimentally, the existence of enantiomorphic pairs, in a geometric crystalline sense, requires the lack of a center of symmetry, which is also a necessary condition for Optical Activity. However, Faraday effects may or may not require a lack of a center of symmetry. The two species of enantiomorphic pairs are distinct as the rotation of the plane of polarization by Optical Activity is a reciprocal phenomenon, while rotation of the plane of polarization by the Faraday effect is a non-reciprocal phenomenon.

From a topological viewpoint, Maxwell's electrodynamics indicates that the concept of Chirality is to be associated with a third rank tensor density of Topological Spin induced by the interaction of the 4 vector potentials $\{\mathbf{A}, \phi\}$ and the field excitations (\mathbf{D}, \mathbf{H}) . The distinct concept of Helicity is to be associated with the third rank tensor field of Topological Torsion induced by the interaction of the 4 vector potentials and field intensities (\mathbf{E}, \mathbf{B}) . The divergence of the topological Spin is equal to the first Poincare Invariant of classical electromagnetism. The divergence of the topological Torsion is equal to the second Poincare Invariant. When either Poincare Invariant is zero, the closed integrals of the Spin (respectively, Torsion) have values whose ratios are rational. That is, both the objects of topological torsion and topological spin can be quantized. However, when the divergence of Topological Torsion is not zero, the second Poincare invariant is not zero, and parity is not conserved.

Specific media are defined by equivalence classes of constitutive constraints between the field intensities and field excitations. It is known that an anisotropic birefringence constitutive constraint has some of the properties of the curvature tensor generated by a torsion-free Riemannian metric space. However, it is impossible to use a real metric to generate Faraday rotation or Optical Activity. Herein the Riemannian concept is broadened to include topological and affine torsion by computing the curvature tensor generated from a Cartan Connection of a Frame Matrix (not a symmetric metric matrix) that defines equivalence classes of differential linearity of manifold neighborhoods. The non-symmetric Frame Matrix admits both a left handed and a right handed realization. The topological analysis indicates that Optical Activity is related to the \mathbf{D} field (and possible translational accelerations), while Faraday rotation is related to \mathbf{B} field (and possible rotational accelerations). These facts lead to the possibility of finding new methods of enantiomorphic separation.

1. Introduction

It is a remarkable result of experimental chemistry, recognized by Pasteur and others, that there can exist enantiomorphic pairs of states of chemical systems that cannot be smoothly mapped into one another, starting from the identity. From a geometrical perspective, right handed quartz and left handed quartz are systems with apparent equivalent energies, yet with decidedly different behavior when interacting with electromagnetic fields. In fact, the rotation of the plane of optical polarization is often used as a tool to distinguish the enantiomorphic isotopes. However, the issue is deeper than that of geometry. There exist topologically equivalent isotopes that cannot be smoothly mapped into one another starting from the identity. Geometrical properties are those subsets of topological properties that depend upon size and shape. However, herein, the category of interest is that subset of topological properties which do not depend upon the geometrical issues of size and shape, and yet are to be associated with enantiomorphic pairs. These distinct topological isotopes (enantiomers) are of different size and shape and are connected by homeomorphisms, but they are not smoothly connected to one another by a map about the identity. The properties and existence of such topological enantiomers is the theme of this article. For example, a right handed Moebius band is topologically equivalent to a left handed Moebius

band, but possibly of different size and distorted shape. The homeomorphism between the two topological isotopes consists of more than one step: the right handed Moebius band can be transformed into a left handed Moebius band by first cutting the band, applying a 360 degree twist, and then reconnecting the cut ends such that points that were initially near to one another remain near to one another. Such a combination of processes is not C2 smooth but can be continuous in the topological sense.

