TOPOLOGY AND TOPOLOGICAL EVOLUTION OF ELECTROMAGNETIC FIELDS AND CURRENTS.

R. M. Kiehn
Physics Department, University of Houston, Houston, Texas, 77004
email: jjkrmk@swbell.net

Abstract

When represented by exterior differential systems of field intensities and potentials, $F - dA = 0$, and field excitations and currents, $J - dG = 0$, it is evident that classical electromagnetism induces a coarse topological structure on the domain of $\{x, y, z, t\}$. Topological domains where the field intensities (and currents) are finite and non-zero cannot be compact without boundary, except for the torus and the Klein bottle. The existence of potentials refines the coarse topological structure and leads to the independent concepts of topological torsion, $A\wedge F$, and topological spin, $A\wedge G$. The exterior derivative (divergence) of these two 3-forms defines the Poincare deformation invariants of an electromagnetic system. When the Poincare invariants vanish, closed integrals of these three forms are deformable evolutionary invariants which are recognized as coherent topological structures in plasmas. Non-zero values of the Poincare invariants are the cause of topological evolution and the creation of topological defects.

1. THE TOPOLOGICAL BASIS OF ELECTROMAGNETISM

The existence of an exterior differential system of constraints [1] on an arbitrary set defines a coarse topology on the domain of independent variables. The classic formalism of electromagnetism is a consequence of the exterior differential system,

$$F - dA = 0, \qquad J - dG = 0, \tag{1.1}$$

representing two fundamental topological constraints imposed on a domain of four independent variables. The topological theory requires the existence of a basis of four fundamental exterior differential forms, $\{A, F, G, J\}$, [2] which can be used to construct the complete Pfaff sequence [3] of forms by the processes of exterior differentiation and exterior multiplication. The Maxwell-Faraday equations are a consequence of the exterior differential system $F - dA = 0$, where A is a 1-form of Action potentials which induce the 2-form, F, of electromagnetic field intensities ($\mathbf{E}$ and $\mathbf{B}$) related to forces. This exterior differential constraint is defined as the postulate of potentials. In detail and on (x, y, z, t), the 1-form of Action

$$A = \Sigma_{k=1}^{3} A_k(x, y, z, t)dx^k - \phi(x, y, z, t)dt = \mathbf{A} \circ d\mathbf{r} - \phi dt. \tag{1.2}$$

generates the 2-form of field intensities, $F = dA$. Subject to the constraint of the postulate of potentials, the 2-form of field intensities, F, becomes:

$$F = dA = \{\partial A_k/\partial x^j - \partial A_j/\partial x^k\}dx^j\wedge dx^k \tag{1.3}$$

$$= F_{jk}dx^j\wedge dx^k = \mathbf{B}_z dx\wedge dy \ldots \mathbf{E}_x dx\wedge dt \ldots$$

where in usual engineering notation,

$$\mathbf{E} = -\partial\mathbf{A}/\partial t - grad\phi, \qquad \mathbf{B} = curl\ \mathbf{A} \equiv \partial A_k/\partial x^j - \partial A_j/\partial x^k. \tag{1.4}$$

The closure of the exterior differential system, $dF = 0$, generates the system of Maxwell-Faraday partial differential equations:

$$ddA = dF = 0 \Rightarrow \{curl\ \mathbf{E} + \partial\mathbf{B}/\partial t = 0, \quad div\,\mathbf{B} = 0\}. \tag{1.5}$$

The component functions ($\mathbf{E}$ and $\mathbf{B}$) of the 2-form, F, transform as covariant tensor of rank 2. The topological constraint that F is exact (the postulate of potentials), implies that the domain of support for the field intensities cannot be compact without boundary, unless the Euler characteristic vanishes. These facts topologically distinguish classical electromagnetism from Yang-Mills field theories. Moreover, the fact that F is subsumed to be exact and C1 differentiable excludes the concept of magnetic monopoles from classical electromagnetic theory on topological grounds.

