Maxwell-Faraday Equations are the same in N $\geq$ 4 dimensions

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1 1,2,3,4.. and the Maxwell Faraday Induction in all dimensions ≥ 4

The Maxwell Faraday system of PDE's forms a nested set in the n $\geq$ 4 space of all PDE's associated with the postulate of potentials, a differential constraint that states that $F-dA=0$. The construction below demonstrates the extraordinary universality of the Maxwell-Faraday induction principle, not just to EM theory, but also to all thermodynamic systems that can be encoded in terms of a 1-form of Action, ain any number of geometric dimensions.

1. Consider the ordered sequence of integers including zero
$$k=\{0;1,2,3,4,5...\} \tag{1}$$
2. Consider the ordered sequence of N independent variables
$$\xi^k=\{t;x,y,z,s...\} \tag{2}$$
3. Map the independent variables into the ordered sequence of N functions
$$A_k=\{\phi(\xi^k);A_1(\xi^k),A_2(\xi^k),A_3(\xi^k),\Lambda(\xi^k)...\}. \tag{3}$$
4. The ordered set of functions will be constrained to behave as a covariant vector field with respect to diffeomorphisms of the independent variables. Consider the topological constraint generated by the exterior differential system
$$\text{The topological constraint} \quad F-dA=0. \tag{4}$$
5. Construct the exterior differential 1-form of Action per unit charge
$$\begin{aligned} Action(\xi^k,d\xi^k) &= A_m(\xi^k)d\xi^m && (5)\\ &= -\phi(\xi^k)dt+A_1(\xi^k)dx+A_2(\xi^k)dy+A_3(\xi^k)dz+\Lambda(\xi^k)ds... && (6)\end{aligned}$$

6. Compute the exterior derivative of the *Action*,as an exact 2-form, also using ordered pairs for the collective indices $\{mn\}$.

$$F = dA = \{\partial A_m/\partial \xi^n d\xi^n\}\hat{}\, d\xi^m \tag{7}$$
$$= \{\partial A_m/\partial \xi^n - \partial A_n/\partial \xi^m\} d\xi^n\hat{}\, d\xi^m = F_{mn} d\xi^n\hat{}\, d\xi^m \tag{8}$$

$$\{mn\} = \{01, 02, 12; 03, 13, 23; 04, 14, 24, 34; 05..... \tag{9}$$

7. Apply the Poincare lemma that $dF = ddA = 0$, again using ordered collective index sets so as to eliminate duplicates.

$$dF = (\partial F_{mn}/\partial \xi^k) d\xi^k\hat{}\, d\xi^n\hat{}\, d\xi^m = M_{\{kmn\}} d\xi^k\hat{}\, d\xi^n\hat{}\, d\xi^m \Rightarrow 0, \tag{10}$$

$$\{kmn\} = \{012; 013, 023, 123; 014, 024, 034, 124, 134, 234; 05.... \tag{11}$$

The Poincare lemma implies that always, no matter what the value of $N \geq 4$, the first 4 coefficients,

$$M_{012} = 0 \tag{12}$$
$$M_{013} = 0 \tag{13}$$
$$M_{023} = 0 \tag{14}$$
$$M_{123} = 0 \tag{15}$$

must vanish.

It now will be shown that these first 4 equations are exactly the format of the Maxwell-Faraday induction equations. They are logical consequences of the abstract theory, and form a nested set of PDE's of the same format no matter what the symbols ξ^m stand for. They are therefor valid in every system of independent coordinate variables that are differentiable.

8. For example

$$M_{123} = 0 \Rightarrow \partial B_{23}/\partial \xi^1 + \partial B_{31}/\partial \xi^2 + \partial B_{12}/\partial \xi^3 = div\, \mathbf{B} = 0 \tag{16}$$

where

$$B_{12} = \partial A_1/\partial \xi^2 - \partial A_2/\partial \xi^1 \simeq B_z. \tag{17}$$

All of the E and B coefficients defined above depend upon only the first 4 potential functions.

9. All of the other terms such as $M_{125} = 0$, lead to additional sets of PDE's that may involve components of the E and B fields, and new variables defined above in terms of partial derivatives of the potentials plus the additional potential functions $\Lambda(\xi^k)$ associated with the higher dimensions. The equations, such as $M_{125} = 0$, add additional PDE,s that must be satisfied by the topological constraint, but the first 4 PDE's, the Maxwell-Faraday set, depend only upon the first four potentials.

10. This logical construction (not of empirical data) of Maxwell-Faraday induction equations implies that these are universal laws, to be obeyed by any system that admits description in terms of a pair 1-form A of potentials with C2 differentiable coefficients. For example, a fluid described by a 1-form of Action admits the Faraday induction law.

11. Any modifications of the system of PDE's implies that the basic axioms of the exterior calculus are bypassed. For this reason I do not like to screw around with modifications to Maxwell's PDE equations, and suspect all theories (such as Yang Mills) that presume to modify Maxwell theory are not topological Electrodynamics, but are certain constrained deviates of the general theory. They may be interesting, but they are not fundamental EM theory.