The objective of this article is to examine those special features of electromagnetic systems that can exhibit topological enantiomers, and to determine how such enantiomers can be created. As all chemical systems are special examples of electromagnetic interactions, the methods to be developed are useful to the understanding of the more constrained (geometrical) features of chemical enantiomorphism. A remarkable, but little appreciated, fact is that the electromagnetic field itself has certain properties that exhibit the topological enantiomorphism mentioned above. Hence a study of these electromagnetic properties (defined below as topological torsion and topological spin) delivers a necessary foundation for the existence, control and modification of the geometric properties of enantiomorphism displayed in modern chemistry.

In modern stereochemistry, Optical Activity and Faraday Rotation have a dominant experimental role. Optical Activity and Faraday Rotation have many similarities, yet they are distinct, different, electromagnetic phenomena. A necessary, but not sufficient, condition for Optical Activity in crystalline structures is the lack of a center of symmetry. This lack of a center of symmetry is often used as the basis for defining "chirality", and, conversely, chirality is often associated with Optical Activity. However, according to Post [1], there exist 3 crystal classes without a center of symmetry (crystal classes 26,27,29) that do not support Optical Activity, hence a lack of a center of symmetry is not sufficient condition for Optical Activity. Note that Optically Active media have the capability of "rotating" the plane of polarization, as linearly polarized light passes through the media, a practical effect used by the wine grower to estimate the sugar content in his grapes.

Similar rotation of the plane of polarization occurs when linearly polarized light passes through Faraday media. However, Faraday effects can exist both in crystalline structures that have a center of symmetry and in crystalline structures that do not have a center of symmetry. Post reports that there are 9 crystalline structures that support both optical Activity and Faraday rotation. What are the intrinsic differences between Optical Activity and Faraday Rotation? In his

book "Formal Structure of Electromagnetics", E. J. Post [1] clearly delineates the differences between Optical Activity and Faraday rotation, and demonstrates solutions to Maxwell's equations for both effects. The crucial result is that Optical Activity is reciprocal and Faraday Rotation is not.

In short, the lack of a center of symmetry, the rotation of the plane of polarization, and the existence of enantiomorphic pairs, are necessary but not sufficient properties to define the concept of chirality. There exist two species of phenomena that exhibit the three properties stated above, one species is "reciprocal" and defines Chirality, and the other species is "non-reciprocal" and defines Helicity. These differences between Chirality and Helicity deserve attention, clarification, and exploitation. Such is the purpose of this article.

1.0.1. Transverse Inbound and Outbound Waves

First consider a complex four vector potential solution to the vector wave equation which propagates as a transverse wave in the $\pm z$ direction with a phase $\theta = \pm kz \mp \omega t$. There are 4 possibilities: The \mathbf{E} field rotates about the z axis in a Right Handed manner as viewed by an observer looking towards the positive z direction, or it rotates in a Left Handed manner. Outbound $\theta = kz - \omega t$ and Inbound $\theta = -kz - \omega t$ waves are to be distinguished.

$$ORH = \left| \begin{array}{c} 1 \\ i \end{array} \right\rangle e^{i(kz-\omega t)} \quad IRH = \left| \begin{array}{c} 1 \\ i \end{array} \right\rangle e^{i(-kz-\omega t)} \quad ORH = \left| \begin{array}{c} 1 \\ -i \end{array} \right\rangle e^{i(kz-\omega t)} \quad ILH = \left| \begin{array}{c} 1 \\ -i \end{array} \right\rangle e^{i(-kz-\omega t)}$$

For media with the symmetries of the Lorentz vacuum, the phase velocities $v = \omega/k$ are the same for all four modes. Addition or subtraction of ORH and OLH produces a Linearly polarized state outbound. Addition or subtraction of IRH and ILH produces a Linearly polarized state inbound.