The Maxwell Ampere equations are a consequence of second exterior differential system,

$$J - dG = 0, \tag{1.6}$$

where G is an N-2 form *density* of field excitations (**D** and **H**, related to sources), and J is the N-1 form of charge-current densities. The partial differential equations equivalent to the exterior differential system are precisely the Maxwell-Ampere equations. In detail, this second postulate, the postulate of field excitations, on a four dimensional domain of independent variables, assumes the existence of a N-2 form density given by the expression,

$$G = G^{34}(x,y,z,t)dx\hat{}dy... + G^{12}(x,y,z,t)dz\hat{}dt... \tag{1.7}$$
$$= \mathbf{D}^z dx\hat{}dy...\mathbf{H}^z dz\hat{}dt...$$

Exterior differentiation produces an N-1 form,

$$J = \mathbf{J}^z(x,y,z,t)dx\hat{}dy\hat{}dt... - \rho(x,y,z,t)dx\hat{}dy\hat{}dz. \tag{1.8}$$

Matching the coefficients of the exterior expression $dG = J$ leads to the Maxwell-Ampere equations,

$$curl\,\mathbf{H} - \partial\mathbf{D}/\partial t = \mathbf{J} \quad and \quad div\,\mathbf{D} = \rho. \tag{1.9}$$

The fact that J is exact leads to the charge conservation law, $dJ = ddG = 0,$ or

$$\partial\mathbf{J}^x/\partial x + \partial\mathbf{J}^y/\partial y + \partial\mathbf{J}^z/\partial z + \partial\rho/\partial t = div_3\mathbf{J} + \partial\rho/\partial t = 0. \tag{1.10}$$

Topological properties are to be recognized as invariants of smooth deformations, and are typically expressed as integrals over closed domains. A deformation invariant is defined as an integral over a closed manifold, $\oint\omega,$ such that the Lie derivative of the closed integral with respect to a singly parametrized vector field, $\beta V^k,$ vanishes, for any choice of parametrization, $\beta(x,y,z,t).$ The integration domain is effectively deformed by the function $\beta(x,y,z,t)$ as the points that make up the integration chain are propagated down the flow lines generated by the vector field, $V.$ Cartan's magic formula [4]

$$L_{\beta\mathbf{V}}(\omega) = \{i(\beta V)d\omega + d\{i(\beta V)\omega\} \Rightarrow 0 \tag{1.11}$$

may be used to test for evolutionary invariance of a system of differential forms. If the Lie derivative vanishes, then the form, $\omega,$ is an absolute (local) invariant of the evolutionary process, $V.$ If the Lie derivative vanishes for the closed integral of the form, for any $\beta,$ then the integral defines a global deformation invariant relative to the process, $V.$ It follows that the closed integral of the 2-form F is such a deformation invariant, a result that demonstrates that the global conservation of flux is a topological property. Similarly, the closed integral of the 3-form, $J,$ is a deformation invariant, and demonstrates that the conservation of charge current is a global topological property.

It is to be emphasized that the topological method presumes that the F and G fields are distinct. No metric, no connection, no constitutive constraint has as yet been applied to the variety of space time, $\{x,y,z,t\}.$ The only constraints are the two postulates, the postulate of potentials, $F - dA = 0,$ and the postulate of field excitations, $J - dG = 0.$

2. TOPOLOGICAL TORSION AND TOPOLOGICAL SPIN

On a domain of four independent variables, the complete Pfaff sequence for a Maxwell system,

$$\{A, F = dA, G, J = dG, A\hat{}F, A\hat{}G, A\hat{}J, F\hat{}F, G\hat{}G\}, \tag{2.1}$$

generates a Cartan topology in which the exterior derivative may be interpreted as a limit point, or closure, operator in the sense of Kuratowski [5]. The sequence contains three 3-forms: the classic 3-form of charge current density, $J,$ and the (apparently novel to many researchers) 3-forms of Spin Current density, $A\hat{}G,$ [6] and Topological Torsion-Helicity, $A\hat{}F$ [7]. For an electromagnetic system, the Action 1-form has the physical dimensions of the flux quantum, $h/e,$ the 2-form, $G,$ has the physical dimensions of charge, $e,$ the 3-form, $A\hat{}G,$ has the physical dimensions of spin, $h,$ and the 3-form $A\hat{}F,$ has the physical dimensions of spin multiplied by the Hall impedance, $(h/e)^2 = h(h/e^2) = hZ_{hall}.$ These last two 3-forms are explicitly dependent upon postulate of potentials, and demonstrate the physical significance of the vector and scalar potentials. These physical 3-form objects are not independent of "gauge". By direct evaluation of the exterior product on $\{x,y,z,t\}$ each 3-form will have 4 components that transform as a tensor of rank 3 (a pseudo vector):

