I. Envelopes and Topological Torsion

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1. Introduction:

In physical systems the existence of an envelope has its most well-known example in the form of Huygens principle: A wave front (in 3D) is the envelope of multiple expanding spherical surfaces whose multiple origins reside on some initial surface. Herein attention is focused on the fact that the envelope is to be associated with the concept of non-uniqueness: at each point on the wave front, there exists not only the wave front surface but also the spherical wavelet surface. The concept of non-uniqueness implies that a parametric point of view of a surface with its unique range is not applicable. This observation focuses attention on implicit representations of curves and surfaces, where non-uniqueness is admissable.

In the theory of implicit surfaces, the criteria of uniqueness - and therefor the existence of a parametric representation - is related to a differential constraint on the neighborhoods in the form of a Pfaffian equation (a 1-form set equal to zero) defining the surface. If the 1-form, A, satisfies the Frobenious criteria of unique integrability, $A\hat{}dA = 0$, then the surface can be uniquely established in the sense that a normal field can be defined by at most two functions, one giving its scale, and the other its direction field. The direction field is a vector with components defined in terms of the partial derivatives of a unique function. That is, $N = \phi d\psi$. In these cases the Pfaff dimension of the 1-form, A,is 2, and the Topological Torsion 3-form $A\hat{}dA$ is null.

On the otherhand, if $A\hat{}dA \neq 0$, the Pfaff dimension is 3 or greater, and non-uniqueness is to be expected. Topological torsion is not exactly the same as the Frenet torsion of space curve, (which is a parametric, not implicit, concept) nor the more subtle Affine torsion of a connection, but like these concepts Topological Torsion is an artifact of three dimensions or more.

First, a few examples of envelopes will be given to demonstrate how the existence of topological torsion is related to the concept of non-uniqueness.

2. A Family of Curves in the Plane (2+1 space)

As mentioned above, the basic idea of an envelope is that there is a *non-uniqueness* criteria lurking somewhere. First consider the concept of a implicit *curve* in the plane given as the "global" zero set of a function $F(x, y)$ of two variables, (x, y). It is important to note that the curve itself is not necessarily a parameterized set, and can consist of multiple components and branches. No direction (of motion) is defined a priori on any particular curve component by the implicit function equation. In order to define a parameterization of the curve (that is, a direction along a curve component), it is necessary to introduce some third variable, or parameter, say s. This parameter s will be defined as the parameter of *orientation* or *directed* arc length, but such a parameter is not of immediate interest.

A family of non-directed (non-oriented, but orientable) curves may be constructed if the implicit function is a function of one or more *other* parameters, such as $\sigma, \lambda, \ldots.$ Then, for example, in the case of a single (family) parameter, $\sigma,$ the global zero set of $F(x, y..\sigma) = 0$ defines an implicit 2-surface in the 2+1 space of variables $\{x, y..\sigma\}$, with an induced differential Pfaffian equation, or 1-form set equal to zero.

$$dF \equiv (\partial F/\partial x)dx + (\partial F/\partial y)dy + (\partial F/\partial \sigma)d\sigma = F_x dx + F_y dy + F_\sigma d\sigma \Rightarrow 0. \quad (1)$$

For an explicit choice of the family parameter, σ, the implicit function defines a curve which may be viewed as the intersection of the surface $F(x, y..\sigma) = 0$ and the plane defined by the value of σ in the space $\{x, y..\sigma\}$. The differential, dF, has components which may be viewed as the direction field normal to the implicit surface in the space, $\{x, y..\sigma\}$. At certain points all components of the direction field vanish. Such a singular point is defined as a critical point of the implicit function. Critical points will be determined by the points of intersection and contact between two implicit surfaces: the selected surface of the family, $F(x, y..\sigma) = 0$, and the surface where all components of the direction field vanish.

Singular points It is important to distinguish between the singular point sets of the implicit function and the stationary points that may exist on the selected surface of the family, $F(x, y..\sigma) = 0$. The singular critical points are where the induced differential form, dF, is identically zero, a constraint which implies that for every differential directional displacement, $d\mathbf{R}$, the differential form vanishes. Each partial derivative of F as a function of $(x, y..\sigma)$ must vanish identically. For points which are not singular points, it is possible to find n-1 (=2 in the example) differential directions for which the 1-form dF vanishes. These n-1 directions define the tangent space to the implicit surface. If all components of the direction field vanish simultaneously, then Cramer's rule implies that the determinant of the Jacobian matrix of the direction field (F_x, F_y, F_σ) must vanish at certain points, $\mathbf{R}_{critical}(x, y, \sigma)$. The condition can be expressed as by the fact that n=3 form must vanish,

$$\Theta = dF_x\hat{}dF_y\hat{}dF_\sigma = \beta(x, y, \sigma)dx\hat{}dy\hat{}d\sigma \Rightarrow 0 \text{ at a singular point.} \tag{2}$$