Next, recall the experimental differences between Optical Activity and Faraday Rotation:

1.0.2. Optical Activity

Consider an optically active fluid (sugar in water) in a cylindrical tube of length L . For Optical Activity, there are also two distinct phase velocities, ω/k_1 and ω/k_2 . Outbound Right Handed (ORH) circularly polarized light propagates with a phase speed equal to the phase speed of Inbound Left Handed (ILH) circularly polarized light. Outbound Left Handed (OLH) circularly polarized light propagates with

a phase velocity different from the phase velocity of Outbound Right Handed polarized light (ORH), but with the same speed as that of Inbound Right Handed (IRH) polarized light. In summary,

$$\text{Optical Activity Phase Velocity, } V_{ORH} = V_{ILH} \neq V_{OLH} = V_{IRH}$$

The wave solutions for optical activity are of the format:

$$\begin{aligned} ORH &= \left| \begin{array}{c} 1 \\ i \end{array} \right\rangle \exp i(k_1 z - \omega t) & IRH &= \left| \begin{array}{c} 1 \\ i \end{array} \right\rangle \exp i(-k_2 z - \omega t) \\ OLH &= \left| \begin{array}{c} 1 \\ -i \end{array} \right\rangle \exp i(k_2 z - \omega t) & ILH &= \left| \begin{array}{c} 1 \\ -i \end{array} \right\rangle \exp i(-k_1 z - \omega t) \end{aligned}$$

The addition of $|ORH\rangle + |OLH\rangle$ produces a Linearly Polarized state propagating outbound, whose plane of polarization rotates. When the two inbound states are added, $|IRH\rangle + |ILH\rangle$, a linearly polarized state is achieved, and its plane of polarization also rotates *but in the opposite direction as the outbound rotation*. In other words, the round trip (outbound+reflection+inbound) motion causes the plane of polarization to return to its initial value. This result defines what is meant by a reciprocal effect. If the plane of polarization of the original linearly polarized light beam suffers a rotation in the amount of θ degrees as it traverses the Optically Active media, when reflected in a mirror, the plane of polarization suffers an negative rotation of θ degrees, as the light beam traverses the media in the reverse direction. The plane of polarization returns to its original state after the round trip. (The sense of Right Handed and Left Handed polarization is determined by an observer looking away from himself.)

1.0.3. Faraday Rotation

Consider a gas of He-Ne in a cylindrical tube of length L. Surround the tube with a coil of wire that will produce a coaxial magnetic field that partially aligns the spins of the gas atoms. For such Faraday media, there are two distinct phase velocities, ω/k_1 and ω/k_2 . Outbound Right Handed (ORH) circularly polarized light propagates with a phase speed equal to the phase speed of Inbound Right Handed (IRH) circularly polarized light. Outbound Left Handed (OLH) polarized light propagates with a phase velocity different from the phase velocity of Outbound Right Handed polarized light (ORH), but with the same speed as that of Inbound Left Handed (ILH) polarized light. In summary,

Faraday Effect Phase Velocity, $V_{ORH} = V_{IRH} \neq V_{OLH} = V_{ILH}$

The wave solutions for the Faraday effect are of the format:

$$\begin{aligned} |ORH\rangle &= \begin{vmatrix} 1 \\ i \end{vmatrix} \exp i(k_1 z - \omega t) & |IRH\rangle &= \begin{vmatrix} 1 \\ i \end{vmatrix} \exp i(-k_1 z - \omega t) \\ |OLH\rangle &= \begin{vmatrix} 1 \\ -i \end{vmatrix} \exp i(k_2 z - \omega t) & |ILH\rangle &= \begin{vmatrix} 1 \\ -i \end{vmatrix} \exp i(-k_2 z - \omega t) \end{aligned}$$

The formulas represent circularly polarized waves. The addition of $|RHO\rangle + |LHO\rangle$ produces a Linearly Polarized state propagating outbound, whose plane of polarization rotates. When the two inbound states are added, $|RHI\rangle + |LHI\rangle$, a linearly polarized state is achieved, and its plane of polarization also rotates *in the same direction as the outbound rotation*. In other words, the round trip (outbound+reflection+inbound) motion does not cause the plane of polarization to return to its initial value. This result defines what is meant by a non-reciprocal effect. If the plane of polarization of the original linearly polarized light beam suffers a rotation in the amount of θ degrees as it traverses the Faraday media, when reflected in a mirror, the plane of polarization suffers an additional rotation of θ degrees, as the light beam traverses the media in the reverse direction. The plane of polarization does not return to its original state, but instead ratchets by 2θ degrees upon completing the round trip. (The sense of Right Handed and Left Handed polarization is determined by an observer looking away from himself.)

