Entropy and Equilibrium in Topological Thermodynamics

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Abstract

The concept of topological entropy is deduced (without statistics) from the fact that Cartan-Hilbert 1-form of Action defines a symplectic system of Pfaff Topological dimension 2n+2. The perfect differential of entropy, dS, is composed of the interior product of the non-canonical components of momentum with the components of the differential velocities. An equilibrium system is a Lagrange submanifold of the 2n+2 topological space, upon which the change in entropy is zero, $dS_{\{equil\}}=0$.

Key words: Topological entropy, irreversible processes, Pfaff topological dimension, Cartan's Magic formula

1 A Topological Perspective

A major objective of topological thermodynamics is to establish a topological, non-statistical, link between thermodynamics and mechanical, electrical, or hydrodynamic physical systems[1]. A particular goal is to develop a method of describing the differences, and how and when such differences occur, between

Equilibrium and non-equilibrium physical systems, and

Reversible and irreversible evolutionary processes acting on such systems.

The methods which have been developed are based upon Cartan's calculus of exterior differential forms [8], [6]. Exterior differential forms are objects, which, in contrast to tensors, are well behaved with respect to differentiable (continuous) mappings that do not have an inverse (and do not preserve topological properties), as well as with respect to diffeomorphisms, which are differentiable invertible continuous mappings (and which preserve topological properties). Evolutionary processes will be defined in terms of the action of

[1] A more detailed development of Non-equilibrium and Irreversible Thermodynamics can be found at http://www22.pair.com/csdc/pdf/thermo2ch.pdf

the Lie differential with respect to vector direction fields acting on differential forms [12]. The Lie differential acting on differential forms is not confined by the diffeomorphic constraints of tensor analysis, and can treat problems of topological change. The method goes beyond the more standard "extremal" techniques based upon the calculus of variations. In most of that which follows, the functions used to define the physical systems will be assumed to be C2 differentiable. The functions that describe processes most often will be assumed to be C2 differentiable as well, but certain C1 (tangential discontinuities) and C0 (shocks) processes are of physical interest.

A fundamental result can be expressed by the statement: "Topological change is a necessary condition for a thermodynamic process to be irreversible". Irreversible processes, related to the arrow of time and the biological aging process, require topological evolution and topological change. Current physical theories that describe evolutionary processes (for example, Hamiltonian or Unitary dynamics) usually are formulated in terms of homeomorphisms that do not permit topological change.

A remarkable achievement of the topological point view of thermodynamics is the ability to define the concept of entropy in an analytic, non-phenomenological way - and without the use of statistics. The concept of thermodynamic entropy has been extremely hard to define, for, like potential energy, classical mechanics does not yield a clear visual picture of "just what is" entropy. Numerous phenomenological constructions have been suggested (such as entropy is a measure of disorder, entropy is the inverse of information, entropy is proportional to area...) but encoding such concepts is difficult. Associated with the concept of entropy is the idea of a system in equilibrium, which, at least in approximation, is recognized from experience. Cold water poured into a hot bath comes to (approximate) equilibrium within a perceptibly short time span. On the other hand, the ability to interchange kinetic energy and potential energy ultimately yields a visual perception of "energy" in terms of the visual image of motion, but there seems to be no "visual" equivalent for "entropy". Moreover, the currently accepted dogma is that entropy always increases on a global scale. The fact that the idea is global should be recognized immediately as a topological, not a geometrical, concept. Much of current physical theory is based upon geometric concepts, and are not appropriate to the understanding of entropy and irreversibility. It is the purpose of that which follows to demonstrate how a topological, not geometric, point of view enables an analytic coding and derivation of the concept of Entropy - without the use of statistics, and without phenomenological formulation.

2 Topological Thermodynamics

The topological view of thermodynamics described herein is based on four axioms.

Axiom 1 *Thermodynamic physical systems can be encoded in terms of a 1-form of covariant Action Potentials,* $A_k(x, y, z, t)$*, on a 4 dimensional abstract* *variety of ordered independent variables, $\{x, y, z, t\}$. The variety supports a volume element $\Omega_4 = dx\hat{}dy\hat{}dz\hat{}dt$.*

Axiom 2 *Thermodynamic processes are assumed to be encoded, to within a factor, $\rho(x, y, z, t)$, in terms of contravariant vector direction fields, $\mathbf{V}_4(x, y, z, t)$.*

Axiom 3 *Continuous topological evolution of the thermodynamic system can be encoded in terms of Cartan's magic formula (see p. 122 in [12]). The Lie differential, when applied to a exterior differential 1-form of Action, $A = A_k dx^k$, is equivalent abstractly to the first law of thermodynamics.*

$$\begin{aligned}
\textit{Cartan's Magic Formula} \quad L_{(\rho\mathbf{V}_4)}A &= i(\rho\mathbf{V}_4)dA + d(i(\rho\mathbf{V}_4)A) && (1)\\
\textit{First Law of Thermodynamics} \quad &: W + dU = Q, && (2)\\
\textit{Inexact 1-form of Heat} \quad L_{(\rho\mathbf{V}_4)}A &= W + dU = Q && (3)\\
\textit{Inexact 1-form of Work} \quad W &= i(\rho\mathbf{V}_4)dA, && (4)\\
\textit{Internal Energy} \quad U &= i(\rho\mathbf{V}_4)A. && (5)
\end{aligned}$$

Axiom 4 *Equivalence classes of systems and processes can be defined in terms of the Pfaff topological dimension.*

In effect, Cartan's methods can be used to formulate precise mathematical definitions for many thermodynamic concepts in terms of topological properties -without the use of statistics or metric constraints. Moreover, the method applies to non-equilibrium thermodynamical systems and irreversible processes, again without the use of statistics or metric constraints.

