A Strong Equivalence Principle

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Abstract

Consider C_n spaces where the exterior differentiation of a vector basis of C2 functions leads to a Cartan matrix of exterior differential 1-forms. A second exterior differentiation leads to a Cartan matrix of curvature 2-forms, which must evaluate to zero, by the Poincare lemma. The Cartan connection matrix can be decomposed in to two parts, one part based on a metric (Christoffel) connection, and the other part on a residue matrix of 1-forms such that $[C] = [\Gamma] + [T]$. The exterior differential of the composite connection must vanish such that the curvature 2-forms produced by the metric component must be balanced by all other matrices of 2-forms coming from the Residue connection and interaction 2-forms generated by exterior products of the $[\Gamma]$ and $[\Gamma]$. The result must balance to zero. If the curvature 2-forms generated by the metric based connection are associated with Gravity, and the rest of the curvature 2-forms are associated with -Inertia, the result is a strong principle of equivalence:

Gravity curvature 2-forms equal Inertial curvature 2-forms.

1. Examples of Physical Vacuums

1.1. Shipov and Absolute Parallelism

The sophomoric idea of a vacuum is that it is a "void" which is "empty", and without internal structure. The idea that the Physical Vacuum could be a "void" which is not "empty" (in the sense that such a void could have internal structures) was brought to my attention by the work of Gennady Shipov [3]. Shipov's research in this area deserves an accolade.

Shipov built his ideas based on the tensor concepts of "Absolute Parallelism", which was an extension of the idea that length of a vector should be an invariant of "parallel" transplantation process. Schouten describes "Absolute Parallelism" (p. 87 [Schouten]) in terms of a space which he defines as an A_n space. The Schouten space admits a connection which is restricted to be symmetric in the lower indices, $C^b_{[am]} = 0$.

Following the tensor methods of Vitale and Weitzenbock, G. Shipov uses a slightly more general definition of a space of "Absolute Parallelism". Shipov studies those spaces where the connection admits the possibility of "affine torsion"; i. e., where certain anti-symmetric portions of the connection coefficients are not zero: $C^b_{[am]} \neq 0$. Warning: Shipov in his book uses the same notation, A_n, as does Schouten for such a space, which can be confusing. In order to distinguish the Schouten A_n(Schouten) space from a Shipov space, the Shipov space will be described as a S_n(Shipov) space. The remarkable advance developed by Shipov is the idea that the physical vacuum is an S_4(Shipov) space of "Absolute Parallelism, which admits internal structure in terms of the anti-symmetric components of a connection, $C^b_{[am]} \neq 0$. This internal structure has been described as "Affine Torsion". The concept of "Affine Torsion" does not depend upon the choice of a (diagonal) symmetric metric. The physical vacuum is not necessarily an empty "Euclidean void", but can have certain substructure properties, which can be interpreted as being not "empty".

1.2. Extended Cartan Parallelism and "Voids"

Herein, emphasis will be place upon a space defined as a "Cartan" C_n space for which it can be shown that:

$$A_n(Schouten) \subset S_n(Shipov) \subset C_n(Cartan). \tag{1.1}$$

A key feature of the "Cartan" C_n space is that the connection is defined in terms of the exterior derivative of a Basis Frame which may or may not be an element of the affine transitive group. For Affine Basis frames, the concept of Affine torsion (if it exists) can be associated with a transitive evolution with *accelerations*. If the velocity field is constant, such that all accelerations are zero, the such Affine spaces do not admit "Affine torsion". Herein it is suggested, extending Shipov's ideas, that the physical vacuum is a *"Cartan" C_n space.* Such spaces can have an internal vector space partition with an internal structure that can be described as a Physical Vacuum, a "void" which is not "empty".

In particular such Cartan Connections can be deduced for the 15 parameter projective group as well as for the 13 parameter Affine group. The Cartan Connections for other groups of Basis Frames can also have anti-symmetric structures, similar in format to the coefficients of Affine torsion, but with entirely different physical meaning. Examples will be given below. In every case, for a Cartan connection, it can be shown that the idea of a "void" can be defined in terms of certain structural equations that are equal to zero (for C2 Basis Frames $[\mathbb{F}]$).

