E8 Rewrite on May 25 1999 from notes of 4/18/86 and May 1990

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I. FERMIONS AND BOSONS ON E8

An effort to understand torsion stabilized energy surfaces of negative Gaussian curvature led this author to note a remarkable correspondence between two group structures on E8. Consider the set of hypercomplex numbers (Q, I, J, K) that obey the rules:

$$Q \cdot Q = 1, \qquad I \cdot I = -1, \qquad J \cdot J = -1, \qquad K \cdot K = -1$$
$$I \cdot J = K, \qquad J \cdot K = I, \qquad K \cdot I = J,$$
$$Q \cdot I = I \cdot Q, \quad Q \cdot J = J \cdot Q, \quad Q \cdot K = J \cdot Q, \tag{1}$$
$$I \cdot J \cdot K = -1$$

A. The Fermions

The symbols (Q, I, J, K) and their negatives can be used to form two group structures on E8. The most famous is the quaternion group whose four symbols obey the additional rules of anti-commutivity.

$$[I \cdot J + J \cdot I] = 0, \quad [J \cdot K + K \cdot J] = 0, \quad [K \cdot I + I \cdot K] = 0 \tag{2}$$

$$[I \cdot J - J \cdot I] = 2K, \quad [J \cdot K - K \cdot J] = 2I, \quad [K \cdot I - I \cdot K] = 2J \tag{3}$$

Physicists recognize the pure imaginary quaternions, iI, iJ, iK as the Pauli spin matrices. The real numbers, complex numbers and quaternions are known to be the only sets capable of forming an associative algebra, but quaternions lose the property of commutivity. Through the connection to Pauli spin matrices, and the anti-commutation rule, quaternions are the natural group representation for Fermions. The traceless representations of the triple I, J, K are associated with completely anti-symmetric sets, the Fermions.

B. The Bosons

What surprised me was the fact that the hypercomplex symbols that satisfy the "angular momentum rules" of cyclic permutation given in the first equation, also may be used to

construct another group on E8. The only difference is that these new symbols satisfy the commutation rules:

$$[I \cdot J - J \cdot I] = 0, \ \ [J \cdot K - K \cdot J] = 0, \ \ [K \cdot I - I \cdot K] = 0 \tag{4}$$

$$[I \cdot J + J \cdot I] = 2K, \ \ [J \cdot K + K \cdot J] = 2I, \ \ [K \cdot I + I \cdot K] = 2J \tag{5}$$

The group is abelian, but the algebra built on the four symbols is not associative. The traceless representations for the triple I,J,K are associated with completely symmetric sets, the Bosons.

Claim 1 *The striking idea is that while both Bosons and Fermions are hypercomplex numbers, Bosons are commutative but not associative, where Fermions are associative, but not commutative.*

This idea that Fermions and Bosons involve a subtle difference on two group representations of E8 helps clarify for me the super-symmetry arguments that E8 x E8 can be used to form useful representations for a system that has a mixture of Bosons and Fermions

C. The group tables

1. E8 Fermion non-abelian but associative group table

$$
\begin{array}{cccccccc}
Q & -Q & I & J & K & -I & -J & -K \\
-Q & Q & -I & -J & -K & I & J & K \\
I & -I & -Q & K & -J & Q & -K & J \\
J & -J & -K & -Q & I & K & Q & -I \\
K & -K & J & -I & -Q & -J & I & Q \\
-I & I & Q & -K & J & -Q & K & -J \\
-J & J & K & Q & -I & -K & -Q & I \\
-K & K & -J & I & Q & J & -I & Q \\
\end{array}
\tag{6}
$$

3

$$
\begin{array}{cccccccc}
Q & -Q & I & J & K & -I & -J & -K \\
-Q & Q & -I & -J & -K & I & J & K \\
I & -I & -Q & K & J & Q & -K & -J \\
J & -J & K & -Q & I & -K & Q & -I \\
K & -K & J & I & -Q & -J & -I & Q \\
-I & I & Q & -K & -J & -Q & K & J \\
-J & J & -K & Q & -I & K & -Q & I \\
-K & K & -J & -I & Q & J & I & -Q
\end{array}
\tag{7}
$$

A scalar representation is available for the Boson set. Consider the symbol $e \cdot e = i$, where e is the square root of the imaginary i. Then the symbols

$$
1, \quad e \cdot e, \quad (e \cdot e \cdot e + e)/2, \quad (e \cdot e \cdot e - e)/2
\tag{8}
$$

form a basis set for the Bosons $\{Q, I, J, K\}$. The quadratic and cubic forms are (strikingly) connected to the limiting format of intrinsic space curves, whose complex equation can be put into the Ricattic form $df/ds = -t/2 - kf + t/2f^2$.

In the limiting case where curvature k dominates $(t \Rightarrow 0)$, the second order equation, deduced from the Ricattic equation, becomes that of simple harmonic oscillator with a quadratic potential. When torsion dominates $(k \Rightarrow 0)$ the intrinsic equation for a space curve becomes a cubic with the standard fold catastrophe.