$$A\hat{}G \Rightarrow Spin\_Current : \mathbf{S}_4 = [\mathbf{A}\times\mathbf{H}+\mathbf{D}\phi, \mathbf{A}\circ\mathbf{D}] \equiv [\mathbf{S},\sigma], \tag{2.2}$$

$$A\hat{}F \Rightarrow Torsion\_Helicity : \mathbf{T}_4 = [\mathbf{E}\times\mathbf{A}+\mathbf{B}\phi, \mathbf{A}\circ\mathbf{B}] \equiv [\mathbf{T},h]. \tag{2.3}$$

Note that the concept of magnetic helicity, $\mathbf{A}\circ\mathbf{B},$ is the fourth component of the Topological Torsion tensor. Further note that in the limit of zero $\mathbf{E}$ field (the perfect plasma), the spatial components of the Torsion tensor are dominated by the magnetic field, $\mathbf{B}$. The lines of the Torsion vector can be knotted, and would give the appearance of "knotted" $\mathbf{B}$ field lines. If the Torsion pseudo-vector is zero, the magnetic field lines cannot be knotted.

The vanishing of the 3-form, $A\hat{}G,$ is a topological constraint on the domain that defines topologically transverse electric (TTE) waves: the vector potential, $\mathbf{A}$, is orthogonal to $\mathbf{D},$ in the sense that $\mathbf{A}\circ\mathbf{D} = 0.$ The vanishing of the second 3-form is a topological constraint on the domain that defines topologically transverse magnetic (TTM) waves: the vector potential, $\mathbf{A}$, is orthogonal to $\mathbf{B},$ in the sense that $\mathbf{A}\circ\mathbf{B} = 0.$ TTM waves do not have knotted magnetic lines. When both 3-forms vanish, the topological constraint on the domain defines topologically transverse (TTEM) waves. For classic real fields this double constraint would require that vector potential, $\mathbf{A},$ is collinear with the field momentum, $\mathbf{D}\times\mathbf{B},$ and in the direction of the wave vector, $\mathbf{k}$. This topological definition of transversality is not the same as the geometrical definition. The topological definition does not depend upon the choice of coordinates. It is conjectured that if both 3-vectors of topological torsion and spin vanish, then the electromagnetic field will have a zero Poynting vector of energy flux. These results are of interest to fiber optics research.

3. THE POINCARE INVARIANTS

The exterior derivatives of the 3-forms of Spin and Torsion produce two 4-forms, $F\hat{}G - A\hat{}J$ and $F\hat{}F,$ whose integrals over closed 4 dimensional domains are deformation invariants for the Maxwell system. These topological objects are related to the conformal invariants of a Lorentz system as discovered by Poincare and Bateman. In the format of independent variables $\{x,y,z,t\},$ the exterior derivative corresponds to the 4-divergence of the 4-component Spin and Torsion pseudo-vectors, $\mathbf{S}_4$ and $\mathbf{T}_4.$ The functions so created define the Poincare conformal invariants of the Maxwell system:

$$\begin{aligned} Poincare\ 1 &= d(A\hat{}G) = F\hat{}G - A\hat{}J \\ &= \{div_3(\mathbf{A}\times\mathbf{H}+\mathbf{D}\phi) + \partial(\mathbf{A}\circ\mathbf{D})/\partial t\}dx\hat{}dy\hat{}dz\hat{}dt \\ &= \{(\mathbf{B}\circ\mathbf{H}-\mathbf{D}\circ\mathbf{E}) - (\mathbf{A}\circ\mathbf{J}-\rho\phi)\}dx\hat{}dy\hat{}dz\hat{}dt \end{aligned} \tag{3.1}$$