In order to make the equations more suggestive to the reader, the symbolism for the variety of independent variables has been chosen to be of the format $\{x, y, z, t\}$, but be aware that no constraints of metric or connection are imposed upon this variety. For instance, it is NOT assumed that the base variety is euclidean.

2.1 The Pfaff Topological Dimension

Perhaps one of the most important topological tools to be used within the theory of continuous topological evolution is the concept of Pfaff topological dimension. The maximum Pfaff dimension is equal to number of independent variables in the base variety, which in this article has been limited to $n = 4$. For a given 1-form of Action, $A = A_k(x, y, z, t)dx^k$ defined on the base variety of $\{x, y, z, t\}$, it is possible to ask what is the irreducible minimum number of independent functions $\theta(x, y, z, t)$ required to describe the topological features that can be generated by the specified 1-form, A. This irreducible number of functions is defined here in as the "Pfaff topological dimension" of the 1-form, A. For example, if

$$\begin{aligned}
A &= A_k dx^k \Rightarrow d\theta(x, y, z, t)_{irreducible}, && (6)\\
\text{such that } A_k &= \partial\theta(x, y, z, t)/\partial x^k, && (7)
\end{aligned}$$

then only one function $\theta(x, y, z, t)$ is required to describe the Action, not four. In this example the irreducible Pfaff topological dimension of the 1-form, A, is 1, although the dimension of base variety is 4. In a sense, the Pfaff topological dimension defines the existence of a domain of "topological" base variables (topological coordinates) as submersions from the original base variety (geometric coordinates) to the irreducible base variety (topological coordinates). Differential forms constructed on the irreducible base variety of functions, are functionally well defined on the original base variety.

Relative to the Cartan topology [2], the "Pfaff topological dimension" can be generated by each of the Pfaffian forms associated with each discipline. The irreducible Pfaff topological dimension for any given 1-form A is readily computed by constructing the Pfaff sequence of forms:

Pfaff sequence

$$\{A, dA, A\hat{}dA, dA\hat{}dA\}. \tag{8}$$

The Pfaff topological dimension is equal to the number of non-zero terms in the Pfaff sequence. For example, if the Pfaff sequence for a given 1-form A is $\{A, dA, 0, 0\}$ in a region $U \subset \{x, y, z, t\}$, then the Pfaff topological dimension of A is 2 in the region, U. The 1-form A, in the region U, then admits description in terms of only two, but not less than 2, independent variables, say $\{u^1, u^2\}$. For a differentiable map φ from $\{x, y, z, t\} \Rightarrow \{u^1, u^2\}$, the exterior differential 1-form defined on the target variety U of 2 pre-geometry dimensions as

$$A(u^1, u^2) = A_1(u^1, u^2)du^1 + A_2(u^1, u^2)du^2, \tag{9}$$

has a functionally well defined pre-image $A(x, y, z, t)$ on the base variety $\{x, y, z, t\}$ of 4 pre-geometric dimensions. This functionally well defined pre-image is obtained by functional substitution of u^1, u^2, du^1, du^2 in terms of $\{x, y, z, t\}$ as defined by the mapping φ. The process of functional substitution is called the pull-back.

$$A(x, y, z, t) = A_k dx^k = \varphi^*(A(u^1, u^2)) = \varphi^*(A_\sigma du^\sigma) \tag{10}$$

It may be true that the functional form of A yields a Pfaff topological dimension equal to 2 globally over the domain $\{x, y, z, t\}$, except for sub regions where the Pfaff dimension of A is 3 or 4. These sub regions represent topological defects in the almost global domain of Pfaff dimension 2. Conversely, the Pfaff dimension of A could be 4 globally over the domain, except for sub regions where the Pfaff dimension of A is 3, or less. These sub regions represent topological defects in the almost global domain of Pfaff dimension 4. Applications of both viewpoints will be described below. The important concept of Pfaff topological dimension also can be used to define equivalence classes of physical systems and processes.

The concept of "Pfaff topological dimension" was developed more than 110 years ago (see page 290 of Forsyth [9]), and has been called the "class"[2] of a

[2] The term "Pfaff topological dimension" (instead of class) was introduced by the present author in order to emphasize the topological foundations of the concept.

differential 1-form in the mathematical literature. More recent mathematical developments can be found in [16]. The method and its properties have been little utilized in the applied world of physics and engineering. Of key importance is the fact that the non-zero existence of the 3-form $A\hat{}dA = A\hat{}F$ of *Topological Torsion*, and the 4-form of *Topological Parity*, $dA\hat{}dA = F\hat{}F$ implies that the Pfaff topological dimension of the region is 3 or 4, respectively. Either value is an indicator that the physical system (in the sub region) is NOT in thermodynamic equilibrium. The concept of *topological parity*, $F\hat{}F$, has its foundations in the theory of Pfaff's problem, with a recognizable 4 dimensional formulation appearing in Forsyth [9] page 100. The idea of *Topological Torsion*, $A\hat{}F$, is associated with the idea of magnetic helicity density, a concept that apparently had its electromagnetic genesis with the study of plasmas in WWII. However, the concept of helicity density is but one component of the four dimensional *Topological Torsion 4 vector*.

Recall that a space curve with non-zero Frenet - Serret torsion does not reside in a two dimensional plane. Non-zero Frenet - Serret torsion of a space curve is an indicator that the *geometrical* dimension of the space curve is at least 3. The fact that the Pfaff *topological* dimension of the 1-form, A, is at least 3, when $A\hat{}F$ is non-zero, is the basis of why the 3-form, $A\hat{}F$, was called "Topological Torsion". The idea of non-zero $A\hat{}F$ also appears in the theory of the Hopf Invariant [4].