1 The Cartan Curvature as a matrix of exterior differential 2-forms must vanish

$$d[\mathbb{C}_{right}] + [\mathbb{C}_{right}]\hat{}[\mathbb{C}_{right}] = 0. \tag{1.2}$$

2 The Cartan Torsion as a vector of exterior differential 2-forms must vanish.

$$d\left|\sigma^k\right\rangle - [\mathbb{C}_{left}]\,\hat{}\,\left|\sigma^k\right\rangle = 0, \tag{1.3}$$
$$\text{where } [\mathbb{F}] \circ \left|dy^k\right\rangle = \left|\sigma^k\right\rangle \tag{1.4}$$

3 The Cartan Quadratic Congruence as a matrix of exterior differential 2-forms must vanish.

$$d[g] - [\mathbb{C}_{right}]^T \circ [g] - [g] \circ [\mathbb{C}_{right}] = 0, \tag{1.5}$$
$$\text{in terms of the congruence } [g] = [\mathbb{F}]^T [\eta] \circ [\mathbb{F}]\,. \tag{1.6}$$

1.3. Covariant vs. Lie differentials

The covariant derivative relative to a connection was invented such that directional differential processes acting on a tensor would create another tensor. Often it is assumed that a "covariant" derivative, based on a linear connection, is necessary to describe physical dynamics, but it can be shown that such an assumption is equivalent to a restriction whereby the dynamics describes only those thermodynamic processes which are adiabatic [2]. Using the Koszul axioms to define a "connection" it becomes apparent that the Covariant directional differential is not equivalent to the Lie directional differential.

$$\text{Covariant differential operator } \nabla_{(V)}\varpi \approx i(V)d\varpi, \tag{1.7}$$
$$\text{Lie differential operator } L_{(V)}\varpi \approx i(V)d\varpi + d(i(V)\varpi). \tag{1.8}$$

The Lie directional differential always produces a tensor when acting on differential forms. The Lie directional differential does not require a specific choice of a connection. The Covariant differential operator is the same as the Lie differential operator when $d(i(V)\varpi) = 0$. Such a case is always true when ϖ is a scalar function. The Covariant directional differential should be superseded by the concept of the Lie directional differential. For a 1-form, the difference between the Lie differential and the covariant differential is the abstract equivalent of thermodynamic Internal Energy. The difference between the Lie differential and the Covariant differential has been described as the source of the Higgs field [1].

1.4. P-Affine vs. W-Affine groups of Basis Frames

This article addresses a problem where by, in much of the physical literature, the concept of an affine space is not in agreement with the historical mathematics literature. In the study of invertible matrices of n^2 elements, an affine matrix is mathematically defined as a *transitive* element of the general linear group, with n-1 constraints. The classic representation, with $(n^2 - n + 1) = 13$ parameters in 4D, is represented by a matrix with $(n - 1)$ zero elements along the bottom row.

$$[P\_Affine] = \begin{bmatrix} F_1^1 & F_2^1 & F_3^1 & F_4^1 \\ F_1^2 & F_2^2 & F_3^2 & F_4^2 \\ F_1^3 & F_2^3 & F_3^3 & F_4^3 \\ 0 & 0 & 0 & F_4^4 \end{bmatrix}. \tag{1.9}$$

This representation is in accord with the Newtonian idea that dynamics of a Particle is based upon a differential "time" which is exact; i.e., where F_4^4 is independent from the spatial coordinates $\{x, y, z\}$. As there is another 13 parameter (but intransitive) group in 4D, the Affine matrix displayed above is herein named the "P-Affine" matrix. P-Affine matrices preserves parallelism, but not the origin. All finite points move for a transitive group.

The second 13 parameter group is named herein as the W-Affine group, and is related to the "transpose" of the P-Affine group. The W-Affine group is *intransitive*. The origin is a local fixed point. The W-Affine group is appropriate for covariant (Wave) phenomena, while the P-Affine group is appropriate for contravariant (Particle) phenomena. A generic format for the W-Affine group is given by the matrix

$$[W - Affine] = \begin{bmatrix} F_1^1 & F_2^1 & F_3^1 & 0 \\ F_1^2 & F_2^2 & F_3^2 & 0 \\ F_1^3 & F_2^3 & F_3^3 & 0 \\ F_1^4 & F_2^4 & F_3^4 & F_4^4 \end{bmatrix}. \tag{1.10}$$

It is easy to show that a product of two different P-Affine matrices remains P-Affine; i.e., the product matrix has the same format and structure of each factor of the product. Similarly, the product of two different W-Affine matrices remains W-Affine. If the 3x3 submatrix is constrained to be symmetric, then both the P_Affine and the W-Affine matrices have distinct 10 parameter subgroups. Note that the Lorentz group is not a P-Affine group, as dt depends upon the spatial differentials, $\mathrm{d}y^a$.