1.0.4. Polar and Axial vectors

Following Schouten [2], Post points out that Faraday Rotation is "generated" by a "W vector", while Optical Activity is generated by a "vector". Under certain constraints, the W vector plays the role of an "Axial" vector, while the "vector" becomes a "polar" vector. Upon reflection, a polar vector changes its sense (determined by the arrow head). Point your finger into a mirror. The image points back at you. The sense of the image is opposite to the sense of the object. For polar vectors with a line of action parallel to the mirror surface, the opposite result is obtained. The sense of the image is the same as the sense of the object. Note the differences of orthogonal and parallel reflections.

A reflected axial vector does not change its sense if the line of action is orthogonal to the mirror. Curl your fingers and align your thumb in a direction orthogonal to the mirror. It does not matter whether the thumb points into or away from the mirror. The sense of the "axial vector" is determined by the curl of the fingers. The sense of the reflected image is the same as the sense of the object. The opposite effect occurs when the line of action of the axial vector is parallel to the reflection surface. The sense, as determined by the curl of the fingers, is opposite to that of the reflected image.

The magnetic field \mathbf{B} and the angular velocity $\mathbf{\Omega}$ are examples of spatial "W vectors". On the other hand, the \mathbf{D} field is a spatial "polar vector" in the sense used by Post. The anti-symmetric spatial components of the covariant field intensity tensor 2-form, $F = dA$, are formed by the spatial "W vector" field \mathbf{B} . The anti-symmetric spatial components of the tensor density, N-2 form, G , where $J = dG$, are formed by the spatial "polar vector" field \mathbf{D} . These facts yield a clue for distinguishing Faraday Rotation and Optical Activity on topological grounds. As will be shown below, Faraday Rotation is to be associated with the concept of Topological Torsion, and Optical Activity is to be associated with the concept of Topological Spin.

1.1. Topological Formulation of Maxwell's Equations.

1.1.1. Exterior Differential Systems

It is known that Maxwell's system of PDE's (without constitutive constraints) can be expressed as an exterior differential system [3] on a variety of independent variables. Exterior differential systems impose topological constraints on a differential variety. For the Maxwell electromagnetic system on a domain $\{x,y,z,t\}$ the two topological constraints have been called the Postulate of Potentials, and the Postulate of Conserved Currents. [4]. These two topological constraints lead to the system of Partial Differential Equations, known as Maxwell's equations, for any coordinate system so constrained. No metric, no connection, nor other restraints of a geometrical nature are required on the 4 dimensional differential variety of independent variables, typically written as $\{x, y, z, t\}$.

$$\text{Postulate of Potentials (an exact 2-form)} \qquad F - dA = 0$$

$$\text{Postulate of Conserved Currents (an exact 3-form)} \qquad J - dG = 0$$

The method of exterior differential systems insures that the description is not only diffeomorphically invariant in form (natural covariance of form with respect to all invertible smooth coordinate transformations), but also the description is functionally well defined with respect to maps which are C^2 continuous, but not necessarily invertible. This statement implies that those exterior differential forms which are defined on a final state variety can be "pulled back" in a functionally well defined manner to an initial state variety, even though the map from initial to final state of coordinate variables is NOT a diffeomorphic coordinate transformation. The inverse mapping need not exist. This result is truly a remarkable property of Maxwell electrodynamics, for it permits the analysis of certain irreversible electrodynamic processes without the use of statistics. The "push forward" process is not functionally well defined when the inverse map does not exist, a fact that demonstrates that topological evolution induces an "arrow of time" [5].