2.2 Physical Systems: Equilibrium, Isolated, Closed and Open

Physical systems and processes are elements of topological categories determined by the Pfaff topological dimension (or class) of the 1-forms of Action, A, Work, W, and Heat, Q. For example, the Pfaff topological dimension of the exterior differential 1-form of Action, A, determines the various species of thermodynamic systems in terms of distinct topological categories. The are two topological thermodynamic categories that are determined by the closure (or differential ideal) of the 1-form of Action, $A \cup dA$, and the closure of the 3-form of topological torsion, $A\hat{}dA \cup dA\hat{}dA$. The first category is represented by a connected Cartan topology, while the second category is represented by a disconnected Cartan topology.

2.2.1 Connected Topology $A\hat{}dA = 0$

1. Equilibrium physical systems are elements such that the Pfaff topological dimension is 1.

2. Isolated physical systems are elements such that the Pfaff topological dimension is 2, or less. Isolated systems of Pfaff dimension 2 need not be in

equilibrium, but do not exchange radiation or mass with the environment.

$$Systems \quad : \quad \text{Pfaff topological dimension when: } A\hat{}dA = 0 \tag{11}$$

$$dA = 0 \quad \textbf{Equilibrium} \text{ - Pfaff topological dimension 1} \tag{12}$$

$$A\hat{}dA = 0 \quad \textbf{Isolated} \text{ - Pfaff topological dimension 2} \tag{13}$$

2.2.2 Disconnected Topology $A\hat{}dA \neq 0$

1. Closed physical systems are elements such that the Pfaff topological dimension is 3. Closed systems can exchange radiation, but not mass, with the environment.
2. Open physical systems are such that the Pfaff topological dimension is 4. Open physical systems can exchange both radiation and mass with the environment.

$$Systems \quad : \quad \text{Pfaff topological dimension when: } A\hat{}dA \neq 0 \tag{14}$$

$$d(A\hat{}dA) = 0 \quad \textbf{Closed} \text{ - Pfaff topological dimension 3} \tag{15}$$

$$dA\hat{}dA \neq 0. \quad \textbf{Open} \text{ - Pfaff topological dimension 4.} \tag{16}$$

When the 3-form of Topological Torsion is zero, the Cartan Topology is a connected topology, and solutions to the Pfaff equation are uniquely determined to within a factor. When the 3-form of Topological Torsion is non-zero, the Cartan topology is a disconnected topology, and the Pfaffian equation has non-unique solutions. Note that these topological specifications as given above are determined entirely from the functional properties of the physical system encoded as a 1-form of Action, A. The system topological categories do not involve a process, which is encoded (to within a factor) by some vector direction field, $\mathbf{V}_4$. However, the process $\mathbf{V}_4$ does influence the topological properties of the work 1-form W and the Heat 1-form Q.

2.3 Reversible and Irreversible Processes

The Pfaff topological dimension of the exterior differential 1-form of Heat, Q, determines important topological categories of processes. From classical thermodynamics "The quantity of heat in a reversible process always has an integrating factor" [10] [14] . Hence, from the Frobenius unique integrability theorem, which requires $Q\hat{}dQ = 0$, all reversible processes are such that the Pfaff dimension of Q is less than or equal to 2. Irreversible processes are such that the Pfaff dimension of Q is greater than 2. A dissipative irreversible topologically *turbulent* process is defined when the Pfaff dimension of Q is 4.

Processes defined by the Pfaff dimension of Q

$$Processes \quad : \quad \text{as defined by the Pfaff dimension of } Q \tag{17}$$

$$Q\hat{}dQ = 0 \quad \textbf{Reversible} \text{ - Pfaff dimension 2} \tag{18}$$

$$d(Q\hat{}dQ) \neq 0. \quad \textbf{Turbulent} \text{ - Pfaff dimension 4.} \tag{19}$$

Note that the Pfaff dimension of Q depends on both the choice of a process, $\mathbf{V}_4$, and the system, A, upon which it acts. As reversible thermodynamic processes are such that $Q\hat{}dQ = 0$, and irreversible thermodynamic processes are such that $Q\hat{}dQ \neq 0$, Cartan's formula of continuous topological evolution can be used to determine if a given process, $\mathbf{V}_4$, acting on a physical system, A, is thermodynamically reversible or not:

Processes defined by the Lie differential of A

$$\left[\begin{array}{l} \textbf{Reversible} \text{ Processes } \rho\mathbf{V}_4 : \; L_{(\rho\mathbf{V}_4)}A\hat{}L_{(\rho\mathbf{V}_4)}dA = 0, \\ \textbf{Irreversible} \text{ Processes } \rho\mathbf{V}_4 : L_{(\rho\mathbf{V}_4)}A\hat{}L_{(\rho\mathbf{V}_4)}dA \neq 0. \end{array}\right] \tag{20}$$

Remarkably, Cartan's magic formula can be used to describe the continuous dynamic possibilities of both reversible and irreversible processes, in equilibrium or non-equilibrium systems, even when the evolution induces topological change, transitions between excited states, and changes of phase, such as condensations.

It is important to note that the direction field, $\mathbf{V}_4$, need not be topologically constrained such that it is singularly parameterized. That is, the evolutionary processes described by Cartan's magic formula are not necessarily restricted to vector fields that satisfy the topological constraints of kinematic perfection, $dx^k - V^k dt = 0$. A discussion of topological fluctuations, where $dx^k - V^k dt = \Delta^k \neq 0$, and an example fluctuation process is described below.