$$\begin{aligned} Poincare\ 2 &= d(A\hat{}F) = F\hat{}F \\ &= \{div_3(\mathbf{E}\times\mathbf{A}+\mathbf{B}\phi) + \partial(\mathbf{A}\circ\mathbf{B})/\partial t\}dx\hat{}dy\hat{}dz\hat{}dt \\ &= \{-2\mathbf{E}\circ\mathbf{B}\}dx\hat{}dy\hat{}dz\hat{}dt \end{aligned} \tag{3.2}$$

For the vacuum state, with $J = 0,$ zero values of the Poincare invariants require that the magnetic energy density is equal to the electric energy density $(1/2\mathbf{B}\circ\mathbf{H} = 1/2\mathbf{D}\circ\mathbf{E})$, and, respectively, that the electric field is orthogonal to the magnetic field $(\mathbf{E}\circ\mathbf{B} = 0).$ Note that these constraints often are used as elementary textbook definitions of what is meant by electromagnetic waves. When either Poincare invariant vanishes, the corresponding closed 3-dimensional integral becomes a topological property in the sense of a deRham period integral [9] with values that have rational ratios. In effect, the deRham period integral defines the concept of topological quantum numbers. For example, when the first Poincare invariant vanishes, the closed integral of the 3-form of spin becomes a deformation invariant with quantized values:

$$if\ \ d(A\hat{}G) = 0, \quad then\ L_{\beta\mathbf{V}} \iiint_{closed} A\hat{}G = L_{\beta\mathbf{V}}(Spin) = 0 \tag{3.3}$$

Similarly, when the second Poincare invariant vanishes, the closed integral of the 3-form of

Torsion-helicity becomes a deformation invariant with quantized values:

$$if\ \ d(A\hat{}F) = 0, \quad then\ L_{\beta\mathbf{V}} \iiint_{closed} A\hat{}F = \ L_{\beta\mathbf{V}}\,(Torsion\_Helicity) = 0 \tag{3.4}$$

It is important to realize that these topological conservation laws are valid in a plasma as well as in the vacuum, subject to the conditions of zero values for the Poincare invariants. On the other hand, topological transitions and irreversible evolution require that the Poincare invariants are not zero.

4. DEFORMATION INVARIANTS AND THE PLASMA

For a Maxwell system, the closed integrals of F and J are always invariants of a continuous process. As the forms that make up the Pfaff sequence define a topology, and if some of the forms are not deformation invariants for a particular process, then that process must involve topological evolution. The topological evolution of G and A are to be examined.

Using Cartan's magic formula, it follows that deformation invariance of the closed integrals of the N-2 form G requires that the admissable evolutionary processes be restricted to those that satisfy the definitions of the classical plasma:

$$L_{\beta\mathbf{V}} \iint_{closed} G = \iint_{closed} i(\beta V)J = 0 \Rightarrow \mathbf{J} = \rho\mathbf{V}. \tag{4.1}$$

This constraint is used to define the "Plasma state" or the "plasma current" in this article. Similar analysis (due to Cartan [8]) indicates that if the closed integral of the 1-form A is to be a deformation invariant, then all such processes will have a Hamiltonian representation. An *ideal or semi-ideal* plasma will be defined as a plasma state for which both the 1-form A and the 2-form G have closed integrals which are deformation invariants with respect to a plasma current. The ideal and semi-ideal plasma processes will obey the plasma master equation, while non-ideal plasma processes will not.

Consider plasma processes that leave the closed integral of Action an evolutionary invariant. By direct evaluation,

$$L_{\rho\mathbf{V}}(\oint A) = \oint i(\rho V)dA = \oint W = \oint\{(\rho\mathbf{E} + \mathbf{J}\times\mathbf{B})_k dx^k + (\mathbf{J}\circ\mathbf{E})dt\} \Rightarrow 0. \tag{4.2}$$

In this equation, the 1-form W is the 1-form of virtual work, defined in terms of the Lorentz force. The equation demonstrates that the concept of a Lorentz force, $\rho\mathbf{E} + \mathbf{J}\times\mathbf{B},$ has a topological foundation. If the integrand is closed, $dW = 0,$ then the process is symplectic. If the integrand is exact (semi-ideal) or zero (ideal) the process has a Hamiltonian representation.