Note that the zero set of the function

$$\beta(x, y, \sigma) = \det[\mathbb{J}(gradF)] = \det \begin{bmatrix} F_{xx} & F_{xy} & F_{x\sigma} \\ F_{yx} & F_{yy} & F_{y\sigma} \\ F_{\sigma x} & F_{\sigma y} & F_{\sigma\sigma} \end{bmatrix} \Rightarrow 0 \tag{3}$$

defines the "surface of singular points" of the function, $F(x, y, s)$. The two (possibly multiple component) surfaces, $\beta(x, y, \sigma) = 0$, and the selected surface, $F(x, y, \sigma) = 0$, have intersections at points when $dF\hat{}d\beta \neq 0$. In three dimensions this object (a 2-form) has components proportional to the Gibbs cross product of the direction field normal to the implicit surface and the direction field normal to the surface of critical points. At points where the intersection 2-form vanishes, the two surfaces can either be disjoint, or have a point of contact. Hence logical intersection of the critical points includes points of surface intersection and points of surface contact. The problem of finding the critical points is a global issue, but given a point it is possible to test to see if it is a critical point using (local) differential methods. If the 3-form Θ never changes sign, there is no implicit surface of critical points. If Θ is zero, then a surface of critical points exists, but this surface may not have intersection with the surface $F(x, y, \sigma) = 0$. An intersection exists producing a curve of critical points when the 2-form $dF\hat{}d\beta$ is not zero.

Envelopes Envelopes are also related to the intersection of two surfaces

generated by the family. The same is true for envelopes. To find an envelope is more difficult than to determine whether or not the envelope exists.

There exist neighborhood directions constraining the displacements dx, dy, ds such that the differential form dF vanishes in those selected (not all) directions. The covariant components of the 1-form dF define the normal direction field to the implicit 2-surface $F(x, y, s) = 0$. Displacements orthogonal to the normal field satisfy the equation $dF = 0$. Note that the zero set of the implicit function creates a surface in 3 dimension space, $\{x, y, s\}$, not a curve. In order to determine a space curve, a second surface must be described, and the intersection of this second surface and the first surface yields the "space curve". For example, the intersection of the surface $F(x, y, s) = 0$ with the plane $s = 0$ determines a curve in the x, y plane. As s varies a family of such curves are produced which may have multiple components. It is assumed that the members of the family can be projected to the $\{x, y\}$ plane.

For a point p on the surface $F(x, y, s) = 0$ (in $\{x, y, s\}$ space), the neighborhood directions that cause dF to vanish are directions orthogonal to the surface normal at the point p. These directions determine the tangent plane to the surface at p. A second surface in 2+1 space can be determined from the original implicit function by differentiation with respect to the family parameter and setting the resulting function to zero: $F_s(x, y, s) = \partial F/\partial s = 0$. The intersection of these two surfaces produces a tortuous curve of perhaps several segments (components) in the space $\{x, y, s\}$. A necessary condition that the two surfaces $F(x, y, s) = 0$ and $F_s(x, y, s) = 0$ have an intersection (simultaneous solution) is established by the requirement that the exterior product of the two 1 forms dF and dF_s does not vanish. On the sets $F(x, y, s) = 0$ and $F_s(x, y, s) = 0$, this requirement reduces to the constraint

$$dF\hat{}dF_s = \{F_x F_{sy} - F_y F_{sx}\}dx\hat{}dy + F_{ss}\{F_x dx + F_y dy\}\hat{}ds \neq 0. \qquad (4)$$

If $F_{ss} \neq 0$, then it is possible to solve for s from the equation of the second surface, $F_s(x, y, s) = 0$. Use this value to eliminate the family parameter in the first equation of the surface. The result is a Function of $\{x, y\}$ only, that defines a curve in the $\{x, y\}$ plane independent from the family parameter. This curve is the envelope.

The surface constraint induces the Pfaffian equation derived from the differential 1-form,

$$dF_s \equiv (\partial^2 F/\partial s\partial x)dx + (\partial^2 F/\partial s\partial y)dy + (\partial^2 F/\partial s\partial s)ds = F_{sx}dx + F_{sy}dy + F_{ss}ds. \tag{5}$$

A necessary condition that the two surfaces $F(x, y, s) = 0$ and $F_s(x, y, s) = 0$ have an intersection (simultaneous solution) is established by the requirement that the exterior product of the two 1 forms dF and dF_s does not vanish. On the sets $F(x, y, s) = 0$ and $F_s(x, y, s) = 0$, this requirement reduces to the constraint

$$dF\hat{}dF_s = \{F_x F_{sy} - F_y F_{sx}\}dx\hat{}dy + F_{ss}\{F_x dx + F_y dy\}\hat{}ds \neq 0. \tag{6}$$

This intersection has a component in the $\{x, y\}$ plane if the first factor does not vanish. Not that (for fixed s) the critical points of $(F = 0) \cap (F_x = 0) \cap (F_y = 0)$ must be excluded. In more simple language, the critical points are where the tangent vector to the surface vanishes, and the points of interest for self intersection and envelopes is where the normal vector to the surface is zero or for then such a point is

The three components of this 2-form on 2+1 space form the components of a contravariant vector, **J**, which is tangent to the curve of intersection. If all three components vanish, then the two surfaces do not intersect. In particular, if

$$\{F_x F_{sy} - F_y F_{sx}\} = 0 \quad and \quad F_{ss} = 0, \tag{7}$$

there is no intersection and no singularity. If

$$[\{F_x F_{sy} - F_y F_{sx}\} = 0 \ and \ F_{ss} \neq 0] \tag{8}$$

or

$$[\{F_x F_{sy} - F_y F_{sx}\} \neq 0 \quad and \quad F_{ss} = 0], \tag{9}$$

there is a singularity, but no envelope.

