Notes on Cartan's structural theory and its application to the study of Topological Defects

R. M. Kiehn

Emeritus, Physics Dept., Univ. Houston
Work started 1989. Updated Aug 3, 2002
http://www.cartan.pair.com
rkiehn2352@aol.com

Abstract

Abstract: Cartan's theory of exterior differential systems is used to define topological structures and defects on a variety. In particular it is demonstrated how a global 1-form of Action on a projective variety induces a course topology on the domain of support. A constructive method is used to define algebraically a frame field at each point of the projective domain. The induced frame of functions permits a Cartan matrix of connection 1-forms to be computed by means of a single C1 differential process. The constructive technique then permits the algebraic evaluation of certain useful geometric and topological objects, such as the matrix of curvature 2-forms, which conventionally require higher order or C2 differential processes. The C1 projective algebraic methods therefore lend themselves to the development of a theory of defects in which the concept of dislocations and disclinations can be put on equal footing. The methods are applied to the problems of non-equilibrium thermodynamics, and the hydrodynamics of irreversible turbulent flows.

0.1. 1. Introduction (parametric vs. implicit methods)

In the theory of surfaces, it is known that some surfaces can be represented by parametric immersions of two variables into a euclidean space of three dimensions, and other surfaces can be represented by implicit projections from a euclidean space of higher dimensions to a lower dimensional set. Some surfaces (such as a sphere) admit both representations to and from the same euclidean space, others do not. A Mobius band admits a parametrization from two parameters to a euclidean space of 3 dimensions, but there is not an implicit function representation for the Mobius band in three dimensional euclidean space. On the other hand, two isolated spheres have an implicit representation as a single function on 3 dimensional euclidean space, but there does not exist a single path connected parametrization that will represent both spherical components. A hyperbolic surface may have a single function implicit representation from a space of 3 euclidean dimensions, but a twisted hyperbolic surface does not.

More complex examples are given by those surfaces which are envelopes of families of simple surfaces. For example, consider the family of spheres of constant radius, a, whose centers are located on the circle of radius b>a. The family has an envelope which is the surface of a torus. In other cases the family is enveloping, such as the cylinder formed by the set of planes tangent to a circle.

Analogous remarks can be made for curves in 3 dimensional euclidean space: sometimes a curve is represented by a single parametrization into a 3 dimensional euclidean space, and sometimes a curve is represented by the intersection of two implicit surfaces in the same euclidean space (consider the intersection of a sphere and a cone with center and vertex at the 3D origin). A family of curves can envelope a surface. For example consider the parallel transport of a straight line in three dimensional space that meets a circle in the plane, but is orthogonal to the plane of the circle. The parallel transport forms an envelope of cylindrical generators. Note that a parallel transport of the closed curves in the xy plane, (the circles) translated along the z axis, also generates the surface of the cylinder.

Now consider the transport of straight lines not in the plane of the circle that meet the circle, but where the transport around the circle does not preserve the parallelism of the straight lines. The envelope of the construction again forms a surface, but now the surface is twisted. The enveloping surface has a set of generators, but the envelope does not give the details of the torsion twist. Such a surface is easily constructed by connecting strings from one circle about the z axis in the xy plane at z = -1 to another circle about the common z axis at z = +1, to form the cylindrical surface of parallel generators. Now twist the two rings relative to one another, a method of transporting the straight line generators which is not a parallel transport mechanism. The strings form a new hyperbolic surface with a torsion twist. The same envelope is generated by a left handed or a right handed twist.

From the open cylinder, bend the straight line generators to form circles and glue the ends together. The resulting structure is the torus, with one set of generators (closed lines) residing in parallel planes and the other set of of circular generators which do not reside in parallel planes. Consider a slightly thickened cylindrical surface made from a piece of rubber tubing. Bend the tubing in the shape of a hollow thickened circular tube and glue the ends together. Then tubing will lie flat on a plane. In another construction, before glueing the ends together, give the torus a twist of 180 degrees. The resulting structure no longer lies flat in the plane. Physically there is deformation energy stored in both the bending and the twist. In one case the torus is flat in the sense that the generating lines (now circular curves) are parallel to one another and planar, but in the twisted case there the generating lines are not in parallel planes.

Topologically, the parametric methods are related to homotopy and the concept of connectivity, and the implicit methods are related to cohomology and the number of components. The parametric method leads to structural features of the system in terms of differential processes. The implicit projective method leads to structural features in terms of algebraic processes. In this article, the latter (and perhaps less familiar) technique of implicit representation is used to construct the differential topology (cohomology) of subspaces that are generated by implicit constraints. These implicit constraints are formulated as exterior differential systems.

The idea is to decipher and catalog such topological properties for physical systems that are defined in terms of of differential constraints imposed on a variety of independent variables. Most physicists are familiar with the concept of Action, its application in the calculus of variations, mechanics, and classical field theory, as well as the quantum theory based upon path integrals with their infamous phase factors. However, it is not well appreciated that the existence, or specification of, an Action 1-form on a domain actually is equivalent to inducing a topological structure on the domain of support. In this article, the details of such an induced topology and the defects it creates will be described. In a sense, the topological defects considered herein are "deviations" of a submanifold from smooth connected euclidean neighborhoods, a deviation which is persistent under continuous deformations or contractions of a domain. In a two dimensional surface, a submanifold of euclidean 3 space, a typical defect will be a "hole" or a "line of self intersection" that represents an obstruction to a smooth eulidean contraction process. However, there can exist topological defects in subspaces of all dimensions.

Consider the case of two concentric circles in a plane. Attach elastic strings along radii from one circle to another. Rotate the outer circle with respect to the inner circle. Everything works nicely until an angle of rotation equal to 90 degrees is reached. (The strings are now tangent to the inner circle. The inner circle (the hole in the annular disc) is an obstruction to parametric rotations beyond 90 degrees. Next assume the circles are in the same plane. The twist can proceed beyond 90 degrees, and things will deform nicely until the 180 degree position is reached. At this point the original cylinder of strings has been deformed to a hyperbolic surface, and then finally to a cone at the 180 degree position. The conical point produces an obstruction to further parametric deformation of the strings.

The domain of interest in this article will be limited to those situations that can be described by manifolds, a constraint which is taken to mean that the subspaces of dimension n to be studied are presumed to be embedded in a euclidean space of some higher dimension (at most of dimension 2n+1 according to Whitney). This method permits the structural equations to be deduced in a constructive algebraic manner, by partitioning the euclidean space according to a parallizeable basis set on the euclidean space.

Another technique utilizes the concept of a global 1-form of Action to describe a physical system on a projective space of n+1 dimensions. The Action 1-form in n+1 dimensional space admits n "associated" (or tangent, or interior, or intrinsic, or horizontal) linearly independent vector fields to be constructed algebraically. The components of the 1-form itself are proportional to the adjoint of the system of n tangent vectors, and can be used to form a basis frame with a global inverse, in the projective space of n+1 dimensions mod the submanifold origin. Both methods will be exploited in this article.

The theory of parametric space curves has its beginnings with the generalization of the Frenet-Serret concept of a moving frame. Recall that a singly parameterized position vector from an origin to a point in euclidean space will sweep out a space curve as the parameter, s, is varied. For certain restrictions of the parametric interval, the position vector may be considered as a homotopy, $[s] \rightarrow \mathbf{R}(s)$. The differential constraint,

$$d\mathbf{R}(s) - \mathbf{T}(s)ds = 0, \tag{0.1}$$

$$\text{with } \mathbf{T}(s) \circ \mathbf{T}(s) = \mathbf{1} \tag{0.2}$$

which is the definition of the unit (kinematic) tangent vector, $\mathbf{T}$, should be viewed as a topological constraint on the path connected neighborhoods of the point, p, defined by the position vector, $\mathbf{R}(s)$. In fact, this specification of a kinematic motion of a point is the first example of an exterior differential system. An exterior differential system defines a topological structure on a domain. More complicated exterior differential systems can not be described kinematically in terms of a single parameterization, s. For example, in string theory, a second parameter is introduced such that the evolution depends upon two parameters (conventionally designated as time and the string extension). The key point is that a path connected basis frame will be deduced from the concept of an immersion (or embedding) in terms of parameters.