3 Physical Systems of Pfaff Topological dimension 3

Consider those physical systems that are represented by 1-forms, A, of Pfaff topological dimension 3. The concept implies that the topological features can be describe in terms of 3 functions and their differentials. For example, if one presumes the fundamental independent base variables are the set $\{P, q, \tau\}$, with an exterior differential volume element consisting of a product[3] of exact 1-forms $\Omega_3 = dP\hat{}dq\hat{}dt$, then a Darboux representation for a physical system could have the appearance:

$$A = Pdq + d\tau. \tag{21}$$

The objective is to use the features of Cartan's magic formula to compute the possible evolutionary features of such a system. The evolutionary dynamics is essentially the first law of thermodynamics.

$$L_{\rho\mathbf{V}}A = i(\rho\mathbf{V})dA + di(\rho\mathbf{V})A) = W + dU = Q. \tag{22}$$

[3] More abstract systems could be constructed from differential forms which are not exact.

The elements of the Pfaff sequence for this Action become:

$$A = Pdq + d\tau \tag{23}$$
$$dA = dP\hat{}dq, \tag{24}$$
$$A\hat{}dA = dP\hat{}dq\hat{}d\tau, \tag{25}$$
$$dA\hat{}dA = 0. \tag{26}$$

Relative to the "ordered position" vector $\mathbf{R} = [P, q, \tau]$, consider the 3 linearly independent orthogonal vector direction fields:

$$\mathbf{V} = [0, 1, 0] \tag{27}$$
$$\mathbf{V}_{\perp} = [1, 0, 0] \tag{28}$$
$$\mathbf{E} = [0,\ 0, 1]. \tag{29}$$

The *extremal* vector $\mathbf{E}$ is the unique eigen vector with eigenvalue zero relative to the anti-symmetric matrix generated by the 2-form, dA. The *associated* vector $\mathbf{V}_{\perp}$ is orthogonal to the q, τ plane.

Next deform the vector direction fields by an arbitrary function, ρ. Then construct the contractions (the internal energy).

$$U_{\mathbf{V}} = i(\rho\mathbf{V})A = \rho P \tag{30}$$
$$U_{\mathbf{V}_{\perp}} = i(\rho\mathbf{V}_{\perp})A = 0 \tag{31}$$
$$U_{\mathbf{E}} = i(\rho\mathbf{E})A = \rho. \tag{32}$$

The linearly independent Work 1-forms for evolution in the direction of the 3 basis vectors,

$$W_{\mathbf{V}} = i(\rho\mathbf{V})dA = -\rho dP \tag{33}$$
$$W_{\mathbf{V}_{\perp}} = i(\rho\mathbf{V}_{\perp})dA = +\rho dq \tag{34}$$
$$W_{\mathbf{E}} = i(\rho\mathbf{E})dA = 0. \tag{35}$$

From Cartan's Magic Formula, $L_{(\mathbf{V})}A = i(\rho\mathbf{V})dA + d(i(\rho\mathbf{V})A) \equiv Q$, it becomes apparent that

$$Q_{\mathbf{V}} = Pd\rho, \qquad dQ_{\mathbf{V}} = dP\hat{}d\rho \tag{36}$$
$$Q_{\mathbf{V}_{\perp}} = +\rho dq, \qquad dQ_{\mathbf{V}_{\perp}} = d\rho\hat{}dq$$
$$Q_{\mathbf{E}} = d\rho \qquad dQ_{\mathbf{E}} = 0 \tag{37}$$

All processes in the extremal direction satisfy the conditions that $Q_{\mathbf{E}}\hat{}dQ_{\mathbf{E}} = 0$. Hence, all extremal processes are reversible. It is also true that evolutionary processes in the direction of the other basis vectors, separately, are reversible, as the 3-form $Q\hat{}dQ$ vanishes. Hence all such *piecewise* continuous processes are thermodynamically reversible.

However, evolution in the direction of smooth combinations of the base vectors may not satisfy the reversibility conditions, $Q\hat{}dQ = 0$. For example, it is possible to consider expansions or rotations in the P, q plane.

$$V_{\text{expansion}} = \mathbf{V}_{\perp} + \mathbf{V}_{V}, \tag{38}$$
$$Q\hat{}dQ = -\rho d\rho\hat{}dP\hat{}dq \tag{39}$$

$$V_{\text{rotation}} = \mathbf{V}_{\perp} - \mathbf{V}_{V}, \tag{40}$$
$$Q\hat{}dQ = +\rho d\rho\hat{}dP\hat{}dq \tag{41}$$

The non-zero value of $Q\hat{}dQ$ for the continuous expansions and rotations are related to the non-zero Godbillon-Vey class [15].

Conclusion 1: *C1 processes formed from segments along the directions of the orthogonal direction fields in 3 D are thermodynamically reversible, but C2 combinations of the orthogonal vector fields are not.*

It should be remarked that for Action 1-forms of odd Pfaff topological dimension, addition of a closed form whose format contains new independent variables does not change the Pfaff topological dimension of the composite. On the other hand, if the original 1-from is renormalized by some factor, then the Pfaff topological dimension does change. In the next section it will be demonstrated that this piecewise equivalence class of processes does not produce a thermodynamically irreversible process. In this sense it may be said that thermodynamic irreversibility is an artifact of dimension 4.