The W-Affine 13 parameter matrices are more suited to rotations and expansions about the fixed point (origin) and do not preserve parallelism. Both groups of Basis Frames admit Cartan Connection matrices $[\mathbb{C}_{right}]$ with a vector of "affine torsion" two forms defined by the equation $[\mathbb{C}_{right}]\,\hat{}\,|dy^a\rangle$. This concept called affine torsion is valid for Basis Frames which are not W-Affine nor P-Affine, but, for example, could be Projective Basis Frames of 15 parameters. Herein it is suggested that the word description of such torsion 2-forms be preceded with an adjective specialized to the group of Basis Frames being considered. For example P-Affine Torsion, and W-Affine Torsion, or Projective Affine Torsion distinguish between the various types of Torsion coefficients. The vector of "Affine" torsion 1-forms, is not the same as the vector of Cartan Torsion 2-forms:

$$\text{Affine torsion } [\mathbb{C}_{right}]\,\hat{}\,|dy^a\rangle \neq \{d\left|\sigma^k\right\rangle - [\mathbb{C}_{left}]\,\hat{}\,\left|\sigma^k\right\rangle\} \text{ Cartan Torsion.} \tag{1.11}$$

2. The fundamentals of the theory

Cartan's concept of the Repere Mobile can be extended to any Basis Frame $[\mathbb{F}]$ as a matrix of functions, $\{y^a\}$, on the domain of definition, such that the determinant of the matrix is not zero. Hence all of the columns of the Basis Frame are linearly independent. In that which follows it will be assumed that the functions that make up the Basis Frame are limited to those functions that are C2 differentiable. The Basis Frame, as a matrix of functions, is an element of the General Linear group, but often the Basis Frame is restricted to belong to some subgroup of the

General Linear group. Often such a constraint is described as a choice of a gauge condition. As the determinant of an allowable Basis Frame is not zero, an inverse matrix $[\mathbb{G}]$ exists, and exterior differentiation of the expression,

$$[\mathbb{F}] \circ [\mathbb{G}] = [\mathbb{G}] \circ [\mathbb{F}] = [\mathbb{I}]\,, \tag{2.1}$$

leads to the idea of differential closure with respect to a matrix of 1-forms defined as the right Cartan Connection:

$$\begin{aligned} d\,[\mathbb{F}] &= [\mathbb{F}] \circ [\mathbb{C}_{right}]\,, \quad \text{where} &(2.2)\\ [\mathbb{C}_{right}] &= -d\,[\mathbb{G}] \circ [\mathbb{F}] = +\,[\mathbb{G}] \circ d\,[\mathbb{F}]\,. &(2.3) \end{aligned}$$

Differential closure implies the topological property that the differential of any basis vector (a column of $[\mathbb{F}]$) is a linear combination of all of the basis vectors (columns) of $[\mathbb{F}]$:

$$\textbf{Differential Closure}\text{: } d\,[\mathbb{F}] = [\mathbb{F}] \circ [\mathbb{C}_{right}]\,. \tag{2.4}$$

Remember, that as matrices of 1-forms, these expressions do not depend upon a choice of coordinate functions.

The right Cartan matrix $[\mathbb{C}_{right}] = [\mathbb{C}(y^a, dy^a)]$ of 1-forms consists of matrix elements $\left[\mathbb{C}_{a\_\text{column}\,index}^{b\_\text{ row}\,index}\right]$. The matrix $\left[\mathbb{C}_a^b\right]$ is a matrix whose elements are differential 1-forms of the type

$$\mathbb{C}_a^b = \sum_m \mathbb{C}_{am}^b(y^n)dy^m. \tag{2.5}$$

This matrix $[\mathbb{C}(y^a, dy^a)]$ is defined on the domain as the Cartan (right) Connection matrix of differential 1-forms. A space with a Connection constructed in such a manner will be defined herein as a Cartan space, C_n(Cartan).