In the classic case of a kinematic constraint, at the point, p, it is conventional to erect a set of basis vectors or a "Basis Frame" (a triad in 3 -space), the elements of which are be used to compose arbitrary vectors at the point. Then as the point p moves along the space curve or path, the Frame of basis vectors also moves. Cartan called this idea the "Repere Mobile", or the theory of the moving Frame.

0.2. 2. The Frenet Basis (parametric subspaces)

0.2.1. The singly parameterized vector field

Even for kinematic singly parameterized constraints, the question arises as to how to define the Basis Frame, and how the Basis Frame at point p is connected to its neighborhood at point $p+ds$. There are two methods of constructing the basis frame that have been used extensively, and a third method defined in this article that has not been used extensively. The first classical method is the method of Frenet-Serret. It is based upon higher order differential processes (and an assumed euclidean inner product). The fundamental assumption of Frenet theory is that there exists a map from single parameter, s, to a "unit tangent" vector, $\mathbf{T}(s)$. The integration to yield the position vector is not required explicitly.

From the unit tangent vector, $\mathbf{T}(s)$, which is assumed to be singly parameterized in terms of the parameter, s, construct the unit normal vector, $\mathbf{N}(s)$, from the formula of constraint:

$$d\mathbf{T}(s) - \kappa\mathbf{N}(s)ds = 0. \tag{0.3}$$

Given $\mathbf{t}(s)$, the differential process constructs a new vector on the euclidean 3 space, whose magnitude relative to a euclidean norm is the curvature function

$\kappa(s)$. In a euclidean space of 2 dimensions, the two vectors $\mathbf{T}(s)$ and $\mathbf{N}(s)$ can be used to form a basis frame. If the path or space curve is in 3 euclidean dimensions, there are two options for constructing the third basis vector of the frame, one option is algebraic and the other is differential. The last vector (third of three) of the 3-dimensional Frenet Basis Frame, is defined as the unit bi-normal, $\mathbf{B}(s)$, which can be constructed either by using the Gibbs cross product formula,

$$\mathbf{B} = \mathbf{T} \times \mathbf{N}, \tag{0.4}$$

or by using a third differential constraint of the form

$$d\mathbf{N}(s) + \kappa\mathbf{T}(s)ds - \tau\mathbf{B}(s)ds = 0 \tag{0.5}$$

The coefficient functions, $\kappa(s)$, defined as the Frenet curvature of the space curve, and $\tau(s)$, defined as the Frenet torsion of the space curve, are chosen to insure that the basis vectors have a unit norm and are orthogonal to one another. However, these useful and interesting constraints are topologically severe.

The resulting complex of the Frenet method gives a method for evaluation of the evolution of the basis frame (a $3x3$ matrix) with respect to the parameter, s.

$$d\begin{bmatrix} \mathbf{T}^x(s) & \mathbf{N}^x(s) & \mathbf{B}^x(s) \\ \mathbf{T}^y(s) & \mathbf{N}^y(s) & \mathbf{B}^y(s) \\ \mathbf{T}^z(s) & \mathbf{N}^z(s) & \mathbf{B}^z(s) \end{bmatrix} = \begin{bmatrix} \mathbf{T}^x(s) & \mathbf{N}^x(s) & \mathbf{B}^x(s) \\ \mathbf{T}^y(s) & \mathbf{N}^y(s) & \mathbf{B}^y(s) \\ \mathbf{T}^z(s) & \mathbf{N}^z(s) & \mathbf{B}^z(s) \end{bmatrix} \circ \begin{bmatrix} 0 & -\kappa ds & 0 \\ +\kappa ds & 0 & -\tau ds \\ 0 & +\tau ds & 0 \end{bmatrix} \tag{0.6}$$

In Cartan's notation

$$\begin{aligned} d\,[F] &= [F] \circ [C]\,, && (0.7) \\ \text{The Frame matrix } [F] &= \begin{bmatrix} \mathbf{T}^x(s) & \mathbf{N}^x(s) & \mathbf{B}^x(s) \\ \mathbf{T}^y(s) & \mathbf{N}^y(s) & \mathbf{B}^y(s) \\ \mathbf{T}^z(s) & \mathbf{N}^z(s) & \mathbf{B}^z(s) \end{bmatrix}, && (0.8) \\ \text{The right Cartan connection matrix } [C] &= \begin{bmatrix} 0 & -\kappa ds & 0 \\ +\kappa ds & 0 & -\tau ds \\ 0 & +\tau ds & 0 \end{bmatrix}. && (0.9) \end{aligned}$$

The anti-symmetric matrix of Frenet curvature and torsion coefficients forms a connection matrix, $[C]$, linearly connecting differentials of the functions that make up the Frame basis vectors. The remarkable result is the differential of any basis vector of the Frame is a linear combination of the original basis vectors. Such a differential process is said to be closed under the constraints. The Cartan matrix of connection 1-forms is anti-symmetric because the Frenet basis frame was constructed to be orthonormal. The determinant of the Frenet Frame has been adjusted to be either plus or minus 1. A linear independent set of basis vectors which are not orthonormal would yield a different form for the Cartan matrix.

The crux of the matter is the construction of the basis frame. There is both the mathematical problem, and the physical problem of how to construct the basis frame. A basis frame can be constructed from the assumption that each point (a position vector) in the 3D space can be singly parametrized in a kinematic sense. Subsequent differential processes can be used to generate a basis frame (over some parametric interval) in terms of the velocity vector (which is tangent to a parameterized curve), the acceleration, and the rate of change of the acceleration (along the curve). In the Frenet case, a special choice is made for the kinematic parametrization (the arc length) such that the "velocity vector" becomes the "unit" tangent vector, a vector field which is homogeneous of degree zero.

0.2.2. The N parameterized vector field in N=3D

Consider the vector $\mathbf{V}(x,y,z) = [U(x,y,z), V(x,y,z), W(x,y,z)]$, and define the unit vector,

$$\mathbf{T}(x,y,z) = [U,V,W]/(U^2+V^2+Z^2)^{1/2} = \mathbf{V}(x,y,z)/(\mathbf{V}\circ\mathbf{V})^{1/2}. \tag{0.10}$$

The scaling denominator, $(\mathbf{V}\circ\mathbf{V})^{1/2}$, is restricted to the Gaussian format, defined as the square root of the sum of squares of the component functions that define the vector field, $\mathbf{V}(x,y,z)$. The differential of the unit vector can be evaluated in terms of the "unit" Jacobian matrix, $[\mathbf{J}_m^k(x)]$, of the mapping to the unit vector, $\mathbf{T}(x,y,z)$:

$$d\,|\mathbf{T}\rangle = [\partial\mathbf{T}^k/\partial x^m]\circ|dx^m\rangle = [\mathbf{J}_m^k(x)]\circ|dx^m\rangle\,. \tag{0.11}$$

The unit Jacobian matrix (or its transpose) has a determinant equal to zero. If it is further assumed that a map exists such that $s \Rightarrow [x(s), y(s), z(s)]$, then

$$d\,|\mathbf{R}\rangle = |dx^m\rangle = |\partial x^m/\partial s\rangle\,ds = |\mathbf{T}\rangle\,ds, \tag{0.12}$$

and it is possible to make a direct comparison to the Frenet construction, by means of the chain rule:

$$d\,|\mathbf{T}\rangle = [\partial \mathbf{T}^k/\partial x^m] \circ |\partial x^m/\partial s\rangle\, ds = [\mathbf{J}^k_m] \circ |\mathbf{T}^m\rangle\, ds = \kappa\,|\mathbf{N}^m(x,y,z)\rangle\, ds. \quad (0.13)$$

The unit vector $\mathbf{T}(x,y,z)$ is the null eigen vector of the transpose of the "unit" Jacobian matrix having eigen value zero:

$$[\mathbf{J}^k_m]^{transpose} \circ |\mathbf{T}^m\rangle = 0. \quad (0.14)$$

Hence it follows that

$$d\,|\mathbf{T}\rangle = \{[\mathbf{J}^k_m] \pm [\mathbf{J}^k_m]^{transpose}\} \circ |\mathbf{T}^m\rangle\, ds = \kappa\,|\mathbf{N}^m(x,y,z)\rangle\, ds. \quad (0.15)$$