1.1.2. Constitutive Constraints

In practical applications, it is possible to impose constraints on the Maxwell system in the form of constitutive relations between the thermodynamically conjugate variables of field intensity (\mathbf{E}, \mathbf{B}) and field excitations (\mathbf{D}, \mathbf{H}). Post has demonstrated that the constitutive tensor (density) has many of the properties of the Riemann tensor [6]. These constraints are NOT necessarily equivalent to the Riemann tensor generated by a Riemannian metric imposed upon the variety $\{x, y, z, t\}$. In many circumstances the equivalence classes of such constitutive constraints can be put into correspondence with the geometrical symmetries of the 32 crystal classes that are used to discriminate between the many different observed physical structures. As mentioned above, a *complex* 6x6 constitutive constraint has been used by Post to delineate between Optical Activity, Faraday Phenomena, Birefringence and Fresnel-Fizeau motion induced effects in electromagnetic signal propagation. The complex constitutive tensor cannot be deduced from a real metric tensor. However it would appear that the constitutive tensor has a constructive definition in terms of a non-symmetric connection.

Indeed, the work of Post, who subsumed a complex constitutive tensor, has been extended [7] to demonstrate the existence of irreducible "quaternion" solutions to the Maxwell system. Quaternion waves cannot be represented by complex functions, which are the usual choice for describing electromagnetic signals. Complex wave solutions generate a 4th order characteristic polynomial for

the phase speed which is doubly degenerate. The wave speeds have only two distinct magnitudes depending upon direction and polarization. For cases where a center of symmetry is not available, and yet the medium supports both Optical Activity and Faraday rotation, the wave solutions can NOT be expressed as complex functions, but can be written as quaternions. The resulting 4th order characteristic polynomial for the wave speeds is not degenerate and has four distinct root magnitudes. The results indicate that the phase propagation speed of light is different for each direction of propagation and for each mode of polarization. The theory has been used to explain the experimental results measured in dual polarized ring laser apparatus.

In contrast, in a medium with the Lorentz symmetries, the characteristic polynomial is 4-fold degenerate; e.g., all polarizations and all directions have the same propagation speed. The result leads to the ubiquitous statement that the speed of light is the same for all observers, which is incorrect for media that do not have the Lorentz symmetries. For Birefringent, or Optically Active, or Faraday media, the characteristic polynomial for phase velocity is doubly degenerate, implying a relationship exists between for the 4 modes of propagation. There exist only two distinct phase velocity magnitudes. The correlation speeds for direction and polarization pairs have been presented above. Faraday rotation and Optical Activity have different propagation direction-polarization handedness correlations. The Faraday rotation is not reciprocal; the rotation induced by Optical Activity is reciprocal.

1.1.3. Topological Three Forms

The classic formalism of electromagnetism is a consequence of a system of two fundamental topological constraints as defined above on a domain of four independent variables. The theory requires the existence of two fundamental exterior differential forms, $\{A, G\}$, where the postulates permit the construction of the differential ideal $\{A, F = dA, G, J = dG\}$. This system of differential forms may be prolonged (by construction all possible exterior products) to yield the Pfaff sequence of forms: $\{A, F, G, J, A \wedge G, A \wedge F, A \wedge G, F \wedge F, F \wedge G, A \wedge J, G \wedge G\}$. On a domain of four independent variables, the complete Pfaff sequence contains three 3-forms: the classic 3-form of charge current density, J , and the (apparently novel to many researchers) 3-forms of Spin Current density, $A \wedge G$, [9] and Topological Torsion-Helicity, $A \wedge F$ [10].