4 Physical Systems of Pfaff dimension 4

Consider those physical systems that are represented by 1-forms, A, of Pfaff dimension 4. For example, if one presumes the fundamental independent base variables are the space time set $\{x, y, z, t\}$, then a representation for physical system would consist of four functions whose arguments are the base variables, and the Action 1-form would have the appearance:

$$A = \Sigma_{k=1}^{3} A_k(x, y, z, t) dx^k - \phi(x, y, z, t) dt. \tag{42}$$

This representation has applicability to the study of electromagnetic systems, where the functions $A_k(x, y, z, t)$ and $\phi(x, y, z, t)$ play the role of the vector and scalar potentials of classic electromagnetic theory [11] .

However, the Darboux theorem says that there exists a map from the set $\{x, y, z, t\}$ into four independent functions, $\{q, \tau, P, H\}$ such that the elements of the Pfaff sequence become:

$$
\begin{aligned}
A &= Pdq - Hd\tau, &(43)\\
dA &= dP\hat{}dq - dH\hat{}d\tau, &(44)\\
A\hat{}dA &= (PdH - HdP)dq\hat{}d\tau, &(45)\\
dA\hat{}dA &= 2dP\hat{}dH\hat{}dq\hat{}d\tau, &(46)
\end{aligned}
$$

Relative to the "position" vector $\mathbf{R} = [q, \tau, P, H]$, consider the 4 linearly independent orthogonal vector direction fields:

$$
\begin{aligned}
\mathbf{V} &= [P, H, 0, 0] &(47)\\
\mathbf{V}_{\perp} &= [-H, P, 0, 0] &(48)\\
\mathbf{T} &= [0, 0, P, H] &(49)\\
\mathbf{T}_{\perp} &= [0, 0, -H, P] &(50)
\end{aligned}
$$

The vectors $\mathbf{V}$ and $\mathbf{V}_{\perp}$ reside in the subspace of $\{x, \tau, 0, 0\}$ and there for will be called "tangent" vectors. The vectors $\mathbf{T}$ and $\mathbf{T}_{\perp}$ reside in the subspace $\{0, 0, P, H\}$ and will be called "normal" vectors These vectors can be used a basis frame for any vector in the 4 dimensional space.

Next construct the contractions

$$
\begin{aligned}
i(\mathbf{V})A &= P^2 + H^2 &(51)\\
i(\mathbf{V}_{\perp})A &= 0 &(52)\\
i(\mathbf{T})A &= 0 &(53)\\
i(\mathbf{T}_{\perp})A &= 0 &(54)
\end{aligned}
$$

and note that the vectors form an orthogonal (but not necessarily normalized) set. Then construct the linearly independent Work 1-forms for evolution in the direction of these 4 basis vectors,

$$
\begin{aligned}
W_{\mathbf{V}} &= i(\mathbf{V})dA = -d(P^2 + H^2)/2 &(55)\\
W_{\mathbf{V}_{\perp}} &= i(\mathbf{V}_{\perp})dA = PdH - HdP &(56)\\
W_{\mathbf{T}} &= i(\mathbf{T})dA = Pdq - Hd\tau = A &(57)\\
W_{\mathbf{T}_{\perp}} &= i(\mathbf{T}_{\perp})dA = -Hdq + Pd\tau. &(58)
\end{aligned}
$$

From Cartan's Magic Formula, $L_{(\mathbf{V})}A = i(\mathbf{V})dA + d(i(\mathbf{V})A) \equiv Q$, it becomes apparent that

$$
\begin{aligned}
Q_{\mathbf{V}} &= +d(P^2+H^2)/2, \qquad dQ_{\mathbf{V}} = 0 & (59)\\
Q_{\mathbf{V}_\perp} &= PdH - HdP, \qquad dQ_{\mathbf{V}_\perp} = 2dP\hat{}dH & (60)\\
Q_{\mathbf{T}} &= Pdq - Hd\tau = A \qquad dQ_{\mathbf{T}} = dP\hat{}dq - dH\hat{}d\tau = dA & (61)\\
Q_{\mathbf{T}_\perp} &= -Hdq + Pd\tau \qquad dQ_{\mathbf{T}_\perp} = -dH\hat{}dq + dP\hat{}d\tau, & (62)
\end{aligned}
$$

and it follows that the three forms $Q\hat{}dQ$ become:

$$
\begin{aligned}
Q_{\mathbf{V}}\hat{}dQ_{\mathbf{V}} &= 0 & (63)\\
Q_{\mathbf{V}_\perp}\hat{}dQ_{\mathbf{V}_\perp} &= 0 & (64)\\
Q_{\mathbf{T}}\hat{}dQ_{\mathbf{T}} &= (HdP - PdH)\hat{}dq\hat{}d\tau = A\hat{}dA & (65)\\
Q_{\mathbf{T}_\perp}\hat{}dQ_{\mathbf{T}_\perp} &= (HdP - PdH)\hat{}dq\hat{}d\tau = A\hat{}dA & (66)
\end{aligned}
$$

The vector $\mathbf{T}$ is known as the Topological Torsion vector (in 4D), and motion in the direction of the Torsion vector (or its orthogonal compliment $\mathbf{T}_\perp$) yield a non-zero value for the 3-form Q. It follows that evolution in the direction of the Torsion vector is thermodynamically irreversible. For any vector with components constructed from $\mathbf{V}$ and $\mathbf{V}_\perp$ with arbitrary functional coefficients, it follows that the Heat 1-form Q satisfies the Frobenius integrability theorem. Only evolutionary processes with components constructed from $\mathbf{T}_\perp$ and $\mathbf{T}$ will represent thermodynamically irreversible processes. The difference between the 3D case and the 4D case is that piecewise evolution in the direction of the basis vectors is always reversible in the 3-D case, but piecewise evolution in the direction of the 4D basis vectors is not always thermodynamically irreversible.