Using the minus sign, it is to be recognized that, in 3D,

$$\begin{aligned} d\,|\mathbf{T}\rangle &= \{[\mathbf{J}^k_m] - [\mathbf{J}^k_m]^{transpose}\} \circ |\mathbf{T}^m\rangle\, ds & (0.16)\\ &= |curl\ \mathbf{T} \times \mathbf{T}\rangle\, ds = \kappa\,|\mathbf{N}^m(x,y,z)\rangle\, ds, & (0.17)\end{aligned}$$

permiting the Frenet unit normal field, $|\mathbf{N}^m(x,y,z)\rangle$, to be described in terms of the parameters, (x,y,z).

$$\kappa\mathbf{N}(x,y,z) = curl\ \mathbf{T} \times \mathbf{T} \quad (0.18)$$

Using the same method as for the Frenet technique, the unit binormal is expressed as

$$\begin{aligned} \kappa\mathbf{B} &= \mathbf{T} \times \kappa\mathbf{N} = \{curl\mathbf{T} - (\mathbf{T} \circ curl\mathbf{T})\mathbf{T}\}, & (0.19)\\ \text{such that } \kappa^2 &= (curl\mathbf{T} \circ curl\mathbf{T}) - (\mathbf{T} \circ curl\mathbf{T})^2 & (0.20)\end{aligned}$$

This is a remarkable hydrodynamic relation between the square of the vorticity of the unit velocity field, $curl\mathbf{T}$, (enstrophy of the unit velocity field), the square of the abnormality (topological torsion) of the unit tangent field ($\mathbf{T} \circ curl\mathbf{T}$), and the square of the Frenet curvature, κ. It is also possible to show that there is also a relationship between the Frenet Torsion, τ, and the helicity of each of the unit basis vectors:

$$2\,\tau = \mathbf{T} \circ curl\mathbf{T} - \mathbf{N} \circ curl\mathbf{N} - \mathbf{B} \circ curl\mathbf{B} \tag{0.21}$$

This expression, above, and the examples, below, point out some of the differences between Topological Torsion of the Velocity Field, $(\mathbf{V} \circ curl\mathbf{V})$, and the Frenet Torsion, τ, of a flow line.

Example 0.1.

$$\begin{aligned}
\textit{Heisenberg Covector } \mathbf{V}(x,y,z) &= [-y,x,1] && (0.22)\\
\textit{Frenet Curvature } \kappa &= \sqrt{x^2+y^2}/(x^2+y^2+1) && (0.23)\\
\textit{Frenet Torsion } \tau &= -1/((x^2+y^2+1) && (0.24)\\
\textit{Topological Torsion } \mathbf{V} \circ curl\mathbf{V} &= -2 && (0.25)\\
\textit{helicityT} \;\; \mathbf{T} \circ curl\mathbf{T} &= -2/((x^2+y^2+1) && (0.26)\\
\textit{helicityN} \;\; \mathbf{N} \circ curl\mathbf{N} &= 0 && (0.27)\\
\textit{helicityB} \;\; \mathbf{B} \circ curl\mathbf{B} &= 0 && (0.28)
\end{aligned}$$

Example 0.2.

$$\begin{aligned}
\textit{Covector } \mathbf{V}(x,y,z) &= [-y,x,z] && (0.29)\\
\textit{Frenet Curvature } \kappa &= \sqrt{(x^2+y^2)(x^2+y^2+2z^2)}/(x^2+y^2+z^2)^{(3/2)} && (0.30)\\
\textit{Frenet Torsion } \tau &= -2z/(x^2+y^2+2z^2) && (0.31)\\
\textit{Topological Torsion } \mathbf{V} \circ curl\mathbf{V} &= -2z && (0.32)\\
\textit{helicityT} \;\; \mathbf{T} \circ curl\mathbf{T} &= -2/(x^2+y^2+z^2) && (0.33)\\
\textit{helicityN} \;\; \mathbf{N} \circ curl\mathbf{N} &= 2z(x^2+y^2)/\{(x^2+y^2+2z^2)(x^2+y^2+z^2)\} && (0.34)\\
\textit{helicityB} \;\; \mathbf{B} \circ curl\mathbf{B} &= 0 && (0.35)
\end{aligned}$$

Example 0.3.

$$\begin{aligned}
\textit{Covector } \mathbf{V}(x,y,z) &= [y,0,x^2+y^2] && (0.36)\\
\textit{Frenet Curvature } \kappa &= 2xy^2/\{(x^2+y^2)^2+y^2\}^{(3/2)} && (0.37)\\
\textit{Frenet Torsion } \tau &= 0 && (0.38)\\
\textit{Topological Torsion } \mathbf{V} \circ curl\mathbf{V} &= -2(x^2-y^2) && (0.39)\\
\textit{helicityT} \;\; \mathbf{T} \circ curl\mathbf{T} &= -2(x^2-y^2)/\{(x^2+y^2)^2+y^2\} && (0.40)\\
\textit{helicityN} \;\; \mathbf{N} \circ curl\mathbf{N} &= -2(x^2-y^2)/\{(x^2+y^2)^2+y^2\} && (0.41)\\
\textit{helicityB} \;\; \mathbf{B} \circ curl\mathbf{B} &= 0 && (0.42)
\end{aligned}$$

Example 0.4.

$$\begin{aligned}
\textit{Spinning Top covector } \mathbf{V}(x,y,z) &= [yz, zx, xy] \\
\textit{Frenet Curvature } \kappa &= x^3(y^2-z^2)+y^3(z^2-x^2)+z^3(x^2-y^2)/\{(xy)^2+(yz)^2+ \\
\textit{Frenet Torsion } \tau &= -3xyz(x^2-y^2)(y^2-z^2)(z^2-x^2)/\lambda \\
\lambda &= \{x^6(y^2-z^2)^2+y^6(z^2-x^2)^2+z^6(x^2-y^2)^2\} \\
\textit{Topological Torsion } \mathbf{V}\circ curl\mathbf{V} &= 0 \\
\textit{helicityT } \ \mathbf{T}\circ curl\mathbf{T} &= 0 \\
\textit{helicityN } \ \mathbf{N}\circ curl\mathbf{N} &= 4xyz(x^2-y^2)(y^2-z^2)(z^2-x^2)/\lambda \\
\textit{helicityB } \ \mathbf{B}\circ curl\mathbf{B} &= 2xyz(x^2-y^2)(y^2-z^2)(z^2-x^2)/\lambda
\end{aligned}$$

0.2.3. The N-1 parameterized vector field in N=3D

In general, the parametric idea is to construct a Frame on a space of dimension N. The Frame will consist of N linearly independent column vectors with N components. For a N-1 dimensional surface, the parametric method presumes a map is given from N-1 parameters into the N dimensional space. The N vectors of the N x N Frame on space N are N^2 functions with arguments in terms of the N-1 parameters that define the N-1 space. The N mapping functions permit the construction of N-1 contravectors on N via the N-1 partial derivatives of the N mapping functions. The question is how to complete the frame by construction of the Nth linearly independent vector.

In order for the Frame matrix to have an inverse, it is necessary that the Frame have a non-zero determinant. It is also convenient that the Nth vector be orthogonal to all of the other vectors of the Frame. This construction can be achieved by adding a column of zeros to the N-1 columns of the pre-Frame. The pre-Frame so constructed is singular and does not have an inverse. However it is algebraically possible to construct the adjoint matrix to the pre-Frame. The transpose of the adjoint is defined as the Grassman adjoint, and its last column is orthogonal to the N-1 vectors previously computed. This "orthogonal - or vertical to the surface - direction field" (to within a factor) will be used as the Nth column vector of the basis Frame. The determinant of the resulting Frame matrix can be adjusted to be some arbitary function, by appropriate choice of the "scaling factor". A particular choice for the scaling factor yields a Frame determinant which is globally equal to 1. Hence, the parametric N-1surface method, using the adjoint orthogonal direction field for the N basis vector, can always be used to construct a basis Frame on the N-space, if the mapping functions from N-1 parameters to the N space are given.