If both

$$[\{F_x F_{sy} - F_y F_{sx}\} \neq 0 \quad and \quad F_{ss} \neq 0] \tag{10}$$

then there is a curve which is an envelope of the family of curves. Note that the envelope condition implies that the primitive function, F, is non-linear in the family parameter, s.

The process can be continued. A cuspoidal point of regression can be determined when the three functions $F, F_s,$ *and* F_{ss} satisfy the equation,

$$dF\hat{}dF_s\hat{}dF_{ss} \neq 0. \tag{11}$$

3. A Family of Surfaces in 3+1 space

The basic idea extends to higher dimensions. An implicit function $\Phi(x, y, z, \sigma) = 0$,does not determine a surface in 3-space, but instead determines a hypersurface in 4 space. For a family of surfaces in three dimensions $\{x, y, z\}$, with a family parameter, σ, the criteria for intersection of $\Phi(x, y, z, \sigma) = 0$ and $\partial\Phi(x, y, z, \sigma)/\partial\sigma = \Phi_\sigma(x, y, z, \sigma) = 0$ becomes

$$\begin{aligned} d\Phi\hat{}d\Phi_\sigma &= \{\Phi_x\Phi_{\sigma y} - \Phi_y\Phi_{\sigma x}\}dx\hat{}dy + \\ &\quad \{\Phi_y\Phi_{\sigma z} - \Phi_z\Phi_{\sigma y}\}dy\hat{}dz + \\ &\quad \{\Phi_z\Phi_{\sigma x} - \Phi_x\Phi_{\sigma z}\}dz\hat{}dx + \\ &\quad \Phi_{\sigma\sigma}d\Phi\hat{}d\sigma \\ &\neq 0 \end{aligned} \tag{12}$$

The first three terms are to be recognized as the components of the cross product,

$$\nabla_{(x,y,z)}\Phi \times \nabla_{(x,y,z)}\Phi_\sigma. \tag{13}$$

The argument is that when either

$$\{\nabla_{(x,y,z)}\Phi \times \nabla_{(x,y,z)}\Phi_\sigma \neq 0 \qquad \text{and} \quad \Phi_{\sigma\sigma} = 0\} \tag{14}$$

,or

$$\{\nabla_{(x,y,z)}\Phi \times \nabla_{(x,y,z)}\Phi_\sigma = 0 \qquad \text{and} \quad \Phi_{\sigma\sigma} \neq 0\}, \tag{15}$$

then the family has an intersection singularity.

When both

$$\nabla_{(x,y,z)}\Phi \times \nabla_{(x,y,z)}\Phi_{\sigma} \neq 0 \;\; and \;\; \Phi_{\sigma\sigma} \neq 0 \tag{16}$$

then there is a surface envelope. Only non-linear family parameters produce envelopes.

4. The edge of regression

The process can be extended to find an edge of regression. In this case it is assumed that the three zero sets $\Phi(x, y, z, \sigma) = 0$, $\Phi_{\sigma}(x, y, z, \sigma) = 0$ and $\Phi_{\sigma\sigma}(x, y, z, \sigma) = 0$ have a common solution. The criteria for solubility for an edge of regression requires that the three form, which is the exterior product of all three differentials, does not vanish:

$$d\Phi\hat{}d\Phi_{\sigma}\hat{}d\Phi_{\sigma\sigma} \neq 0. \tag{17}$$

The spatial components of this expression require that

$$(\nabla_{(x,y,z)}\Phi \times \nabla_{(x,y,z)}\Phi_{\sigma}) \bullet (\nabla_{(x,y,z)}\Phi_{\sigma\sigma}) \neq 0 \tag{18}$$

for the existence of an (cuspoidal) edge of regression.

5. Examples of Envelopes of families of surfaces.

A. Spheres moving along x axis: The cylindrical canal surface.

Consider the function

$$\Phi = (x - \sigma)^2 + y^2 + z^2 - 1 \tag{19}$$

with a zero set which represents a family of unit spheres with centers at $\sigma = ct$ moving along the x axis.

$$\Phi_{\sigma} = \partial\Phi/\partial\sigma = -2(x - \sigma), \qquad \Phi_{\sigma\sigma} = +2. \tag{20}$$

The 2-form $dA = -d\Phi\hat{}d\Phi_{\sigma} \Rightarrow \{4zdx\hat{}dz + 4ydx\hat{}dy\}$ at $d\sigma = 0$, is non-zero, and $\Phi_{\sigma\sigma} \neq 0$. From another point of view, $\nabla_{(x,y,z)}\Phi \times \nabla_{(x,y,z)}\Phi_{\sigma} = 0\mathbf{i} - 4z\mathbf{j} + 4y\mathbf{k}$. Therefor the necessary conditions for the existence of an envelope are valid. Solving for σ from $\Phi_\sigma = 0$ and substituting in $\Phi = 0$, leads to the equation of the envelope,

$$y^2 + z^2 - 1 = 0 \tag{21}$$

The envelope is a cylinder of radius 1, with the x axis as the axis of rotational symmetry. The 3-form $d\Phi\hat{}d\Phi_\sigma\hat{}d\Phi_{\sigma\sigma}$ vanishes so there is no edge of regression..