The work of the preceding paragraphs leads to the conclusion:

Conclusion 2: *Thermodynamic irreversibility is an artifact of topological Pfaff dimension 4.*

It is of some interest to recognize that the 1-form $HdP - PdH$ has a rotational polar coordinate representation

$$
HdP - PdH = (P^2 + H^2)\delta(\Theta) \tag{67}
$$

under the mapping

$$
\begin{aligned}
H &= \sqrt{(H^2+P^2)}\cos\Theta & (68)\\
P &= \sqrt{(H^2+P^2)}\sin\Theta. & (69)
\end{aligned}
$$

The 1-form

$$
\delta(\Theta) = (HdP - PdH)/(P^2 + H^2) \qquad d(\delta(\Theta) = 0 \tag{70}
$$

is not an exact 1-form, but it is closed mod the origin of H and P. Such closed but not exact forms are called harmonic forms and have values when integrated over closed cycles with rational ratios.

It should be remarked that for Action 1-forms of even Pfaff topological dimension, addition of a closed form whose format contains new independent variables does change the Pfaff topological dimension of the composite. If the closed 1-form is a function is a function of the original 2n+2 variables, it does not change the Pfaff topological dimension of the even dimensional form. On the other hand, if the original 1-from is renormalized by some factor, then the Pfaff topological dimension does not change.

5 The Cartan-Hilbert Action 1-form.

The topological view establishes a rather precise mathematical definition of what is meant by an equilibrium physical system. Closed or Open physical systems have a disconnected Cartan Topology, and are of Pfaff dimension 3 or 4. The topological difference between a connected and a disconnected topology is a sufficient topological property which can be used to distinguish an equilibrium system from a non-equilibrium physical system. This concept is based on the Frobenius unique integrability theorem, which is valid for an equilibrium system, but not for (most) non-equilibrium systems. However, the concept of equilibrium is more subtle. Bamberg and Sternberg (p. 795 [3]) suggest that a thermodynamic equilibrium state corresponds to a solution of a Lagrangian submanifold structure to an exterior differential system (in 4D). In 4D, the Lagrangian submanifold of a symplectic manifold generated by a 2-form, dA, is a 2 dimensional submanifold upon which the 2-form dA vanishes. Of more interest to this article is how such a submanifold structure may be viewed in terms of the limit set of topological fluctuations in arbitrary dimension.

The starting point of a topological analysis begins with those physical systems that can be encoded in terms of the Cartan-Hilbert [7] 1-form of Action, which at first glance involves 3n+1 independent variables, $(x^k, v^k,\ p_k,\ t)$:

$$A = L(x^k, v^k, t)dt + p_k(dx^k - vdt). \tag{71}$$

$L(x^k, v^k, t)$ is a Lagrange function associated with the physical system. It is assumed that the p_k are Lagrange multipliers, and need not be "canonical" momenta. That is $(\partial L/\partial v^k - p_k) = \varpi_k \neq 0$, where the ϖ_k are the non-canonical components of momenta.

Although it appears that the Cartan-Hilbert Action involves 3n+1 variables, it turns out by direct computation that the Pfaff topological dimension (determined from the Pfaff sequence) is 2n+2. The top Pfaffian in the Pfaff sequence formed from A, and dA and their interior products (eq. 8) has the form of a 2n+2 dimensional volume element:

$$(dA)^{n+1} = (n+1)!\{\Sigma_{k=1}^{n}(\partial L/\partial v^k - p_k)dv^k\}\hat{}dp_1\hat{}...dp_n\hat{}dq^1\hat{}..dq^n\hat{}dt. \tag{72}$$

It follows that the expression in brackets can be represented by a perfect differential, dS :

$$(n+1)!\{\Sigma_{k=1}^{n}(\partial L/\partial v^k - p_k)dv^k\} = \Sigma_{j=1}^{n}\varpi_j dv^j \Rightarrow dS. \quad (73)$$

The exact 1-form, dS, is a candidate for the differential of the Entropy function, S, such that the Top Pfaffian becomes a 2n+2 volume element:

$$(dA)^{n+1} = dS\hat{}dp_1\hat{}...dp_n\hat{}dq^1\hat{}..dq^n\hat{}dt. \quad (74)$$

The 2-form, dA, generates a symplectic manifold structure.

Conclusion 3: *As the 2n+2 form represents a volume element, the coefficient of the top Pfaffian has a representation as a perfect differential of a function, S, which has terms which are not functionally dependent upon* $(x^k,\ p_k,\ t)$. *The differential of entropy function S is explicitly dependent upon the differentials of velocity* dv^k *and the non-canonical components of momentum* $(\partial L/\partial v^k - p_k)$.

Definition: *The function S whose differential is the 1-form*

$$dS = (n+1)!\{\Sigma_{k=1}^{n}(\partial L/\partial v^k - p_k)dv^k\} \quad (75)$$

is defined as the topological entropy.

The even dimensional 2n+2 form represents an orientable volume element, and once an orientation has been fixed (say +1), and evolution is constrained to maintain the volume element as a 2n+2 dimensional element with fixed sign, the change in the entropy, dS, must be of one sign. So entropy, if it changes globally, can be only of one sign (chosen to be positive in the historic literature). Also, as dA is presumed to be non-degenerate, then the differential, dS, can not be zero on the 2n+2 dimensional symplectic space.