From the Frame matrix it is possible to deduce what are known as the Cartan torsion and Cartan curvature properties of the N-1 space. However, a fundamental theorem states that the Cartan torsion and the Cartan curvature of the N space (which has a global invertible C2 Frame field) is zero. In the next section, the Cartan torsion and the Cartan curvature of subspaces of an N space with a Global Frame field will be defined and computed. The N-1 subspace, in general, will have both a N-1 vector of translational (affine) torsion 2-forms and a N-1 covector of rotational expansion torsion 2-forms, as well as a N-1 x N-1 matrix of curvature 2-forms. Parametric surface Frames, however, are always free from affine torsion. For certain choices of the scaling function, the Frame field can be adjusted such that the rotation expansion torsion coponents vanish as well.

0.3. 3. The Cartan Basis (Fiber Bundles the easy way)

Another method for defining a basis frame was (essentially) pulled out of thin air by Cartan. It is subsumed that at the point p of a space there exists a basis frame **F** and the only evolutionary paths that are admissable to the theory are to be those where the basis frames in the nearby neighborhood can connected to the frame at point p by means of a *group* of continuous transformations (most often a Lie group). In effect no assumption is made about a metric in such a space. The fundamental constraint is the existence of a connection on the domain, not a metric. This restriction to a Lie group forces the spaces under consideration to be parallizeable manifolds. The functions that make up the basis frame may be functions of more than one parameter (usually k). However, the essential assumption is that the basis frame of functions has a global inverse. From this constraint,

$$[F] \circ \left[F^{-1}\right] = [F] \circ [G] = [G] \circ [F] = [1]\,, \tag{0.51}$$

it is possible to differentiate and apply the Leibniz rule to obtain

$$d\,[F] = [F] \circ [C(right)] = [F] \circ \{-\,[dG] \circ [F]\}. \tag{0.52}$$

Hence, given a Frame of functions, $[\mathbf{F}]$, that is an element of a group with an inverse, construct the (right) Cartan matrix of connection 1-forms according to the formula

$$\text{right Cartan connection } [C(right)] = \{-[dG] \circ [F]\} = \{[G] \circ [dF]\}. \tag{0.53}$$

The (right) Cartan matrix is now a matrix of 1-forms, $[C]$, constructed from the differentials of the functions that make up the matrix inverse, and every element is well defined. The important point is, again, that the differentials of any basis vector in the Cartan system is a linear combination of the basis vectors; i.e., the process of differentiation is closed.

It is apparent that there is another representation for the differential of the basis frame:

$$d[F] = [C(left)] \circ [F] = \{-[F] \circ [dG]\} \circ [F] \tag{0.54}$$

$$\text{left Cartan connection } [C(left)] = \{-[F] \circ \{[dG]\} = \{d[F] \circ [G]\} \tag{0.55}$$

The left Cartan connection is the similarity transform of the right Cartan connection, but is, indeed, distinct.

$$[G] \circ [C(left)] \circ [F] = [C(right)]. \tag{0.56}$$

The fact that there are two Cartan connection matrices is often confused in the literature, especially when the tensor index notation is used.

It is possible that the Cartan motivation was based upon situations when two spaces of equal dimension are diffeomorphically mapped: $\{x^k\} \Rightarrow \{\xi^a(x^m)\}; \{dx^k\} \Rightarrow \{d\xi^a(x^m, dx^k)\}$. Then it is natural to construct the Frame matrix $[F]$ in terms of the Jacobian matrix $[J(x^m)]$ of the mapping. The functions that make up the Frame matrix have arguments in terms of the independent variables of the initial state.

$$\left|dx^k\right\rangle \Rightarrow |d\xi^a\rangle = [\partial\xi^a(x^m)/\partial x^n] \circ |dx^n\rangle = [J^a_n(x^m)] \circ |dx^n\rangle \approx [F^a_n(x^m)] \circ |dx^n\rangle \tag{0.57}$$

It is apparent that the differentials on the final state $|d\xi^a\rangle$ can be used to retrodict the differentials $|dx^n\rangle$ on the initial state, by means if the inverse mapping, $[G^n_a(x^m)]$. However, even if the spaces are not diffeomorphically mapped, it is presumed that there exists a well defined matrix such that, locally, differentials $|dx^n\rangle$ on the initial state are linearly mapped into differential 1-forms $\varpi^a(x^m, dx^m)$ on the final state,

$$\left|dx^k\right\rangle \Leftarrow [G^n_a(x^m)] \circ |\varpi^a\rangle \tag{0.58}$$

$$\text{or } [F^a_m(x^m)] \circ |dx^m\rangle \Rightarrow |\varpi^a\rangle \tag{0.59}$$

or differential 1-forms $|d\xi^a\rangle$ on the final state have a preimage $|\sigma^m\rangle$ on the initial state.

$$[F^a_m(x^m)] \circ |\sigma^m\rangle \Rightarrow \left|d\xi^b\right\rangle \tag{0.60}$$

$$\text{or } |\sigma^m\rangle \Leftarrow [G^m_b(x^m)] \circ \left|d\xi^b\right\rangle \tag{0.61}$$

For an arbitrary Frame matrix, the induced 1-forms, $\sigma^k(x^m, dx^k)$, need not be exact.

A physical mechanism for defining the "position" or the existence of an absolute origin is lacking in the Cartan assumption. Furthermore it is not clear that the "origin" is located in the space of interest, but might actually be a point in some higher dimensional space, in the sense of a projection. For example, consider the projective plane where the perspective point is not in the plane at all. (The projective case will be studied in detail below.)

The column vector of differential 1-forms, $|\sigma^a\rangle$, representing the differential position vector in terms of the basis frame is well defined algebraically in terms of the matrix product. These Cartan assumptions lead to the exterior differential system:

$$d\,|\mathbf{R}^a_k\rangle - [\mathbf{F}^a_b] \circ \left|\sigma^b\right\rangle = 0 \tag{0.62}$$

$$d\,[\mathbf{F}^a_b] - [\mathbf{F}^a_c] \circ [\mathbf{C}^c_a] = 0. \tag{0.63}$$

The question remains, what determines the frame $[\mathbf{F}]$, and what determines the position vector $\mathbf{R}$? It is not clear that an arbitrary space admits a frame $[\mathbf{F}]$ which is an element of a continuous group, nor is it clear that the position vector can be defined uniquely. If the k dimensional space is euclidean, then the Cartan equations of a differential system defined above indeed may be satisfied. The common choice for the frame matrix is the basis of orthogonal unit vectors. The Cartan matrix in this case is anti-symmetric. This choice is a severe restriction on the topology of the domain.

Suppose that the Cartan differentials are exact, and all of the functions in the matrices are differentiable. Then take the exterior derivative of both equations to yield, (using the Poincare lemma which states that $dd\,|\mathbf{R}\rangle = 0$, and $dd\,[\mathbf{F}] = 0$):

$$\begin{aligned} dd\,|\mathbf{R}_k^a\rangle &= [\mathbf{F}] \circ \{d\,|\sigma\rangle + [\mathbf{C}]\,\hat{}\,|\sigma\rangle\} = [\mathbf{F}] \circ |\tau\rangle \Rightarrow 0. && (0.64) \\ |\tau\rangle &= \{d\,|\sigma\rangle + [\mathbf{C}]\,\hat{}\,|\sigma\rangle\} = \text{vector of Cartan Torsion 2-forms.} && (0.65) \end{aligned}$$

$$\begin{aligned} dd\,[\mathbf{F}_b^a] &= [\mathbf{F}] \circ \{d\,[\mathbf{C}] + [\mathbf{C}]\,\hat{}\,[\mathbf{C}]\} = [\mathbf{F}] \circ [\Theta] \Rightarrow 0. && (0.66) \\ [\Theta] &= \{d\,[\mathbf{C}] + [\mathbf{C}]\,\hat{}\,[\mathbf{C}]\} = \text{ matrix of Cartan Curvature 2-forms.} && (0.67) \end{aligned}$$

These eqautions are often called Cartan's structural equations. However, as the Frame matrix $[\mathbf{F}]$ is invertible (criteria of linear independence of the basis vectors), and it is presumed that the functions that make up the Frame are C2 differentiable, these equations imply that

$$|\tau\rangle = \text{vector of Cartan (right) Torsion 2-forms} \Rightarrow 0, \tag{0.68}$$

$$[\Theta] = \text{ matrix of Cartan (right) Curvature 2-forms} \Rightarrow 0. \tag{0.69}$$

These facts prove the theorem:

Theorem 0.5. *Invertible C2 Frame fields* $[\mathbf{F}]$ *have zero Cartan (right) Torsion and zero Cartan (right) Curvature.*

Note that the matrix dot $\circ$ product symbol has been replace by the wedge product symbol $\hat{}$ to remind one that the matrix elements of the connection are 1-forms, and the standard matrix product of matrix elements has to preserve the exterior product of factors.