The Basis Frame may be viewed as a map of a vector of differentials $|dy^m\rangle$ on the domain to a range of exterior differential one forms $\left|\sigma^k\right\rangle$ on the range, with coordinate functions $\{x^k\}$:

$$[\mathbb{F}(y^n)] \circ |dy^m\rangle \Rightarrow \left|\sigma^k\right\rangle \tag{2.6}$$

The vector array of 1-forms $\left|\sigma^k\right\rangle$ need not be exact, nor closed, nor integrable. However, if the vector array $\left|\sigma^k\right\rangle$ is exact, such that $\left|\sigma^k\right\rangle \Rightarrow \left|dx^k\right\rangle$, then the Basis Frame consists of those functions that compose the Jacobian matrix $\left[\mathbb{J}_a^k(y^m)\right]$ of partial derivatives of the coordinate mapping functions ϕ^k from domain y^a to range x^k. Working backwards, assume that the coordinate map exists:

$$\phi \quad : \quad \{y^a\} \Rightarrow \{x^k\} = \phi^k(y^m), \tag{2.7}$$

$$d\phi \quad : \quad |dy^a\rangle \Rightarrow \left|dx^k\right\rangle = \left[\partial\phi^k(y^m)/\partial y^a\right] \circ |dy^a\rangle\,, \tag{2.8}$$

$$= \quad \left[\mathbb{J}^k_a(y^m)\right] \circ |dy^a\rangle = [\mathbb{F}] \circ |dy^a\rangle \Rightarrow \left|dx^k\right\rangle. \tag{2.9}$$

The coefficients of the Cartan matrix $[\mathbb{C}_{right}]$ are such that for an arbitrary basis frame, the difference

$$\textbf{Cartan Affine Torsion coefficients} \quad \mathbb{C}^b_{[am]} = \mathbb{C}^b_{am} - \mathbb{C}^b_{ma} \tag{2.10}$$

need not be zero. If the Cartan matrix is evaluated relative to a Basis Frame $\left[\mathbb{J}^k_a(y^n)\right]$, which is a Jacobian matrix of C2 functions, then the antisymmetric difference vanishes for C2 functions,

$$\left[\partial^2\phi^b(y^n)/\partial y^a\partial y^m - \partial^2\phi^b(y^n)/\partial y^m\partial y^a\right] = coefficient\ of\ \mathbb{C}^b_{[am]} \Rightarrow 0. \tag{2.11}$$

Classically the difference, if not zero, has been defined as the components of "affine" torsion. The coefficients of the Affine Torsion can be combined with differentials to produce the vector of "Affine Torsion" of 2-forms:

$$\textbf{Vector of "Affine Torsion" 2-forms} \quad \left|\Sigma_{affine\_torsion}\right\rangle = [\mathbb{C}_{right}]\,\hat{}\,|dy^m\rangle\,. \tag{2.12}$$

This vector of 2-forms is **not** the same as the vector of Cartan Torsion 2-forms. (See the Section below on the Left Cartan Connection.) Again the historical wording is somewhat misleading, for the "affine torsion" concept is also to be associated with Basis Frames that are not elements of the Affine group, such as the Projective group. The formula (see below) for the vector of 2-forms which define Cartan Torsion (not Affine torsion) is given by the expression:

$$\textbf{Vector of "CartanTorsion" 2-forms} \quad \left|\Sigma_{Car\tan\_torsion}\right\rangle = \{d\left|\sigma^k\right\rangle - [\mathbb{C}_{left}]\,\hat{}\,\left|\sigma^k\right\rangle\}. \tag{2.13}$$

The two definitions are not equivalent.

The historical use of the word "affine" is unfortunate, for it could be interpreted that the admissible Basis Frame is required to be an element of the Affine subgroup of the General Linear group. This constraint that the Basis Frame is an element of the Affine group is not required for a C_n(Cartan) space. All of these concepts described above are independent from a "choice" of coordinates.

A second exterior differentiation of the Differential Closure expression yields the equation,

$$dd\,[\mathbb{F}] = [\mathbb{F}] \circ \{[\mathbb{C}_{right}]\,\hat{}\,[\mathbb{C}_{right}] + d\,[\mathbb{C}_{right}]\}, \tag{2.14}$$

where the bracketed factor is defined as the Cartan Curvature matrix of 2-forms, $[\Theta(y^a, dy^a)]$:

Cartan Curvature 2-forms $\left[\Theta_{Car\tan\_curvature}\right] = [\mathbb{C}_{right}]\,\hat{}\,[\mathbb{C}_{right}] + d\,[\mathbb{C}_{right}]\,.$ (2.15)

As the Basis Frame $[\mathbb{F}]$ is presumed to be constructed from C2 functions,

$$dd\,[\mathbb{F}] = 0 = [\mathbb{F}] \circ \{[\mathbb{C}_{right}]\,\hat{}\,[\mathbb{C}_{right}] + d\,[\mathbb{C}_{right}]\}, \tag{2.16}$$

then due to linearity, it follows that the Cartan Curvature matrix of 2-forms, $[\Theta]$ must vanish,

$$\text{In a Cartan C}_n \text{ space:} \quad [\Theta] = [\mathbb{C}_{right}]\,\hat{}\,[\mathbb{C}_{right}] + d\,[\mathbb{C}_{right}] \Rightarrow 0. \tag{2.17}$$

This statement is defined as Cartan's second equation of structure. All of the above depends only upon the existence of a Basis Frame as a linear form (when acting on differentials), and is independent of any metric constraint.