Conclusion 4: *Hence the fact that global changes in entropy must be of one sign > 0 is an artifact of topological orientability of the symplectic 2n+2 manifold.*

Next consider subspaces of the Symplectic 2n+2 space. In particular consider a Lagrangian submanifold, which must be dimension n+1. By definition, on the Lagrangian submanifold (of dimension n+1) of the Symplectic space (of dimension 2n+2), the 2-form dA must vanish. The 2-form can be written as:

$$dA = dS\hat{}dt + \{dp_k - \partial L/\partial x^k dt\}\hat{}(\Delta q^k) \Rightarrow 0. \quad (76)$$

Observe that the immersion ψ of the configuration space with differentials $\{dq^1\hat{}..dq^n\hat{}dt\}$ into the top Pfaffian space $\{dS\hat{}dp_1\hat{}...dp_n\hat{}dq^1\hat{}..dq^n\hat{}dt\}$, defines a Lagrangian submanifold when the pullback of the 2-form dA vanishes. The

2-form dA has expression given by the equation above. Consider the case where the immersion into the 3n+1 space is such that the pullback of $(\Delta q^k) \Rightarrow 0$.

$$\psi : (q^1, .., q^n, t) \Rightarrow (S(q, p, t, v), p_1, ..., p_n, q^1, .., q^n, v^1, .., v^n, t) \tag{77}$$

Then the 2-form has a pullback realization such that

$$\psi^*(dA) = dS\hat{}dt \Rightarrow 0 \text{ for a Lagrange submanifold.} \tag{78}$$

The Pfaff topological dimension of the constrained 1-form of Action is then 2 on configuration space, and induces a connected Cartan Topology. The 2-form vanishes when the entropy is a constant:

Conclusion 5: $dS(q, p, t, v) \Rightarrow 0$ *implies equilibrium.*

It is also remarkable to note that $dS = 0$, when the momenta are canonically defined, such that

$$\{\partial L/\partial v^k - p_k\} \Rightarrow 0 \supset dS = \{\Sigma_{k=1}^n(\partial L/\partial v^k - p_k)dv^k\} \Rightarrow 0. \tag{79}$$

The topological concept of entropy is explicitly dependent upon the existence of *non-canonical* momenta, ϖ_j.

6 An Irreversible Example: The Sliding Bowling Ball

6.1 The Observation

Consider a bowling ball given an initial amount of translational energy and rotational energy. Assume the angular momentum and the linear momentum are orthogonal to themselves and also orthogonal to the ambient gravitational field. Then place the bowling ball, subject to these initial conditions, in contact with the bowling alley. Initially, it is observed that the ball slips or skids, dissipating its linear and angular momentum, until the No-Slip condition is achieved. Note that it is possible for the angular momentum or the linear momentum to change sign during the irreversible phase of the evolution. The dynamical system representing the evolutionary process is irreversible until the No-Slip condition is reached. Thereafter, the dynamical system is reversible, and momentum is conserved.

6.2 The Analysis

Assume that the physical system may be represented by a 1-form of Action constructed from a Lagrange function:

$$L = L(x, \theta, v, \omega, t) = \{\beta m(\lambda\omega)^2/2 + mv^2/2\} \tag{80}$$

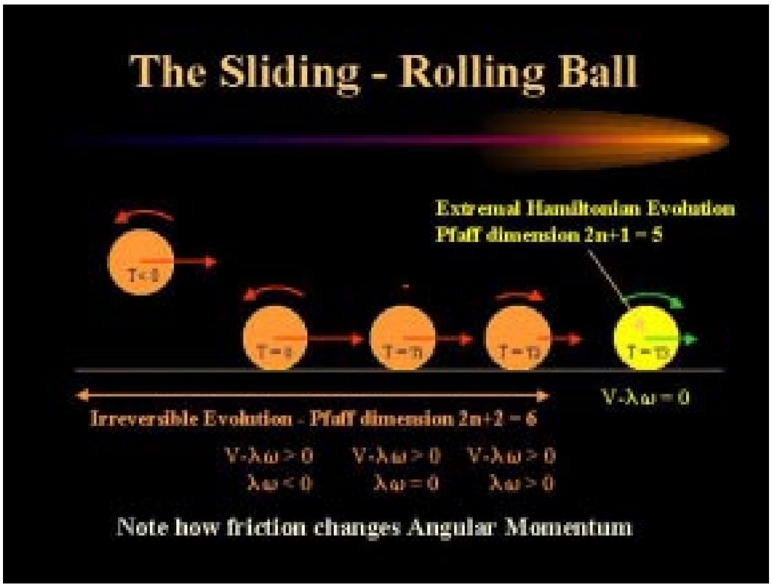

Figure 1:

The constants are: m=mass, β = moment of inertial factor (2/5 for sphere), λ = effective "radius" of the object, the moment of inertia = $\beta m\lambda^2$.

Let the topological constraints be defined anholonomically by the Pfaffian system:

$$\{dx - vdt\} \Rightarrow 0, \quad \{d\theta - \omega dt\} \Rightarrow 0, \quad \{dx - \lambda d\theta\} \Rightarrow 0 \tag{81}$$

Define the constrained 1-form of Action as

$$A = L(x, \theta, v, \omega, t)dt + p\{dx - vdt\} + l\{d\theta - \omega dt\} + ms\{\lambda d\theta - dx\} \tag{82}$$

where $\{p, l, s\}$ are Lagrange multipliers. Rearrange the variables to give (in the language of optimal control theory) a pre-Hamiltonian action:

$$A = (-p - ms)dx + (l + \lambda ms)d\theta - \{pv + l\omega - L\}dt. \tag{83}$$

It is apparent that the Pfaff dimension of this Action 1-form is 2n+2 = 6. The Action defines a symplectic manifold of dimension 6.