0.3.1. Cartan Torsion, Topological Torsion, Affine Torsion

It needs to be made clear that when the Cartan Torsion 2-forms vanish, it is still possible that the closure 2-forms, $d\,|\sigma\rangle$, need not be zero. It is the closure 2-forms that determine the existence of Affine Torsion. In otherwords, the Cartan Torsion 2-forms consist of two parts, one part is the conventional Affine Torsion (closure) 2-forms, and the second part is due to the term $[\mathbf{C}]\,\hat{}\,|\sigma\rangle$. If (as in a space of absolute parallelism) the Cartan Torsion 2-forms vanish, then it is not necessarily true that the Affine Torsion 2-forms need to vanish. It is also possible that even though the Cartan Torsion 2-forms, $\{d\,|\sigma\rangle + [\mathbf{C}]\,\hat{}\,|\sigma\rangle\}$, vanish, the Topological Torsion need not be zero, as long as $\langle\sigma|\,\hat{}\,[\mathbf{C}]\,\hat{}\,|\sigma\rangle \neq 0$

Suppose the base space is not euclidean. Tensor analysis gives a clue that in Riemannian spaces, the space may be curved, such that if the operations specified above are carried out (relative to the Levi Cevita connection deduced from the metric), then the RHS of the second Cartan structural equation is not equal to zero. Instead, the second structural equation takes the form

$$\{d\,[\mathbf{C}] + [\mathbf{C}]\,\hat{}\,[\mathbf{C}]\} = [\mathbf{\Theta}]\,, \tag{0.70}$$

where $[\mathbf{\Theta}]$ is defined as the matrix of curvature 2-forms. The assumption is then made that the second structural equation is valid even though the Cartan matrix $[\mathbf{C}]$ is not a connection deducible from the metric. Then in 1922, Cartan further presumed that the first structural equation may also be not equal to zero but could be written in the form

$$\{d\,|\sigma\rangle + [\mathbf{C}]\,\hat{}\,|\sigma\rangle\} = \left|\mathbf{\Sigma}^k\right\rangle. \tag{0.71}$$

The non-zero RHS of this (first) structural equation is defined as the vector of Cartan Torsion 2-forms.

0.3.2. Justification of the Cartan Intuition.

How can these assertions above be justified? The answer follows by considering the space of interest to be a differentiable manifold which is a subspace of dimension n of a "Frame space" of higher dimension, say $n+p$. Use the Whitney embedding theorem if necessary to generate a "Frame space", for the euclidean space always admits a global basis as a $(n+p)\times(n+p) = N\times N$ matrix frame $[F]$. The subspace of dimension n, will be defined as the horizontal base space, and the subspace of dimension p will be defined as the vertical space of the fibers.

For purposes at the moment, how the Framed space is generated is not important. It is assumed that the Frame of fundtions exists, and can be partitioned. The exterior differential system is given by the equations:

$$d\,|R\rangle - [F]\circ|\sigma\rangle = 0 \tag{0.72}$$

$$d\,[F] - [F]\circ[C(right)] = 0. \tag{0.73}$$

The equations express the property of differential closure of any base vector of the Frame, and of the position vector to a point p in the space. However, these equations of differential closure of the base vectors may not be valid on the subspace of dimension n. Displacements along vectors of the subspace may not be constrained to produce closure. The differentials of the base or horizontal vectors that define the partitioned subspace need not be constrained to remain within the subspace.

In terms of the embedded global basis frame, $[F]$, all of Cartan theory can be developed by repeated application of the exterior derivative to the embedded exterior differential system, and then separating the resulting equations into their linearly independent partitioned parts. For example

$$d\,[F]\,\hat{}\,|\sigma\rangle + [F] \circ d\,|\sigma\rangle \Rightarrow [F] \circ \{[C]\,\hat{}\,|\sigma\rangle + d\,|\sigma\rangle\} = 0 \tag{0.74}$$

$$d\,[F]\,\hat{}\,[C] + [F] \circ d\,[C] \Rightarrow [F] \circ \{[C]\,\hat{}\,[C] + d\,[C]\} = 0. \tag{0.75}$$

As the basis frame $[F]$ is not zero, the linear independence condition requires that the bracket factors (which now consist of 2-forms) must vanish.

$$\{[C]\,\hat{}\,|\sigma\rangle + d\,|\sigma\rangle\} = 0 \tag{0.76}$$

$$\{[C]\,\hat{}\,[C] + d\,[C]\} = 0. \tag{0.77}$$

These are the two fundamental structural equations of Cartan (with 2-form coefficients) for Framed (for example, Eucildean) spaces of dimension $(n+p)$.

The next step is to partition the basis frame into n horizontal (or tangent or interior) column vectors, $\mathbf{e}_\alpha$, and p vertical (or exterior or orthogonal) column vectors, $\mathbf{n}_p$. Each of these vectors has $N = n + p$ components. The concept of magnitudes is not defined. These direction fields of these vectors could be modified (scaled) by arbitrary non-zero functions. The objective is to study the effects of the partition on the n dimensional subspace. The partition produces structural equations related to the subspace where, in general, on both the interior (and exterior) subspaces there are three (not two) fundamental structural equations involving 2-forms. In contrast to the Frame space, the partitioned 2-form equations of structure on the subspaces are such that the RHS of each structural equation is not necessarily zero. It is these non-zero features that define the concepts of curvature, and two species of Cartan Torsion, on the subspace. Cartan, evidently from intuition, developed two of the structural equations involving curvature and the concept of Cartan Torsion of the kind from his experience with exterior differential forms. The structural equation involving curvature could be deduced from tensor analysis built on a metric on the subspace, but the structural equation for what has become known as Cartan Torsion was not deducible from differentiable processes on the metric of the subspace. Using the Levi-Civita method, the concept of connection coefficients could be deduced from differential processes on the subsumed metric, but such connection coefficients, Γ^i_{jk}, (these Levi-Civita 3-index symbols do not transform as a tensor) are symmetric in the lower pair of indices. The concept of affine torsion (which forms part of the Cartan Torsion) presumes that a connection with anti-symmetric components can be imposed on a domain in some manner independent from the metric. As will be demonstrated below, when surfaces are defined parametrically, such affine torsion does not exist for C2 differentiable functions.

For a long period of time after Cartan's discovery, the concept of a space with torsion was ignored as an oddity. As Brillouin remarked [10], "If one does not admit the symmetry of the coefficients of Γ^i_{jk} one obtains the twisted spaces of Cartan, spaces which have been scarcely used in physics to the present, but which seem to be called to an important role." Kondo and Bilby applied the concepts of spaces with affine torsion to the study of dislocation defects in crystals. However, there is another type of Cartan torsion, defined herein as the Cartan torsion of circulation, or twist-expansion, that exists even when the affine torsion vanishes. This second concept of torsion is related to "expansions" or extensions in the adjoint or normal field directions. For example, does a cylindrical shape twist as it expands along the axis of rotation? A spring in the form of a helix exhibits this phenomena, as do many materials that are chiral. This second type of Torsion can appear in parametrically defined surfaces (although affine torsion does not) and depends upon the fact the the adjoint field need not be scaled by a factor related to a euclidean norm.

It is important to realized that none of the basis vectors that make up a frame field are necessarily "normalized". In fact, the imposition of global normalization is a constraint that eliminates certain topological features, or limits the domain of definition. For example, the normal field of a surface must vanish on a cuspoidal edge of regression.