B. Expanding spheres moving along the x-axis: The Mach cone.

Consider the function

$$\Phi = (x - k\sigma)^2 + y^2 + z^2 - \sigma^2 \tag{22}$$

with a zero set which represents a family of expanding spheres of radius σ with centers at $k\sigma$ moving along the x axis. When $k > 1$ the translational speed exceeds the expansion speed (of, say, sound, where $\sigma = ct$)

$$\Phi_\sigma = \partial\Phi/\partial\sigma = -2k(x) + 2(k^2 - 1)\sigma, \qquad \Phi_{\sigma\sigma} = +2(k^2 - 1). \tag{23}$$

The 2-form $dA = -d\Phi\hat{}d\Phi_\sigma \Rightarrow \{4zkdx\hat{}dz + 4ykdx\hat{}dy\}$ for $d\sigma = 0$, and is non-zero, and $\Phi_{\sigma\sigma} \neq 0$.. Therefor the necessary conditions for the existence of an envelope are valid. Solving for σ from $\Phi_\sigma = 0$ and substituting in $\Phi = 0$, leads to the equation of the envelope,

$$(k^2 - 1)(y^2 + z^2) - x^2 = 0 \tag{24}$$

which is a cone (the Mach cone), with a symmetry axis as the x axis, and an aperature

$$tan\theta = \sqrt{1/(k^2 - 1)}. \tag{25}$$

The 3-form $d\Phi\hat{}d\Phi_\sigma\hat{}d\Phi_{\sigma\sigma}$ vanishes so there is no edge of regression.

C. Concentric Spheres

Consider the function

$$\Phi = x^2 + y^2 + z^2 - \sigma^2 \tag{26}$$

with a zero set which represents a family of unit spheres with variable radii, $\sigma = ct$, and centered on the origin.

$$\Phi_\sigma = -2\sigma, \qquad \Phi_{\sigma\sigma} = -2. \tag{27}$$

The 2-form $dA = -d\Phi\hat{}d\Phi_\sigma = 0$ for $d\sigma = 0$. Therefor the necessary conditions for the existence of an envelope are not valid. The family of surfaces do not intersect as

$$\nabla_{(x,y,z)}\Phi \times \nabla_{(x,y,z)}\Phi_\sigma = 0 \qquad . \tag{28}$$

The 1-form A is integrable. The 3-form $d\Phi\hat{}d\Phi_\sigma\hat{}d\Phi_{\sigma\sigma}$ vanishes so there is no edge of regression.

D. Spheres with a common point of tangency on the x axis.

Consider the function

$$\Phi = (x - \sigma)^2 + y^2 + z^2 - \sigma^2 \tag{29}$$

with a zero set which represents a family of spheres of various radii and with centers along the x axis.

$$\Phi_\sigma = \partial\Phi/\partial\sigma = -2x, \qquad \Phi_{\sigma\sigma} = 0. \tag{30}$$

The 2-form $dA = -d\Phi\hat{}d\Phi_\sigma \neq 0$ for $d\sigma = 0$. Therefor the *necessary* condition for the intersection singularity exists, but the subsidiary condition $\Phi_{\sigma\sigma} \neq 0$ is not satisfied. The singularity is the point where all the spheres have a common tangent, $\{x = 0,\ y^2 + z^2 = 0\}$. The envelope **does not exist** because the subsidiary condition $\Phi_{\sigma\sigma} \neq 0$ is not valid.

E. Spheres with a common circle of intersection.

Consider the function

$$\Phi = (x-\sigma)^2 + y^2 + z^2 - (a^2+\sigma^2) \tag{31}$$

with a zero set which represents a family of spheres with centers along the x axis.

$$\Phi_\sigma = \partial\Phi/\partial\sigma = -2(x), \qquad \Phi_{\sigma\sigma} = 0. \tag{32}$$

The 2-form $dA = -d\Phi\hat{}d\Phi_\sigma \neq 0$ for $d\sigma = 0$. Therefor the *necessary* condition for the intersection singularity exists, but the subsidiary condition $\Phi_{\sigma\sigma} \neq 0$ is not satisfied. However, the singularity exists as the circle of radius a in the x=0 plane: $\{x=0,\ \ y^2+z^2=a^2\}$. The envelope **does not exist** because of the subsidiary condition $\Phi_{\sigma\sigma} \neq 0$ is not valid.

F. The Jacobian cubic characteristic polynomial.

Consider the cubic polynomial

$$\Phi(X,Y,Z;\sigma) = \sigma^3 - X\sigma^2 + Y\sigma - Z \tag{33}$$

with

$$\Phi_\sigma(X,Y,Z;\sigma) = \partial\Phi/\partial\sigma = 3\sigma^2 - X2\sigma + Y \tag{34}$$

and similarly,

$$\Phi_{\sigma\sigma}(X,Y,Z;\sigma) = 6\sigma - 2X \tag{35}$$

For real σ, the zero set, $\Phi(X,Y,Z;\sigma)=0$, may be viewed as a family of 3 dimensional hypersurfaces in the space $(X,Y,Z;\sigma)$ with the family parameter, σ. It is useful to determine if the family of surfaces has an envelope. The envelope will be determined by the intersection of the two hypersurfaces, $\Phi(X,Y,Z;\sigma)=0$, and $\Phi_\sigma(X,Y,Z;\sigma)=0$. The envelope is a two dimensional surface independent from the parametrization, and is regular if $\Phi_{\sigma\sigma}(X,Y,Z;\sigma)\neq 0$.