$$\text{The 3-form of Charge Current density} \quad J = dG$$

The 3-form of Topological Spin density $S = A \wedge G$

The 3-form of Topological Torsion $T = A \wedge F$

In most elementary descriptions of electromagnetic theory, the 3-forms of Spin and Torsion are ignored. By direct evaluation of the exterior product, and on a domain of 4 independent variables, each 3-form will have 4 components that can be symbolized (in engineering format) by the 4-vector arrays

$$\text{Spin} - \text{Current} : \mathbf{S}_4 = [\mathbf{A} \times \mathbf{H} + \mathbf{D}\phi, \mathbf{A} \circ \mathbf{D}] \equiv [\mathbf{S}, \sigma],$$

and

$$\text{Torsion} - \text{vector} : \mathbf{T}_4 = [\mathbf{E} \times \mathbf{A} + \mathbf{B}\phi, \mathbf{A} \circ \mathbf{B}] \equiv [\mathbf{T}, h],$$

which are to be compared with the four construction components of the charge current 4-vector density:

$$\text{Charge} - \text{Current} : \mathbf{J}_4 = [\mathbf{J}, \rho].$$

1.1.4. Topological Invariants

The closed integral of each of the three 3-forms is a deformation invariant (hence a topological property) if the selected 3-form is closed in an exterior derivative sense ($dJ = 0$, $dS = 0$, $dT = 0$ respectively). For example, for any 3-form, J , such that $dJ = 0$, the Lie derivative of the closed integral relative to an arbitrary process path denoted by βV is given by the expression,

$$L_{\beta \mathbf{V}} \int_{\text{closed}} (J) = \int_{\text{closed}} \{i(\beta V)d(J) + d(i(\beta V)(J))\} = \int_{\text{closed}} \{0 + d(i(\beta V)(A \wedge G))\} = 0.$$

The zero result is interpreted by the statement "the closed integral is a deformation invariant" for the process can be deformed by any non-zero function $\beta(x, y, z, t)$, and the integral is unchanged.

As the charge current 3-form, J , is closed by construction, ($dJ = ddG = 0$) it follows that its closed integral is always a deformation invariant. The result leads to another ubiquitous statement known in electromagnetic theory as the "Conservation of electric charge". It is not equivalent to the quantization of charge. The additional topological constraints of closure imply that the exterior derivative of each of the three forms is empty (zero). By direct computation,

such a constraint of differential closure leads to the Poincare invariants for the electromagnetic system.

$$\begin{aligned}
\text{Poincare 1} &= d(A \wedge G) = F \wedge G - A \wedge J \\
&= \{div_3(\mathbf{A} \times \mathbf{H} + \mathbf{D}\phi) + \partial(\mathbf{A} \circ \mathbf{D})/\partial t\} dx \wedge dy \wedge dz \wedge dt \\
&= \{(\mathbf{B} \circ \mathbf{H} - \mathbf{D} \circ \mathbf{E}) - (\mathbf{A} \circ \mathbf{J} - \rho\phi)\} dx \wedge dy \wedge dz \wedge dt
\end{aligned}$$

$$\begin{aligned}
\text{Poincare 2} &= d(A \wedge F) = F \wedge F \\
&= \{div_3(\mathbf{E} \times \mathbf{A} + \mathbf{B}\phi) + \partial(\mathbf{A} \circ \mathbf{B})/\partial t\} dx \wedge dy \wedge dz \wedge dt \\
&= \{-2\mathbf{E} \circ \mathbf{B}\} dx \wedge dy \wedge dz \wedge dt
\end{aligned}$$

For a (vacuum) state, with $J = 0$, zero values of the Poincare invariants require that the magnetic energy density is equal to the electric energy density ($1/2\mathbf{B} \circ \mathbf{H} = 1/2\mathbf{D} \circ \mathbf{E}$), and, respectively, that the electric field is orthogonal to the magnetic field ($\mathbf{E} \circ \mathbf{B} = 0$). Note that these constraints often are used as elementary textbook definitions of what is meant by electromagnetic waves. The possible values of the topological quantities, as deRham period integrals [11], form rational ratios and topological quantum numbers. These quantum numbers should NOT be considered as topological "charge". Electromagnetic (topological) charge is related to the two dimensional closed integrals of G , not the three dimensional closed integrals described above.