If the virtual Work 1-form is exact, $W = d\Theta,$ then the Lorentz force is represented by a spatial gradient, $\rho\mathbf{E} + \mathbf{J}\times\mathbf{B} = \nabla\Theta.$ The function $\Theta(x,y,z,t)$ is a Bernoulli-Casimir function, and acts as the generator of a symplectic Hamiltonian flow. The (non-unique) Bernoulli-Casimir function is an evolutionary invariant for each process path, but is not necessarily a constant over the domain. It follows that the gradient of the Bernoulli-Casimir function is transverse to the $\mathbf{B}$ field only when the second Poincare invariant vanishes, for

$$\rho\,\mathbf{E}\circ\mathbf{B} = \nabla\Theta\circ\mathbf{B} \quad and \quad \rho\,\mathbf{E}\circ\mathbf{V} = \nabla\Theta\circ\mathbf{V}. \tag{4.3}$$

If the Ohmic assumption is made for the plasma process, $\mathbf{J} = \rho\mathbf{V} = \sigma(\mathbf{E} + \mathbf{V}\times\mathbf{B})$, then the symplectic condition leads to a thermopower format of the type,

$$\mathbf{J} = (1/\rho\sigma)grad(kT), \tag{4.4}$$

when it is subsumed that the Bernoulli-Casimir function is related to temperature. It would appear that for plasma motion along the $\mathbf{B}$ field lines, there can exist a classical dynamo action to produce an $\mathbf{E}$ field collinear with the magnetic field. Such a topological result should have application to the understanding of jets from rapidly rotating neutron stars.

If the Work 1-form is closed, but not exact, then the Lorentz force must have zero curl. By using the Maxwell-Faraday equation, this topological constraint becomes equivalent to the plasma master equation:

$$-\partial\mathbf{B}/\partial t + curl(\mathbf{V}\times\mathbf{B}) = -\nabla\ln\rho\times(\mathbf{E} + \mathbf{V}\times\mathbf{B}). \tag{4.5}$$

However all such symplectic plasma processes as defined in this section are thermodynamically reversible, although they may not conserve mechanical energy.

5. EVOLUTIONARY PROCESSES WITH FROZEN-IN LINES

It is of some interest to examine the evolution of the differential forms that make up an electromagnetic system relative to Plasma processes. The method is to construct the Lie derivative with respect a plasma process, $\mathbf{J} = \rho\mathbf{V}$, of all forms that make up the electromagnetic Pfaff sequence. The objective is to study the topological evolution of objects that depend upon the evolution of the potentials. This study will lead to the conditions for when the points that make up a vector line evolve into points that make up the same line. Such field lines are said to be "frozen-in", relative to an evolutionary process. It may be shown [10] that for any contravariant vector field $\mathbf{Z}$ there exists a 3-form defined as $Z = i(\mathbf{Z})dx\hat{}dy\hat{}dz\hat{}dt$. The requirement that the tangent lines to $\mathbf{Z}$ be frozen-in relative to a process, $\mathbf{V}$, is given by the statement that the form Z be conformal relative to the process :

$$L_{(\mathbf{V})}Z = i(V)dZ + d(i(V)Z) = \Gamma(x,y,z,t)Z \tag{5.1}$$

The lines that make up the tangents to the charge current 3-form are "frozen-in" by the plasma process, $\mathbf{J} = \rho\mathbf{V}$. However, the lines of the 3-form of Torsion are not frozen-in with respect to a plasma current, without further topological constraints. Neither are the lines of the 3-form of Spin frozen-in with respect to plasma currents. For those systems where the second Poincare invariant vanishes it follows that the lines which are tangent to the 3-form $A\hat{}F$ must have zero divergence. These lines can only start and stop on boundary points, or they are closed on themselves. As the electromagnetic current is exact, any three dimensional domain of support for a finite plasma current cannot be compact without a boundary. If the lines of plasma current start and stop on boundary points, then the lines of torsion can form closed loops that link the current lines. It is the concept of linkages that is of interest to the theory of magnetic knots.