For a given vector field, $\mathbf{V}(x,y,z)$, the similarity invariants of the Jacobian matrix of $\mathbf{V}$ with respect to (x,y,z) can be used to determine the explicit form for the functions, $\{X(x,y,z),Y(x,y,z),Z(x,y,z)\}$. The similarity invariants of a given Jacobian matrix, $[\mathbb{J}]$, are given by formulas:

$$X(x,y,z) = trace\,[\mathbb{J}]\,, \tag{36}$$

$$Y(x,y,z) = trace\,[\mathbb{J}]^{adjoint}\,, \tag{37}$$

$$Z(x,y,z) = det\,[\mathbb{J}]\,. \tag{38}$$

The cubic polynomial is then the Cayley-Hamilton characteristic polynomial of the Jacobian matrix, which always exists for the 3x3 matrix,

$$Characteristic\ Polynomial\ of\ \ [\mathbb{J}] \Rightarrow \sigma^3 - X\sigma^2 + Y\sigma - Z = 0, \tag{39}$$

with the "family parameter" $\sigma(X, Y, Z)$ playing the role of the eigenvalues for the matrix, $[\mathbb{J}]$. From classic theory,

$$\begin{aligned} X &= \sigma_1 + \sigma_2 + \sigma_3 && (40)\\ Y &= \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1 && (41)\\ Z &= \sigma_1\sigma_2\sigma_3, && (42) \end{aligned}$$

where $\sigma_1, \sigma_2, \sigma_3$ are the eigenvalues of $[\mathbb{J}]$, and can be viewed as functions on (x, y, z).

Solving for σ from $\Phi_\sigma = 0$ leads to two roots, σ_+ and σ_-. Substitution into the primary cubic $\Phi = 0$ leads to two envelop equations, $\Phi_+ = 0$, and $\Phi_- = 0$, depending on which root is chosen for the substitution. There are two envelope sheets with each envelope function independent of the parameter, σ. The product of the two functions leads to a new function which is precisely the historical Cardano function, with values that determine the root structure of the polynomial, and whose zero set determines the degenerate root structure, or the edge of regression of the two envelope sheets.

$$Cardano\ = \Phi_+ \cdot \Phi_- = -4(X^2 - 3Y)^3 + (2X^3 - 9XY + 27Z)^2. \tag{43}$$

This derivation of the Cardano formula is remarkable in that it is based upon the theory of envelopes in a space of 3 coordinates and 1 family parameter.

The intersection of the two gradient surfaces is given by the expression,

Another way to specify the envelope is to construct the 1-form A in a space of 3+1 variables,

$$A = \Phi_x dx + \Phi_y dy + \Phi_z dz ... = d\Phi - \Phi_\sigma d\sigma = d(\Phi - \sigma\Phi_\sigma) + \sigma d\Phi_\sigma, \qquad (44)$$

(which by construction is not explicitly dependent only upon displacement, $d\sigma$). This 1-form may not be globally exact, as $dA = -d\Phi_\sigma\hat{} d\sigma \neq 0$,necessarily. In fact, this 1-form, A, need not be uniquely integrable, for globally $A\hat{} dA = -d\Phi\hat{} d\Phi_\sigma\hat{} d\sigma \neq 0$, necessarily. If the 2-form $d\Phi\hat{} d\Phi_\sigma = 0 \mod d\sigma$, then no envelope exists, and the Topological Torsion of the 1-form vanishes, $A\hat{} dA = 0$. In other words, the 1-form A defined above does not satisfy the Frobenius criteria of unique integrability, when an envelope exists. Moreover, the space exhibits Topological Torsion. This result is pleasing, for the concept of an envelope intuitively implies non-uniqueness.

For the cubic polynomial it is known that the Cardano function not only separates the domains for which the eigenvalues are real or complex, but also the zero set of the Cardano function, when it has an intersection with the envelope determines a curve upon which there can exist repeated roots. The edge of regression for the two sheets of the envelope is precisely such a curve of repeated roots. The tangent vector to the curve which is the edge of regression is given by solving for $\sigma = X/3$ from the equation, $\Phi_{\sigma\sigma}(X, Y, Z; \sigma) = 0$. Substitution of this value for sigma into the equation for the position vector in $[X, Y, Z]$ space, $\mathbf{J} =[X, Y, Z]$ leads to tangent vector at the edge of regression, $\mathbf{J} =[1, 2X/3, X^2/9]$. A plot of the double sheeted Cardano envelope appears in Figure 1.

The 2-form $J = -d\Phi\hat{} d\Phi_\sigma \Rightarrow \{dy\hat{} dz - 2\sigma dx\hat{} dz + \sigma^2 dx\hat{} dz\}$ for $d\sigma = 0$, and is non-zero, and $\Phi_{\sigma\sigma} \neq 0$. Therefor the conditions for the existence of an envelope are valid. The tangent vector to the characteristic curve is given by $\mathbf{J} = [1, 2\sigma, \sigma^2]$. The envelope consists of two sheets which join at the edge of regression. As the 3-form, $d\Phi\hat{} d\Phi_\sigma\hat{} d\Phi_{\sigma\sigma} \Rightarrow -2dx\hat{} dy\hat{} dz$ is non-zero for $d\sigma = 0$, and $\Phi_{\sigma\sigma\sigma} \neq 0$, an edge of regression exists. For the cubic polynomial it is known that the Cardano function not only separates the domains for which the eigenvalues are real or complex, but is also the surface upon which there can exist repeated roots. The edge of regression is precisely such a curve of repeated roots. The tangent vector to the curve which is the edge of regression is given by solving for $\sigma = x/3$ from the equation, $\Phi_{\sigma\sigma}(x, y, z; \sigma) = 0$. Substitution of this value into the equation for $\mathbf{J}$ leads to tangent vector at the edge of regression, $\mathbf{J} =[1, 2x/3, x^2/9]$. A plot of the Cardano envelope appears in Figure 1. Note the edge of regression.