1.1.5. Field Momentum, Propagation direction, and the 4-Vector Potential

The 4 vector potential $A_4 = [\mathbf{A}, \phi]$ is different from the 4 dimensional propagation vector $k_4 = [\mathbf{k}, \omega]$; in many cases the two vectors are not even proportional (although they can be). In the language of differential forms, it must be recognized that there is a difference between the 1-form, $A = A_x dx + A_y dy + A_z dz - \phi dt$, and the 1-form, $k = k_x dx + k_y dy + k_z dz - \omega dt$. The integral of k defines the phase $\theta = \int k$. The propagation 1-form k is defined from the equations that generate the singular solutions [7] to Maxwell's equations:

$$k \wedge F = 0, \quad \text{and} \quad k \wedge G = 0.$$

Note that these equations for singular solutions are derived from the 3-forms $k^\wedge F$ and $k^\wedge G$ constrained to be zero. The 3-form $k^\wedge F$ is not necessarily the same as the 3-form, $A^\wedge F$. Similarly, the 3-form $k^\wedge G$ is not necessarily the same as the 3-form, $A^\wedge G$. These singular solutions are always transverse in a geometrical sense that the wave 3 vector \mathbf{k} is in the direction of the field momentum, $\mathbf{D} \times \mathbf{B}$. Both the \mathbf{D} vector and the \mathbf{B} vector are orthogonal to the wave vector, \mathbf{k} , that generates singular solutions.

The 1-forms, k , that satisfy the equations

$$\text{Associated 1-forms } k \quad \{k^\wedge F = 0, k^\wedge G = 0\}$$

are defined as the "associated" 1-forms relative to F and G . The 1-forms, k , that satisfy the equations

$$\text{Extremal 1-forms } k \quad \{k^\wedge dF = 0, k^\wedge dG = 0\}$$

are defined as "extremal" 1-forms relative to F and G . 1-forms such that

$$\text{characteristic 1-forms } k \quad \{k^\wedge F = 0, k^\wedge G = 0, k^\wedge dF = 0, k^\wedge dG = 0\},$$

are defined as the "characteristic" 1-forms, k . These 1-forms k are dual to the associated, extremal, and characteristic vector fields, V , which satisfy the equations

$$\text{Associated vector fields } V \quad \{i(V)A = 0, i(V)G = 0\}$$

or

$$\text{Extremal vector fields } V \quad \{i(V)dA = 0, i(V)dG = 0\},$$

or

$$\text{Characteristic vector fields } V \quad \{i(V)A = 0, i(V)G = 0, i(V)dA = 0, \text{ and } i(V)dG = 0\},$$

respectively.

The concepts of Spin Current and the Torsion vector have been utilized hardly at all in applications of classical electromagnetic theory. Just as the vanishing of the 3-form of charge current, $J = 0$, defines the topological domain called the vacuum, the vanishing of the two other 3-forms will refine the fundamental topology of the Maxwell system. Such constraints permit a definition of transversality to be made on topological (rather than geometrical) grounds. If both $A^\wedge G$ and $A^\wedge F$ vanish, the vacuum state supports topologically transverse modes only

(TTEM). Examples lead to the conjecture that TTEM modes do not transmit power, a conjecture that has been verified when the concept of geometric transversality (TEM) and topological transversality (TTEM) coincide. A topologically transverse magnetic (TTM) mode corresponds to the topological constraint that $A \hat{F} = 0$. A topologically transverse electric mode (TTE) corresponds to the topological constraint that $A \hat{G} = 0$. Examples, both novel and well-known, of vacuum solutions to the electromagnetic system which satisfy (and which do not satisfy) these topological constraints are given [12]. The ideas should be of interest to those working in the field of Fiber Optics. Recall that classic waveguide solutions which are geometrically and topologically transverse (TEM \equiv TTEM) do not transmit power [13]. However, in [14] an example vacuum wave solution is given which is geometrically transverse (the fields are orthogonal to the field momentum and the wave vector), and yet the geometrically transverse wave transmits power at a constant rate: the example wave is not topologically transverse as $A \hat{F} \neq 0$.