6. TOPOLOGICAL THERMODYNAMICS AND IRREVERSIBILITY

To understand what is meant by thermodynamic irreversibility, realize that Cartan's magic formula of topological evolution is equivalent to the first law of thermodynamics.

$$L_{(\mathbf{v})}A = i(\mathbf{V})dA + d(i(\mathbf{V})A) = W + dU = Q. \tag{6.1}$$

A is the "Action" 1-form that describes the thermodynamic system. $\mathbf{V}$ is the vector field that defines the evolutionary process. W is the 1-form of (virtual) work. Q is the 1-form of heat. If Q is zero, the process V is locally adiabatic. From classical thermodynamics, a process is irreversible when the heat 1-form Q does not admit an integrating factor. From the Frobenius theorem, the lack of an integrating factor implies that $Q\hat{}dQ \neq 0$. Hence a simple test may be made for any process, $\mathbf{V}$, relative to a physical system described by an Action 1-form, A:

$$\textit{If}\;\; L_{(\mathbf{v})}A\hat{}L_{(\mathbf{v})}dA \neq 0 \textit{ then the process is irreversible.} \tag{6.2}$$

The fundamental issue is that thermodynamic irreversibility is to be associated with topological change. [11]

This topological definition of thermodynamic irreversibility implies that the three categories of plasma processes described in section 4 {symplectic, Hamiltonian (semi-ideal) or extremal (ideal) }, $\subset \mathbf{S}$, are reversible $(as\; L_{(\mathbf{S})}dA{=}dQ = 0)$. However, for evolution in the direction of the Torsion vector, $\mathbf{T}_4$, direct computation demonstrates that the fundamental equations lead to a conformal evolutionary process, a process which is thermodynamically irreversible.

For an electromagnetic system, the topological torsion 3-form, $A\hat{}dA$, induces the torsion pseudo vector

$$\mathbf{T}_4 = \{(\mathbf{E}\times\mathbf{A} + \mathbf{B}\phi);\mathbf{A}\circ\mathbf{B}\} \equiv \{\mathbf{S},h\}. \tag{6.3}$$

If $div_4\mathbf{T} = -2\,\mathbf{E}\circ\mathbf{B} \neq 0,$ the electromagnetic 1-form, $A,$ defines a domain of Pfaff dimension four. Such domains cannot support topologically transverse magnetic waves $(as\ A\hat{}F \neq 0)$. Evolutionary processes (including plasma currents) that are proportional to the Torsion psuedo-vector are thermodynamically irreversible, if $\sigma = \mathbf{E}\circ\mathbf{B} \neq 0$. By direct calculation, it is a remarkable fact that any evolution in the direction of the Torsion pseudo vector leaves the Action 1-form conformally invariant; that is,

$$L_{(\gamma\mathbf{T}_4)}A = i(\gamma\mathbf{T}_4)dA + di(\gamma\mathbf{T}_4)A = \gamma(\mathbf{E}\circ\mathbf{B})A + 0 = \sigma A \tag{6.4}$$

In addition, the torsion pseudo vector on a domain of 4 variables is transverse to the 1-form of Action, as $A\hat{}(A\hat{}F) = 0.$ Evolution in the direction of the Torsion vector is not Hamiltonian, unless the second Poincare invariant vanishes. Indeed,

$$Q\hat{}dQ = L_{(\gamma\mathbf{T}_4)}A\hat{}L_{(\gamma\mathbf{T}_4)}A\hat{}L_{(\gamma\mathbf{T}_4)}dA = \sigma^2 A\hat{}F \neq 0. \tag{6.5}$$

Hence a non-zero Torsion pseudo vector defines irreversible evolutionary paths, if the second Poincare invariant does not vanish.