What was missed by the intuitive technique, and that which is exemplified by the constructive method exploited herein, is that the concepts of Cartan Torsion and Cartan Curvature can be computed in two ways: the first, more familiar method, is in terms of second order *differential* processes acting on the interior partitions of the Cartan matrix for a given basis frame; the second method is by *algebraic* processes on exterior partitions of the Cartan matrix for a given basis frame. In addition, a third structural equation of torsion 2-forms can be defined, in a sense dual to those torsion 2-forms deduced from the assumption of a position vector.

To exemplify these concepts consider the basis frame written in partitioned form as:

$$[F] = \begin{bmatrix} \mathbf{e}_\alpha & \mathbf{e}_\beta & .. & \mathbf{n} \end{bmatrix} = \begin{bmatrix} \mathbf{e}^1_\alpha & \mathbf{e}^1_\beta & ... & \mathbf{n}^1 \\ \mathbf{e}^2_\alpha & \mathbf{e}^2_\beta & ... & \mathbf{n}^2 \\ ... & ... & ... & ... \\ \mathbf{e}^n_\alpha & ... & ... & \mathbf{n}^n \end{bmatrix}. \tag{0.78}$$

$$\text{The contravariant "horizontal" vector(s) } |\mathbf{e}_\alpha\rangle = \left| \begin{array}{c} \mathbf{e}^1_\alpha \\ \mathbf{e}^2_\alpha \\ ... \\ \mathbf{e}^{n1}_\alpha \end{array} \right\rangle \tag{0.79}$$

$$\text{The contravariant "vertical" vector(s) } |\mathbf{n}\rangle = \left| \begin{array}{c} \mathbf{n}^1 \\ \mathbf{n}^2 \\ ... \\ \mathbf{n}^n \end{array} \right\rangle \tag{0.80}$$

The differential position vector expanded in terms of the partitioned frame as:

$$d\mathbf{R} = [F] \circ \left| \begin{array}{c} \sigma^\alpha \\ \sigma^\beta \\ ... \\ \omega \end{array} \right\rangle = \begin{bmatrix} \mathbf{e}_\alpha & \mathbf{e}_\beta & .. & \mathbf{n} \end{bmatrix} \circ \left| \begin{array}{c} \sigma^\alpha \\ \sigma^\beta \\ ... \\ \omega \end{array} \right\rangle = \mathbf{e}_\alpha \sigma^\alpha + \mathbf{e}_\beta \sigma^\beta + ... + \mathbf{n}\omega \tag{0.81}$$

(The method of constructing or defining the basis frame is deferred until the next section.) This explicit example considers only one vertical or exterior basis vector, $\mathbf{n}$, which is constructed to be orthogonal to the horizontal vectors, $\mathbf{e}_\alpha$. However there could be many such vertical vectors. The general and corresponding partition of the Cartan connection matrix $[C]$ of 1-forms becomes

$$d\,[F]=[F]\circ[C]=[F]\circ\begin{bmatrix}\Gamma_{\alpha}^{\alpha} & \Gamma_{\beta}^{\alpha} & ... & \gamma^{\alpha}\\ \Gamma_{\alpha}^{\beta} & \Gamma_{\beta}^{\beta} & ... & \gamma^{\beta}\\ ... & ... & ... & ...\\ h_{\alpha} & h_{\beta} & ... & \Omega\end{bmatrix}=[F]\circ\begin{bmatrix}[\Gamma] & |\gamma\rangle\\ \langle h| & \Omega\end{bmatrix} \tag{0.82}$$

The $n\times n$ submatrix of 1-forms represented by $[\Gamma]$ will be defined as the interior connection coefficients on the base space of horizontal vectors. Note that these 1-forms range over the n interior variables, and the p exterior variables (or external parameters). There is still some ambiguity for the choice of the basis frame $[F]$ on the N=n+p dimensional Framed space, but a constructive procedure will be developed below in terms of a projective geometry based in the existence of a global 1-form of Action. It is here where contact with physical theories can be made.

To proceed, presume that the basis set $[F]$ is given. Then rewrite the structural equations for the euclidean space in terms of the partition. The equations for the exterior differential system can be partitioned into two parts: the first part relating to the k horizontal vectors (of $k+p$ components).

$$d\mathbf{e}_k=\mathbf{e}_k\Gamma_j^k+\mathbf{n}h_j \tag{0.83}$$

and a second part relating to (in this example) the $p=1$ vertical vector(s), $\mathbf{n}$,

$$d\mathbf{n}=\mathbf{e}_k\gamma^k+\mathbf{n}\Omega. \tag{0.84}$$

The differential position vector can be partitioned as

$$d\mathbf{R}=\mathbf{e}_k\sigma^k+\mathbf{n}\omega. \tag{0.85}$$

The components of the partitioned Cartan right connection matrix differential 1-forms.

When the horizontal vector space and the vertical vector space are transversal in the sense that $\mathbf{e}_k\circ\mathbf{n}=0$, the interpretation of the various coefficients becomes more transparent. The factor Ω represents the change of $\mathbf{n}$ in the direction of $\mathbf{n}$, while the factor γ^k represents the change of $\mathbf{n}$ in the direction of the $\mathbf{e}_k$. The factor h_j represents the change in the $\mathbf{e}_k$ in the direction of $\mathbf{n}$. Note that if the adjoint vector $\mathbf{n}$ is constrained to have no change in the direction of $\mathbf{n}$, then the factor Ω vanishes. When the position vector is constrained to remain in the "surface" of horizontal (tangent vectors), as in Gauss Weingarten theory, the 1-form ω must vanish.

Assuming that the functions that make up the basis vectors and the position vector are C1 differentiable, then the exterior derivative of these three sets of equations must vanish. Most of the features of the theory can be developed by successive application of the exterior derivative to the above equations, followed by algebraic substitution of the closure relations defined by the exterior differential system. As the Frame matrix and the Cartan matrix are partitioned relative to the horizontal (tangent, or interior, or associated) vectors $\mathbf{e}_k$ and the vertical (or exterior, or orthogonal) vectors, $\mathbf{n}$, the Poincare lemma breaks up into linearly independent factors, each of which must vanish separately. The results are given by structural equations (sums over repeated up and down indices):

$$dd\mathbf{R} = \mathbf{e}_k\{d\left|\boldsymbol{\sigma}^k\right\rangle + [\boldsymbol{\Gamma}^k_m]\hat{}\ |\boldsymbol{\sigma}^m\rangle + \left|\boldsymbol{\gamma}^k\hat{}\omega\right\rangle\} + \mathbf{n}\{d\omega + \Omega\hat{}\omega + \langle\mathbf{h}_m|\ \hat{}\ |\boldsymbol{\sigma}^m\rangle\} = 0 \quad (0.86)$$

$$dd\mathbf{e}_j = \mathbf{e}_k\{d[\boldsymbol{\Gamma}^k_j] + [\boldsymbol{\Gamma}^k_m]\hat{}[\boldsymbol{\Gamma}^m_j] + \left|\boldsymbol{\gamma}^k\right\rangle\ \hat{}\ \langle\mathbf{h}_j|\} + \mathbf{n}\{d\langle\mathbf{h}_j| + \Omega\hat{}\ \langle\mathbf{h}_j| + \langle\mathbf{h}_m|\ \hat{}[\boldsymbol{\Gamma}^m_j]\} = 0 \quad (0.87)$$

$$dd\mathbf{n} = \mathbf{e}_k\{d\left|\boldsymbol{\gamma}^k\right\rangle + [\boldsymbol{\Gamma}^k_m]\hat{}\ |\boldsymbol{\gamma}^m\rangle + \left|\boldsymbol{\gamma}^k\hat{}\Omega\right\rangle\} + \mathbf{n}\{d\Omega + \Omega\hat{}\Omega + \langle\mathbf{h}_m|\ \hat{}\ |\boldsymbol{\gamma}^m\rangle\} = 0. \quad (0.88)$$

Each of the bracket {} factors must vanish, for by hypothesis the basis vectors are linearly independent (and non-zero). The outcome (due to the partition) is six structural equations, three related to the "interior domain" of the partition and three related to the "exterior" domain of the partition. Each bracket factor is composed of 2-forms.