2. Right handed vs Left handed evolution

2.0.6. Connections

In spaces (such as Finsler spaces) which may or may not be Riemannian, the topological concept of differential neighborhoods that are linearly connected implies the existence, over the domain, of a matrix of functions $[F_a^k(q^b)]$ that will linearly map 1-forms (linear combinations of differentials $|\omega^a(q, dq)\rangle$) into 1-forms (linear combinations of differentials $|\sigma^k(q, dq)\rangle$).

$$[F_a^k(q^b)] \circ |\omega^a(q, dq)\rangle \Rightarrow |\sigma^k(q, dq)\rangle.$$

The map between differentials is linear, but the matrix elements are not domain constants (non-linearity is built in). The columns of the matrix of functions forms a matrix of basis vectors over the domain which may be used to express any tensorial properties. The matrix of basis vectors defines what Cartan called the Repere Mobile, or the moving frame, for its values change as a point p moves along some curve in the domain. If the mapping of 1-forms is integrable,

$$\text{integrable basis : } [F_a^k(q^b)] \circ |dq^a\rangle \Rightarrow |dx^k\rangle, \text{ implies } x^k = f^k(q^b)$$

then there exist functions whose differentials are exact, and the mapping is said to be holonomic. Often it is presumed that the functional mapping exists (and as

such is called a coordinate transformation under certain additional constraints) and the Frame matrix is deduced by the differential operations to produce the Jacobian matrix of the transformation. However, there are other ways to impose or deduce a Frame matrix on a domain.

The fundamental question of a connection is related to the differential neighborhood properties of the Frame matrix. As the Frame matrix, by definition of its domain, has a non-zero determinant, then it admits an inverse matrix, and by differential and algebraic processes the (right Cartan) connection matrix $[C_R]$ (or the left Cartan matrix $[C_L]$) can be constructed:

$$\begin{aligned} d[F] &= -[F] \circ [dG] \circ [F] \\ &= -[C_L] \circ [F] \\ &= +[F] \circ [C_R]. \end{aligned}$$

The matrix elements of $[C_R]$ and $[C_L]$ are differential 1-forms. These matrix elements do not necessarily vanish nor are they necessarily equal. It is to be noted that the left and the right Cartan matrices are (anti-) similarity transforms of one another.

The reason that there are two distinct methods for constructing the linear mapping is based upon the fact that a matrix, whose determinant is non - zero, always has two representations, a left handed and a right handed representation. The two representations always consist of either a (right handed) product of a unitary matrix times a Hermitean matrix, or the (left handed) product of a Hermitean matrix times a unitary matrix. Only if the original matrix is "normal" (such that the product of itself times its Hermitian conjugate is equal to the product of its Hermitian conjugate times itself) will the right and left handed product representations be degenerately the same. [8] It is this handedness property of topological neighborhoods that captures the features of electromagnetic charge distributions of molecules and crystals that exhibit enantiomorphic states.

In the electromagnetic situation, the constitutive map is often considered to be (within a factor) a linear mapping between two six dimensional vector spaces. As such the constitutive map can have both a right or a left handed representation, implying that there are two topologically equivalent states that are not smoothly equivalent about the identity.

In the geometric situation, the matrix elements of frame matrix, $[F_a^k(q^b)]$, are not constants. The two mechanisms (right and left handed) for neighborhood

expressions imply that the connection is not generated from a symmetric metric. Such connections are said to admit torsion. In short, the concept of an affine connection is more general than the Hermitian (symmetric) offered by the Christoffel symbols (which are generated from a metric). Any domain which is parallelizable will support a linear connection of differentials.

Examples (to follow)

3. References

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