By direct application of the exterior derivative, it follows that

$$L_{(\mathbf{T}_4)}(A\hat{}F) = 2\sigma A = 2(\mathbf{E}\circ\mathbf{B})A\hat{}F. \tag{6.6}$$

Hence, evolution along the direction of the torsion vector freezes-in the lines of the torsion pseudo vector in space time, but the process is irreversible unless the second Poincare invariant is zero.

Recall that the definition of a plasma current, $J,$ is equivalent to an evolutionary process such that $L_{(J)}G = 0.$ Hence consider a plasma current which is also in the direction of the Torsion vector. Then

$$L_{(J)}A\hat{}G = (L_{(J)}A)\hat{}G + A\hat{}L_{(J)}G = (L_{(\gamma\mathbf{T}_4)}A)\hat{}G + A\hat{}L_{(J)}G \tag{6.7}$$
$$= \gamma\bullet(\mathbf{E}\circ\mathbf{B})\ A\hat{}G + 0 = \sigma A\hat{}G$$

Hence for plasma motions in the direction of the (possibly dissipative) torsion vector, both the "lines" of the Spin vector are "frozen-in" and the lines of the Torsion vector are "frozen-in". Such "frozen-in" objects can be used to give a topological definition of deformable coherent structures in a plasma. Moreover, as the evolutionary process causes the frozen-in structures to deform and decay, it is conceivable that evolution could proceed to form stationary (not stagnant) states (where $\mathbf{E}\circ\mathbf{B} \Rightarrow 0),$ such that the frozen-in field line structures become local deformation invariants, or topological defects. Electromagnetic coherent structures are evolutionary deformable (and perhaps decaying) domains of Pfaff dimension 4, which form stationary states of topological defects (including the null state) in regions of Pfaff dimension 3, where $\mathbf{E}\circ\mathbf{B} = 0$.

Reprise

The computations required to display examples of the topological theory described above can be algebraically formidable. Accordingly, a symbolic mathematics program based on Maple, which can be used to evaluate the Torsion and Spin pseudo vectors as well as other interesting field quantities, can be found at http://www.uh.edu/~rkiehn/pdf/cyclide1.pdf.

A more detailed article with many example computations, exhibiting time dependent wave solutions in the vacuum which have both torsion and spin, as well as a non-zero second Poincare invariant, can be found at http://www.uh.edu/~rkiehn/pdf/emtopolb.pdf

Many other applications of Cartan's exterior calculus can be found at http://www.uh.edu/~rkiehn

References

[1] R. L. Bryant, S. S. Chern, R. B. Gardner, H. L. Goldschmidt, P. A. Griffiths, *Exterior Differential Systems*, Springer Verlag (1991)

[2] R. M. Kiehn, G. P. Kiehn, R. B.Roberds, "Parity and Time-reversal Symmetry Breaking, Singular Solutions", Phys Rev A, **43**, (1991) pp. 5665

[3] R. M. Kiehn, "Topological Torsion, Pfaff Dimension and Coherent Structures", in: H. K. Moffatt and T. S. Tsinober eds, Topological Fluid Mechanics, Cambridge University Press, (1990), pp. 449-458 .

[4] J. E. Marsden and T. S. Riatu, *Introduction to Mechanics and Symmetry*, Springer-Verlag, (1994

p.122
[5] S. Lipschutz, *General Topology*, Schaum, New York, (1965) p. 88.
[6] R. M. Kiehn, ”Periods on manfolds, quantization and gauge”, J. of Math Phys **18**, no. 4, (1977) p. 614
[7] R. M. Kiehn, ”Are there three kinds of superconductivity” Int. J. Mod. Phys B Vol. **5** no. 10 (1991) pp.1779-1790
[8] E. Cartan, *Lecons sur les invariants integrauxs*, Hermann, Paris (1958).
[9] G. deRham, *Varietes Differentiables*, Hermann, Paris (1960)
[10] R. M. Kiehn, ”Intrinsic hydrodynamics with applications to space time fluids”, Int. J. Engng Sci **13**, (1975) pp. 941-949
[11] R. M. Kiehn ”Retrodictive Determinism”, Int. J. of Eng. Sci. **14**, (1976) p. 749