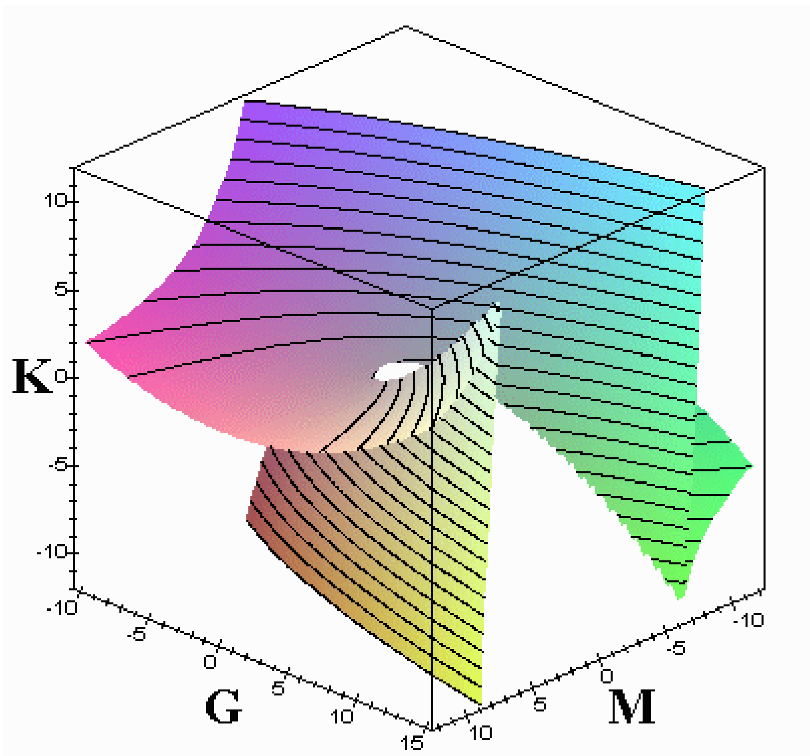

Figure 1: **Cardano surface. Note edge of regression**

The plot is a universal plot, where the coordinates are the similarity invariants, the Mean curvature (M), the Gauss curvature (G) , and the Determinant (K), of the Jacobian matrix of the associated cubic polynomial.

The Cardano function (or envelope) can be constructed as a tangential developable based on the curve whose tangent vector is given by $\mathbf{J} =[1, 2x/3, x^2/9]$. A point on the Cardano surface is given by $\mathbf{X} = \mathbf{R} \pm \lambda\mathbf{J}$ with 1 sheet of the envelope determined by positive motion along the edge of regression and the other sheet of the envelope determined by motion in the opposite direction. It is important to note that the neighborhoods are not "time reversal invariant", although the edge of regression is "time reversal invariant". This property of the trajectory neighborhoods is due to the fact the edge of regression has torsion in the sense of Frenet.

These concepts have utility in thermodynamics, for the Gibbs equilibrium surface of a Van der Waals gas is a function which is cubic in its family parameter. The Spinodal Line of the Gibbs surface is an edge of regression (and is determined by the condition that the Gauss curvature vanish). The Binodal line is a line of self intersection. The critical point is where both the mean curvature and the Gauss curvature of the surface vanish.

Thermodynamically, the Spinodal line is the edge of regression of the Gibbs surface for a Van der Waals gas. The observation above regarding time reversal invariance implies that motion along the spinodal line in the direction of the crictical point is stable in one direction, but unstable in the other.

1. Summary

In each of the examples above, the criteria for an envelope to exist requires that the 2-form $d\Phi\hat{\ }d\Phi_\sigma$ does not vanish for $d\sigma = 0$ (σ is constrained to be a constant). Consider the 1-form A in a space of 3+1 variables,

$$A = \Phi_x dx + \Phi_y dy + \Phi_z dz... = d\Phi - \Phi_\sigma d\sigma = d(\Phi - \sigma\Phi_\sigma) + \sigma d\Phi_\sigma, \qquad (45)$$

(which by construction is not explicitly dependent only upon displacement, $d\sigma$). This 1-form may not be globally exact, as $dA = -d\Phi_\sigma\hat{\ }d\sigma \neq 0$,necessarily. In fact, this 1-form, A, need not be uniquely integrable, for globally $A\hat{\ }dA = -d\Phi\hat{\ }d\Phi_\sigma\hat{\ }d\sigma \neq 0$, necessarily. If the 2-form $d\Phi\hat{\ }d\Phi_\sigma = 0 \bmod d\sigma$, then no envelope exists, and the Topological Torsion of the 1-form vanishes, $A\hat{}dA = 0$.