Consider the three interior structural equations rewritten as

$$d\left|\boldsymbol{\sigma}^k\right\rangle + [\boldsymbol{\Gamma}^k_m]\hat{}\ |\boldsymbol{\sigma}^m\rangle = \omega\hat{}\ \left|\boldsymbol{\gamma}^k\right\rangle \equiv \left|\boldsymbol{\Sigma}^k\right\rangle \quad (0.89)$$

$$d[\boldsymbol{\Gamma}^k_j] + [\boldsymbol{\Gamma}^k_m]\hat{}[\boldsymbol{\Gamma}^m_j] = -\left|\boldsymbol{\gamma}^k\right\rangle\ \hat{}\ \langle\mathbf{h}_j| \equiv \left[\boldsymbol{\Theta}^k_j\right] \quad (0.90)$$

$$d\left|\boldsymbol{\gamma}^k\right\rangle + [\boldsymbol{\Gamma}^k_m]\hat{}\ |\boldsymbol{\gamma}^m\rangle = \Omega\hat{}\ \left|\boldsymbol{\gamma}^k\right\rangle \equiv \left|\boldsymbol{\Psi}^k\right\rangle \quad (0.91)$$

On the left hand side of each structural equation, given the interior connection coefficients, $[\boldsymbol{\Gamma}^k_m]$, differential processes are used to construct the vector or matrix arrays of 2-forms. The vector array $\left|\mathbf{\Sigma}^k\right\rangle$ is defined as the translation *Cartan* torsion 2-forms on the interior, or horizontal vector, subspace. These objects have been used to analyze the concept of dislocation defects in crystals. The matrix array $\left[\mathbf{\Theta}^k_j\right]$ is defined as the curvature 2-forms on the interior, or horizontal vector, subspace . These concepts appear in the classical literature of Cartan (and others). What is new from the construction presented herein, is that these arrays of subspace 2-forms also can be computed algebraically, without the need for another differentiation. Moreover, the third and new interior structural equation yields another vector array of rotational *expansion-twist* torsion 2-forms, $\left|\mathbf{\Psi}^k\right\rangle$. This latter array can represent the concept of disclinations in liquid crystals. Each array of torsion 2-forms depends upon distinct and different 1-forms, ω and Ω. If the 1-form ω vanishes (as it does for all parametrically described surfaces) then there does not exist any two forms of affine torsion on the subspace. If the 1-form Ω vanishes then there does not exist any two forms of expansion-twist torsion on the subspace.

Consider the case of 1 vertical direction field and N horizontal direction fields. Orthogonality requires that the vertical direction field be proportional to Grassman adjoint vector field of the N horizontal vectors. If the scaling factor of the vertical direction field is chosen to be $\pm 1/(\mathbf{n}\circ\mathbf{n})^{(1/2)}$, then the 1-form Ω on the N+1 space Frame Field vanishes, but the determinant of the Frame is $(\mathbf{n}\circ\mathbf{n})$, which is not necessarily a constant. If the scaling factor of the vertical direction field is chosen to be $1/(\mathbf{n}\circ\mathbf{n})$, then the 1-form Ω is not zero, and the determinant of the Frame matrix is globally 1. The bottom line is that if the scaling factor is chosen such that $\Omega = 0$, then the expansion-twist torsion on the subspace vanishes.

One physical objective of this article is to associate the 2-forms $\left|\mathbf{\Sigma}^k\right\rangle$ with shears of (affine) translation of parallel planes, and the 2-forms $\left|\mathbf{\Psi}^k\right\rangle$ with shears of expansion twists. A second physical objective is to associate the structural equation

$$d[\mathbf{\Gamma}^k_j] + [\mathbf{\Gamma}^k_m]\hat{}[\mathbf{\Gamma}^m_j] = -\left|\boldsymbol{\gamma}^k\right\rangle \hat{} \langle\mathbf{h}_j| \tag{0.92}$$

with an extension of the Einstein field equations that would be valid on non-Riemannian spaces.

$$[\mathbf{G}^k_j] \Longleftarrow d[\mathbf{\Gamma}^k_j] + [\mathbf{\Gamma}^k_m]\hat{}[\mathbf{\Gamma}^m_j] = -\left|\boldsymbol{\gamma}^k\right\rangle \hat{} \langle\mathbf{h}_j| \Rightarrow \left[\mathbf{T}^k_j\right]. \tag{0.93}$$

An important feature of the Einstein Ansatz is that the divergence of the Ricci tensor for a metric space is zero, hence the divergence of the stress energy tensor must be zero, which is pleasing on physical grounds. What is remarkable herein, is that when the Cartan matrix of connection 1-forms is antisymmetric, (the case of an orthogonal frame field) the exterior derivative of the matrix of curvature 2-forms, $\left[\boldsymbol{\Theta}_j^k\right]$, vanishes (see below), which implies that both the differential construction of the curvature 2-forms and the algebraic construction of the curvature 2-forms are closed. Locally there exists a set of 1-forms (the potentials) that generate the curvature 2-forms. The topology of the subspace will be dictated by the cohomology of the curvature 2-forms. From the deRham theorems, this closed integrals are quantized (have values whose ratios are rational). This result does not depend upon an interior metric, but does depend upon the group structure of the connection.

In the formula above, on the left is the construction based upon interior geometry of the "connection" and differential processes, and on the right is the stress energy tensor computed algebraically from the exterior features of the embedded system.

The three exterior structural equations are

$$d\omega + \Omega\hat{}\omega = -\langle \mathbf{h}_m | \hat{} | \boldsymbol{\sigma}^m \rangle \equiv L \tag{0.94}$$

$$d\langle \mathbf{h}_j | + \Omega\hat{}\langle \mathbf{h}_j | = -\langle \mathbf{h}_m | \hat{}[\boldsymbol{\Gamma}_j^m] \equiv \langle \mathbf{J} | \tag{0.95}$$

$$d\Omega + \Omega\hat{}\Omega = -\langle \mathbf{h}_m | \hat{} | \boldsymbol{\gamma}^m \rangle \equiv S \tag{0.96}$$

The physical significance of these structural equations has yet to be determined. However, it is important to recognize that the 2-form S is always exact.

The method developed above indicates that there exist a number of exact 2-form structures. Recall that each exact 2-form is an evolutionary deformable integral invariant, that can be used to establish a topological conservation law. These exact 2-forms can be read off from the structural equations:

$$[\boldsymbol{\Gamma}_m^k]\hat{}\,|\boldsymbol{\sigma}^m\rangle - \omega\hat{}\left|\boldsymbol{\gamma}^k\right\rangle = -d\left|\boldsymbol{\sigma}^k\right\rangle \tag{0.97}$$

$$[\boldsymbol{\Gamma}_m^k]\hat{}[\boldsymbol{\Gamma}_j^m] + \left|\boldsymbol{\gamma}^k\right\rangle\,\hat{}\,\langle \mathbf{h}_j| = -d[\boldsymbol{\Gamma}_j^k] \tag{0.98}$$

$$[\boldsymbol{\Gamma}_m^k]\hat{}\,|\boldsymbol{\gamma}^m\rangle - \Omega\hat{}\left|\boldsymbol{\gamma}^k\right\rangle = -d\left|\boldsymbol{\gamma}^k\right\rangle \tag{0.99}$$

$$\Omega\hat{}\omega + \langle \mathbf{h}_m| \hat{} |\boldsymbol{\sigma}^m\rangle = -d\omega \tag{0.100}$$

$$\Omega\hat{}\, \langle \mathbf{h}_j| + \langle \mathbf{h}_m| \hat{} [\boldsymbol{\Gamma}_j^m] = -d\, \langle \mathbf{h}_j| \tag{0.101}$$

$$\Omega\hat{}\Omega + \langle \mathbf{h}_m| \hat{} |\boldsymbol{\gamma}^m\rangle = -d\Omega \tag{0.102}$$

It is to be observed that the structural equations above represent exterior differential systems that define topological properties of the domain.

