

RMK Rewrite of /02/03/90

FERMIONS AND BOSONS ON E8

A recent effort to understand torsion stabilized energy surfaces of negative Gaussian curvature led this author to note a remarkable correspondence between two group structures on E8. Consider the set of hypercomplex numbers (Q,I,J,K) that obey the rules:

$$Q*Q = 1; \quad I*I = -1, \quad J*J = -1, \quad K*K = -1 \quad 1.$$

$$I*J = K, \quad J*K = I, \quad K*I = J$$

$$Q*I = I*Q, \quad Q*J = J*Q, \quad Q*K = J*Q$$

$$I*J*K = -1$$

The symbols (Q,I,J,K) and their negatives can be used to form two group structures on E8. The most famous is the quaternion group whose four symbols obey the additional rules of anti-commutivity.

$$[I*J + J*I] = [J*K + K*J] = [K*I + I*K] = 0 \quad 2.$$

Physicists recognize the pure imaginary quaternions, iI, iJ, iK as the Pauli spin matrices. The real numbers, complex numbers and quaternions are known to be the only sets capable of forming an associative algebra, but quaternions lose the property of commutivity. Through the connection to Pauli spin matrices, and the anti-commutation rule, quaternions are the natural group representation for Fermions. The traceless representations of the triple I,J,K are associated with completely anti-symmetric sets, the Fermions.

What surprised me was the fact that the hypercomplex symbols that satisfy the "angular momentum rules", 1, also may be used to construct another group on E8. The only difference is that these new symbols satisfy the commutation rules:

$$\{I*J - J*I\} = \{J*K - K*J\} = \{K*I - I*K\} = 0 \quad 3.$$

The group is abelian, but the algebra built on the four symbols is not associative. The traceless representations for the triple I,J,K are associated with completely symmetric sets, the Bosons.

The striking idea is that while both Bosons and Fermions are hypercomplex numbers, Bosons are commutative but not associative, where Fermions are associative, but not commutative.

This idea that Fermions and Bosons involve a subtle difference on two group representations of E8 helps clarify for me the super-symmetry arguments that E8 x E8 can be used to form useful representations for a system that has a mixture of Bosons and Fermions

The two group tables for E8 are given below.

Q	-Q	I	J	K	-I	-J	-K
-Q	Q	-I	-J	-K	I	J	K
I	-I	-Q	K	-J	Q	-K	J
J	-J	-K	-Q	I	K	Q	-I
K	-K	J	-I	-Q	-J	I	Q
-I	I	Q	-K	J	-Q	K	-J
-J	J	K	Q	-I	-K	-Q	I
-K	K	-J	I	Q	J	-I	-Q

E8 FERMION - QUATERNION (NON ABELIAN) GROUP TABLE

Q	-Q	I	J	K	-I	-J	-K
-Q	Q	-I	-J	-K	I	J	K
I	-I	-Q	K	J	Q	-K	-J
J	-J	K	-Q	I	-K	Q	-I
K	-K	J	I	-Q	-J	-I	Q
-I	I	Q	-K	-J	-Q	K	J
-J	J	-K	Q	-I	K	-Q	I
-K	K	-J	-I	Q	J	I	-Q

E8 BOSON (ABELIAN) GROUP TABLE

A scalar representation is available for the Boson set. Consider the symbol $e^*e = i$, where e is the square root of the imaginary i .

Then the symbols

$$1, \quad e^*e, \quad (e^*e^*e + e)/2, \quad \text{and} \quad (e^*e^*e - e)/2$$

form a basis set for the Bosons. The quadratic and cubic forms are strikingly connected to the limiting format of intrinsic space curves, whose second order complex equation can be put into the form

$$d f/ds = t/2 - (t + k) f + t/2 f^2$$

In the limiting case where curvature k dominates ($t = 0$), the equation becomes that of simple harmonic oscillator with a quadratic potential. When torsion dominates ($k = 0$) the intrinsic equation for a space curve becomes a cubic with the standard fold catastrophe.