When the Topological Torison does not vanish, $A\hat{}dA \neq 0$,then there exists more than one solution function to the equation $A = 0$ (non-uniqueness), and therefore the failure of the Frobenius unique integrability criteria leads to the possibility of an envelope. The conclusion to be reached is that:

The existence of topological torsion is necessary for the existence of an envelope.

2. The General Theory

The necessary and sufficient conditions for an envelope of a family of functions parameterized by σ are given by the exterior diferential system:

$$d\Phi\hat{}d\Phi_\sigma \neq 0 \;\; and \;\; \Phi_{\sigma\sigma} \neq 0. \tag{46}$$

Consider the 1-form A in a space of 3+1 variables,

$$A = \Phi_x dx + \Phi_y dy + \Phi_z dz ... = d\Phi - \Phi_\sigma d\sigma, \tag{47}$$

(which by construction is not explicitly dependent only upon displacement, $d\sigma$). This 1-form may not be globally exact, as $dA = -d\Phi_\sigma\hat{}d\sigma \neq 0$,necessarily. In fact, this 1-form, A, need not be uniquely integrable, for globally $A\hat{}dA = -d\Phi\hat{}d\Phi_\sigma\hat{}d\sigma \neq 0$, necessarily. If A satisfies the Frobenious integrability theorem, $A\hat{}dA = 0$, and then there exits a globally unique function, $\Theta(x, y, z; \sigma)$ such that the zero set of $\Theta(x, y, z; \sigma)$ defines a hypersurface of dimension 3 and is a solution to the Pfaffian equation, $\lambda A = 0$. For such integrable cases, A is reducible to the format, $A = \beta d\Theta$.

When $H = A\hat{}dA \neq 0$,then there exists more than one solution function to the equation $A = 0$ (non-uniqueness), and therefore the failure of the Frobenius criteria uniqueness leads to the possibility of an envelope. On the otherhand, if the Frobenious condition is valid, the topological torsion, H, is zero, and the criteria for the existence of an envelope is not satisfied.

In 2+1 space, the first criteria for an envelope depends upon the possibility that the two surfaces $\Phi = 0$ and $\Phi_\sigma = \partial\Phi/\partial\sigma = 0$, for fixed values of σ, have an intersection. The criteria that an intersection exists is given by the differential form statement that $d\Phi\hat{}d\Phi_\sigma \neq 0$. The curve that represents this intersection of the two surfaces is usually called the "characteristic" curve. This characteristic curve in the plane is obtained from the solutions to the subsidiary equations $d\mathbf{R} - \mathbf{J}ds = 0$ where $\mathbf{J} = \{\nabla\Phi \times \nabla\Phi_\sigma\}$ is the tangent vector to the curve of intersection projected to the x,y plane. The initial conditions of this characteristic curve are not arbitrary; they must be adjusted such that the tangent vector resides on the intersection of the two surfaces, $\Phi = 0$ and $\Phi_\sigma = 0$. The characteristic curve is a very special curve selected out of the vector field, $\mathbf{J}$. The gradient operations are with respect to the three variables $\{x, y, \sigma\}$

$$\mathbf{J} = +\{\Phi_y\Phi_{\sigma\sigma} - \Phi_\sigma\Phi_{\sigma y}\}\mathbf{i} - \{\Phi_x\Phi_{\sigma\sigma} - \Phi_\sigma\Phi_{\sigma x}\}\mathbf{j} + \{\Phi_x\Phi_{\sigma y} - \Phi_y\Phi_{\sigma x}\}\mathbf{k} \quad (48)$$

As $\Phi_\sigma = 0$, the tangent vector to the curve of intersection becomes

$$\mathbf{J} = +\{\Phi_y\Phi_{\sigma\sigma}\}\mathbf{i} - \{\Phi_x\Phi_{\sigma\sigma}\}\mathbf{j} + \{\Phi_x\Phi_{\sigma y} - \Phi_y\Phi_{\sigma x}\}\mathbf{k} \quad (49)$$

such that if $\Phi_{\sigma\sigma} = 0$, then the tangent vector to the enveloping curve has no components in the two dimensional subspace of {x,y}. In this situation, the envelope is not "visible" and has no extension when projected to the x,y plane.

The same argument works in higher dimensions. The basic idea is that if for a singly parametrized function, $\Phi(x, y, z...; \sigma)$ on a space of N+1 dimensions, the 1-form $A = d\Phi - \Phi_\sigma d\sigma$ is not necessarily globally integrable, a fact which implies non-uniqueness of the solution to the Paffian equation, $A = 0$. The concept of non-uniqueness admits to the possibility of finding an envelope of dimension N-2 which is independent from the parameter, σ. For suppose $\Phi_\sigma = 0$ defines a set of dimension N-1 which intersects with the N-1 set $\Phi = 0$, to produce a set of dimension N-2. In order for an envelope to exist, the non-uniqueness argument implies as a necessary condition that the 2 form $d\Phi\hat{}d\Phi_\sigma$ cannot vanish, and a sufficient condition for non-uniqueness as $A\hat{}dA = -d\Phi\hat{}d\Phi_\sigma\hat{}d\sigma \neq 0$. This result implies that the condition for the existence of an envelope in three spatial dimensions and one parametric dimension requires that

$$A\hat{}dA \neq 0 \Rightarrow \nabla_{(x,y,z)}\Phi \times \nabla_{(x,y,z)}\Phi_\sigma \neq 0. \quad (50)$$

In 3+1 space, the "envelope" is the 2 dimensional *surface* of intersection of the two 3 dimensional sets, $\Phi = 0$ and $\Phi_\sigma = 0$, subject to the constraint that $\Phi_{\sigma\sigma} \neq 0$.