0.4. 4. The Bianchi identities and other conserved 3-forms

An application of the exterior derivative to the equations of the set of section 3 leads to constraints on systems of 3-forms as further necessary conditions. Exterior differentiation of each of the brackets yields the system of 3-form equations:

$$d\,|\boldsymbol{\Sigma}\rangle + [\boldsymbol{\Gamma}]\hat{}\, |\boldsymbol{\Sigma}\rangle = [\boldsymbol{\Theta}]\, \hat{}\, |\boldsymbol{\sigma}\rangle \tag{0.103}$$

$$d\,[\boldsymbol{\Theta}] + [\boldsymbol{\Gamma}]\hat{}\, [\boldsymbol{\Theta}] = [\boldsymbol{\Theta}]\, \hat{}\, [\boldsymbol{\Gamma}] \tag{0.104}$$

$$d\,|\boldsymbol{\Psi}\rangle + [\boldsymbol{\Gamma}]\hat{}\, |\boldsymbol{\Psi}\rangle = [\boldsymbol{\Theta}]\, \hat{}\, |\gamma\rangle \tag{0.105}$$

with similar expressions for the exterior curvature components of the structural equations (the parts that depend upon Ω,not Γ). The 3-form equations lead to exact 3-form structures and topological deformation invariants in terms of the integrals of

$$[\boldsymbol{\Gamma}]\hat{}\, |\boldsymbol{\Sigma}\rangle - [\boldsymbol{\Theta}]\, \hat{}\, |\boldsymbol{\sigma}\rangle = -d\,|\boldsymbol{\Sigma}\rangle \tag{0.106}$$

$$[\boldsymbol{\Gamma}]\hat{}\, [\boldsymbol{\Theta}] - [\boldsymbol{\Theta}]\, \hat{}\, [\boldsymbol{\Gamma}] = -d\,[\boldsymbol{\Theta}] \tag{0.107}$$

$$[\boldsymbol{\Gamma}]\hat{}\, |\boldsymbol{\Psi}\rangle - [\boldsymbol{\Theta}]\, \hat{}\, |\gamma\rangle = -d\,|\boldsymbol{\Psi}\rangle \tag{0.108}$$

The equation (4.2) involving $d\,[\boldsymbol{\Theta}]$ is the equivalent to the Bianchi identities in classical tensor analysis.

The process of repetitive exterior differentiation of these expressions can proceed until the totality of necessary compatibility conditions on a system of

N+1 forms is generated. From then on, the process of exterior differentiation leads to no new information.

0.5. 5. Some Examples: Parametric Surfaces:

Parametric surfaces may be viewed as a map from N-1 parameters into a space (often euclidean) of dimension N. The position vector $\mathbf{R}(u,v)$ to the surface has N components with each component a function of N-1 parameters. The partial derivatives of the position vector form a set of N-1 linearly independent (tangent) vectors $\mathbf{e}_k$ of dimension N. There exists a unique adjoint vector, $\mathbf{N}$, algebraically constructed with components proportional to the (N-1) by (N-1) determinants of the "tangent" vector components. The N contravariant column vectors form a basis frame, $[F]$. No concept of distance (metric) has been established. Each column vector of the basis frame could be multiplied by an arbitrary function. In particular, the adjoint vector, $\mathbf{N}$, could be scaled by an arbitrary function of the parameters, $\mathbf{n} \rightarrow \mathbf{N}/\rho(u,v)$. In such a case the determinant of the basis frame becomes a complicated algebraic expression in the components of the tangent vectors. For a Monge surface, the Frame matrix is computed from the position vector, $\mathbf{R}(u,v) = [u, v, Z(u,v)], as$

$$[F] = \begin{bmatrix} 1 & 0 & \partial Z(u,v)/\partial u/\rho(u,v) \\ 0 & 1 & \partial Z(u,v)/\partial v/\rho(u,v) \\ (\partial Z(u,v)/\partial u & \partial Z(u,v)/\partial v & 1/\rho(u,v) \end{bmatrix}, \tag{0.109}$$

The determinant of the Frame matrix is

$$\det [F] = \{1 + (\partial Z(u,v)/\partial u)^2 + (\partial Z(u,v)/\partial v)^2\}/\rho(u,v) \tag{0.110}$$

and is never zero for real functions, and $\rho(u,v) > 0$. Hence the Cartan matrix exists globally. The 1-forms that make up the differential of the position vector are given by the expression:

$$\left[F^{-1}\right] \circ \left| \begin{matrix} dx \\ dy \\ ... \\ dz \end{matrix} \right\rangle \Rightarrow \left| \begin{matrix} \sigma^\alpha \\ \sigma^\beta \\ ... \\ \omega \end{matrix} \right\rangle = \left| \begin{matrix} du \\ dv \\ ... \\ 0 \end{matrix} \right\rangle . \tag{0.111}$$

Note that the 1-form, $\omega \Rightarrow 0$, vanishes identically for any parametrized surface (Monge or otherwise). Hence, the affine torsion 2-forms,

$$-\omega\hat{}\left|\boldsymbol{\gamma}^k\right\rangle \equiv \left|\boldsymbol{\Sigma}^k\right\rangle = |\mathbf{0}\rangle$$

vanish for all parametric surfaces. Dislocation defects do NOT admit a description in terms of parametric surfaces.

Ruled surfaces are generated by "straight" lines, and are special cases of parametric surfaces. Therefore, ruled surfaces can have rotational twisted torsion, but not translational affine torsion.

Consider the one form ϖ constructed from the components of the adjoint field: $\varpi = \{N_x dx + N_y dy + N_z dz\}/\rho \equiv B_u(u,v)du + B_v(u,v)dv$. As the pullback form only involves two variables, there always exists an integrating factor such that the components of the adjoint field are proportional to a gradient field. (This result is not true in higher dimensions.)

If the scaling function for the adjoint field is chosen such that

$$\rho(u,v) = [+(N1)^2 + (N2)^2 + ...]^{1/2} \tag{0.112}$$

then the parametrized surfaces the expansion-twist 1-form vanishes, $\Omega(u,v,du,dv) \Rightarrow 0$. It follows that

$$d\left|\boldsymbol{\gamma}^k\right\rangle + [\boldsymbol{\Gamma}_m^k]\hat{}\,|\boldsymbol{\gamma}^m\rangle = -\Omega\hat{}\left|\boldsymbol{\gamma}^k\right\rangle \equiv \left|\boldsymbol{\Psi}^k\right\rangle \Rightarrow 0. \tag{0.113}$$

The Torsion 2-forms of the second type, $\left|\boldsymbol{\Psi}^k\right\rangle$, vanish for parametric representations which normalize the surface adjoint field with the quadratic norm. If the surface vector is everywhere quadratically normalizable (without expansion), then parametric methods cannot describe disclination defects.

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Parametric N-1surfaces in N space support a frame matrix which is always affine Torsion free, and can be adjusted such as to have zero adjoint or expansion with twist Torsion. Not only are the affine Torsion coefficients (which depend upon the existence of little ω - see notes) equal to zero for parametric surfaces, but also the adjoint or twisted Torsion coefficients (which depend upon the existence of $\Omega(u,v,du,dv)$ in the right Cartan connection - see notes) can be made to vanish. If the scaling (or renormalization) factor of the adjoint normal field is equal to $1/\pm$(square root of the sum of squares of the components of the normal field), then $\Omega(u,v,du,dv)$ is zero and the "rotational torsion" terms are zero. However, the Frame determinant depends upon the sign of the square root chosen, and the system has two distinct solutions not connected components. If the renormalization factor is equal to 1/(sum of squares of the components of the normal field), then $\Omega(u, v, du, dv)$ is not zero, but the Frame matrix is unimodular with determinant equal to plus 1 globally. There is one connected component.

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Tangential developable in 3D

Scrolls in 3D

1. IMPLICIT SURFACES and Cartan's 1-form of Action.

LAter.

1.1. References

Leon Brillouin, "Tensors in Mechanics and Elasticity", Academic Press, NY, 1964 p.93

See
http://www22.pair.com/csdc/pdf/parametric.pdf
for examples of the Repere Mobile used for parametric surfaces.
See
http://www22.pair.com/csdc/pdf/implinor.pdf
for examples of the Repere Mobile used for implicit surfaces.