For simplicity, assume initially that two of the Lagrange multipliers (momenta) are defined canonically; e.g.,

$$p = \partial L/\partial v \Rightarrow mv, \quad l = \partial L/\partial \omega \Rightarrow \beta m\lambda^2\omega \tag{84}$$

which implies that

$$A = (mv - ms)dx + (\beta m\lambda^2\omega + \lambda s)d\theta - \{-mv^2/2 - \beta m(\lambda\omega)^2/2\}dt. \tag{85}$$

The volume element of the symplectic manifold is given by the expression

$$6Vol = 6m^3\beta\lambda^2\{v-\lambda\omega\}dx\hat{}d\theta\hat{}dv\hat{}d\omega\hat{}ds\hat{}dt = dA\hat{}dA\hat{}dA \tag{86}$$

The symplectic manifold has a singular subset upon which the Pfaff dimension of the Action 1-form is 2n+1 = 5. The constraint for such a contact manifold is precisely the no-slip condition:

$$\{v-\lambda\omega\} \Rightarrow 0 \tag{87}$$

On the 5 dimensional contact manifold there exists a unique extremal (Hamiltonian) field which (to within a projective factor) defines the conservative reversible part of the evolutionary process. As this unique extremal vector satisfies the equation

$$i(\mathbf{V})dA = 0, \tag{88}$$

it is easy to show that dynamical systems defined by such vector fields must be reversible in the thermodynamic sense. (As $dQ = d(i(\mathbf{V})dA) = 0$ for all Hamiltonian or symplectic processes, it follows that $Q\hat{}dQ = 0$.)

However, on the 6 dimensional symplectic manifold, there does not exist a unique extremal field, nor a unique stationary field, that can be used to define the dynamical system. The symplectic manifold does support vector fields, **S**, that leave the Action integral invariant, but these vector fields are not unique in the sense that that they depend on an arbitrary gauge addition to the 1-form of Action that may be required to satisfy initial conditions.

There does exist a unique torsion field (or current) defined (to within a projective factor, σ) by the 6 components of the 5 form,

$$\text{Topological Torsion} = A\hat{}dA\hat{}dA \tag{89}$$

This unique vector, **T**, independent of gauge additions, has the properties that

$$L_{(\mathbf{T})}A = \Gamma \cdot A \quad and \quad i(\mathbf{T})A = 0. \tag{90}$$

This "Torsion" vector field satisfies the equation

$$L_{(\mathbf{T})}A\hat{}L_{(\mathbf{T})}dA = Q\hat{}dQ \neq 0. \tag{91}$$

Hence a dynamical system having a component constructed from this unique Torsion vector field becomes a candidate to describe the initial irreversible decay of angular momentum and kinetic energy.

Solving for the components of the Torsion vector for the bowling ball problem leads to the (unique) decaying dynamical system:

$$dv/dt = m^3\beta\lambda^2\{-\beta\lambda^2\omega^2 - 2\lambda v\omega + v^2\} \tag{92}$$

$$d\omega/dt = m^3\lambda^2\{-\beta\lambda^2\omega^2 + 2\beta\lambda v\omega + v^2\} \tag{93}$$

$$ds/dt = m^3\beta\lambda^2\{-\beta\lambda^2\omega^2 - v^2 - 2(\lambda\omega - v)s\} \tag{94}$$

This is a Volterra system generated on a Finsler space (see p.205 [1]).

It is to be noted that the non-canonical "symplectic momentum" variables, defined by inspection from the constrained 1-form of Action lead to the momentum map:

$$P_x \doteq m(v - s), \qquad P_\theta \doteq m(\beta\lambda^2\omega + s\lambda). \tag{95}$$

Substitution in terms of the momentum variables leads to the generic form (p. 31 [17], also see [13]) for the 1-form of Action:

$$A = P_x dx + P_\theta d\theta - Hdt \tag{96}$$

where H is an independent variable on the 6-dimensional manifold. The H map is given by the expression for energy where v and ω are eliminated in terms of the P_x and the P_θ.

$$H = (mv^2/2 + \beta m(\lambda\omega)^2/2) \Rightarrow 1/2m[(P_x/m + s)^2 + \beta\lambda(\frac{P_\theta/m\lambda - s}{\beta\lambda})^2] \tag{97}$$

Note that $v = \partial H/\partial P_x$ and $\omega = \partial H/\partial P_\theta$. Each component of "canonical momenta" decays with the same rate in the canonical domain.

7 Summary

The topological method leads to a precise evolutionary definition of an entropy function, S, as an integral of the Pfaffian expression, $(n+1)!\{\Sigma_{k=1}^{n}(\partial L/\partial v^k - p_k)dv^k\}$. The continuous change of entropy during a process can never be negative, in order to preserve the orientation of the 2n+2 symplectic manifold generated by the 2-form, dA, representing the limit points of the Action 1-form. Equilibrium is defined by the n+1 dimensional Lagrangian submanifold of the 2n+2 dimensional symplectic space, upon which dA,hence dS, is zero.

It is remarkable that the symplectic systems of irreducible topological dimension 2n+2 seem to solve the Boltzmann - Loschmidt - Zermelo paradox of why *canonical* Hamiltonian mechanics does not seem to be able to describe the decay to an equilibrium state, and why the usual (extremal) methods of Hamiltonian mechanics do not give any insight into the concept of Pressure, Temperature, Entropy or the Gibbs free energy. It is extraordinary that answers to these 150 year old paradoxes of physics seem to follow, without recourse to statistics, or geometric properties of metric or connection, if one utilizes a topological perspective. The interpretation of the fact that the top Pfaffian (for a physical system that can be encoded by a Cartan-Hilbert 1-form of Action) is of dimension 2n+2 and not 3n+1 is, at present, not complete. The implication is that there must exist (3n+1)-(2n+2) = n-1 topological invariants in these systems.

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