The edge of regression The surface function may be non-linear in the parameter σ, such that it is possible to compute $\Phi = 0$, $\Phi_\sigma = 0$, and $\Phi_{\sigma\sigma} = 0$ to find a simultaneous intersection of the three N-1 sets to produce in this case a 1 dimensional line. For the intersection to be non empty it is necessary that the three form $d\Phi_\sigma\hat{}d\Phi_{\sigma\sigma}\hat{}d\Phi \neq 0$.

As the function $\Phi_\sigma(x, y, \sigma) = \partial\Phi/\partial\sigma$ induces second 1-form given by the expression,

$$d\Phi_\sigma = \Phi_{\sigma x}dx + \Phi_{\sigma y}dy + \Phi_{\sigma\sigma}d\sigma. \tag{51}$$

it is possible to construct the 1-form,

$$\omega = \Phi_{\sigma\sigma}d\Phi - \Phi_\sigma d\Phi_\sigma = \Phi_{\sigma\sigma}d\Phi - d(\Phi_\sigma^2/2), \tag{52}$$

which is not only independent from explicit variations in $d\sigma$,but also is in the classical Darboux format. Note that the 2-form constructed as

$$\begin{aligned}(d\Phi - \Phi_\sigma d\sigma)\hat{}(d\Phi_\sigma - \Phi_{\sigma\sigma}d\sigma) &= d\Phi\hat{}d\Phi_\sigma + (\Phi_\sigma d\Phi_\sigma - \Phi_{\sigma\sigma}d\Phi)\hat{}d\sigma &(53)\\ &= d\Phi\hat{}d\Phi_\sigma + \omega\hat{}d\sigma. &(54)\end{aligned}$$

Then $d\omega = d\Phi_{\sigma\sigma}\hat{}d\Phi$, and $\omega\hat{}d\omega = -\Phi_\sigma d\Phi_\sigma\hat{}d\Phi_{\sigma\sigma}\hat{}d\Phi$ is a three form in 4 dimensional space. When $\omega\hat{}d\omega \neq 0$, it is possible to find a common intersection of the three equations, $\Phi = 0$, $\Phi_\sigma = 0$, and $\Phi_{\sigma\sigma} = 0$, represented as a non-zero three form in $\{x,y,z,\sigma\}$. The components of this contravariant vector density may be used to compute a tortuous curve which projects to 3 dimensional space as the 1-dimensional curve representing the edge of regression of the two dimensional surface envelope.

3. Dynamical Systems

First subsume the existence of a function $H(x, y, z; p)$ such that $V(x, y, z; p) := \partial H/\partial p$. The condition that $H(x, y, z; p) = 0$ may be viewed as a family of surfaces paramterized by p. The two criteria,

$$H(x, y, z; t) = 0 \quad and \quad V(x, y, z; t) = \partial H/\partial p = H_p = 0, \tag{55}$$

may be interpreted as the conditions that establish the existence of a envelope to the family of surfaces, $H(x, y, z; p) = 0$. The envelope will be smooth if $\partial^2 H/\partial p^2 = H_{pp} \neq 0$. The envelope will have an edge of regression at points where $\partial^2 H/\partial p^2 = 0$, but only if the three form $dH^\wedge dH_p^\wedge dH_{pp}$ at constant p is not zero. Using the equation, $\partial^2 H/\partial p^2 = 0$ to determine p, the remaining two equations determine the position vector to the edge of regression as a curve generated by the vector which is equal to the cross product of ∇H and ∇H_p. When the three form vanishes, the edge of regression has a self intersection.

Now consider the origin of the function H. Consider the fact that Jacobian matrix of an arbitrary vector field (as a dynamical system) always generates its Cayley Hamilton characteristic polynomial equation, defined as $V(x, y, z; p, s..) = 0$, where p is the complex field of matrix eigenvalues of the Jacobian matrix, and s represents possible other parameters. This characteristic polynomial may be considered as a family of implicit surfaces in the space of coordinates {x,y,z; p.. } for each specific choice of the complex parameter, p. When the eigen values, p, are renormalized to dimensionless variables in terms of the similarity invariant scales of the Jacobian matrix, this characteristic polynomial equation will become equivalent to an equation of state, which is in fact a projective equivalent of a van der Waals gas. It can be shown that the renormalized eigenvalue parameter has the properties of a complex thermodynamic "molar density". It follows that every 3 dimensional dynamical system has an equivalent representation as a Van der Waals gas.

From thermodynamics it is known that the equation of state is an incomplete decription of a thermodyamic system, in that it represents only one of the partial derivatives of the primitive thermodynamic potential, $H(x, y, z; p, s..)$. The equation of state is given by the equation

$$H_p = V = \partial H(x, y, z; p, s..)/\partial p = 0. \tag{56}$$

The equation of state and the zero set of the function, H form the necessary conditions for the existence of an envelope of H in terms of the parameter, p. The criteria that the envelope be smooth in the variable p requires that $H_{pp}(x, y, z; p) \neq 0$. Hence when $H_{pp}(x, y, z; p) = 0$, the failure of surface smoothness is given by an edge of regression, or a self intersection.