However, from the Basis Frame, it is also possible to construct a quadratic form from the law of differential closure. Let the matrix $[g]$ be defined in terms of a congruent transformation of a some Sylvestor signature matrix $[\eta]$ (plus or minus unity along a diagonal), using the Basis Frame and its transpose for the congruence:

$$[g] = [\mathbb{F}]^T \circ [\eta] \circ [\mathbb{F}]\,. \tag{2.18}$$

It then follows that for $d\,[\eta] = 0$,

$$d\,[g] = [\mathbb{C}_{right}]^T \circ [g] + [g] \circ [\mathbb{C}_{right}]\,, \tag{2.19}$$

or

$$d\,[g] - [\mathbb{C}_{right}]^T \circ [g] - [g] \circ [\mathbb{C}_{right}] \Rightarrow 0. \tag{2.20}$$

The format of the preceding formula mimics the formula that defines Absolute Parallelism in classical tensor treatments

The properties of a C_n(Cartan) space require that

1. The Cartan Curvature matrix of 2-forms vanishes.

$$\left[\Theta_{curvature\_2-forms}\right] = [\mathbb{C}_{right}]\,\hat{}\,[\mathbb{C}_{right}] + d\,[\mathbb{C}_{right}] \Rightarrow 0. \tag{2.21}$$

2. The Cartan Torsion vector of 2-forms must vanish.

$$\left|\Sigma_{Car\tan\_torsion\_2-forms}\right\rangle = \{d\left|\sigma^k\right\rangle - [\mathbb{C}_{left}]\,\hat{}\,\left|\sigma^k\right\rangle \Rightarrow 0 \tag{2.22}$$

This statement does not preclude the C_n(Cartan) from have a non-zero vector of "Affine" torsion 2-forms.

3. The system satisfies the definition of "Absolute Parallelism"

$$d\,[g] - [\mathbb{C}_{right}]^T \circ [g] - [g] \circ [\mathbb{C}_{right}] \;\Rightarrow\; 0, \tag{2.23}$$

$$\text{if } d\,[\eta] \;=\; 0. \tag{2.24}$$

This statement is valid for all Cartan Basis Frames (the generalized Repere Mobile) which are subgroups of (or are equivalent to) the full General Linear group, including the projective group of $(n^2 - 1)$ parameters.

The P-Affine space has a connection that appeals to kinematic definitions, and includes possible P_Affine torsion coefficients that are related to kinematic acceleration. The W-Affine space has W-Affine torsion antisymmetries that appeal to the format of electromagnetic forces per unit source.

2.1. The Left Cartan Connection

The concept of a Basis Frame $[\mathbb{F}]$ with inverse $[\mathbb{G}]$ also leads to a formulation of the "left" Cartan connection $[\mathbb{C}_{left}]$:

$$d\,[\mathbb{F}] \;=\; [\mathbb{C}_{left}] \circ [\mathbb{F}]\,, \tag{2.25}$$

$$[\mathbb{C}_{left}] \;=\; d\,[\mathbb{F}] \circ [\mathbb{G}] = -\mathbb{F} \circ d\,[\mathbb{G}]\,, \quad \text{such that} \tag{2.26}$$

$$d[\mathbb{G}] \;=\; [\mathbb{G}] \circ [-\mathbb{C}_{left}]\,. \tag{2.27}$$

It follows by exterior differentiation of the expression,

$$|dy^m\rangle = [\mathbb{G}(y^n)] \circ \left|\sigma^k\right\rangle, \tag{2.28}$$

such that

$$d\,|dy^m\rangle = 0 = [\mathbb{G}]\,\{d\left|\sigma^k\right\rangle - [\mathbb{C}_{left}]\,\hat{}\,\left|\sigma^k\right\rangle\}. \tag{2.29}$$

By linearity, the bracketed factor as a vector of 2-forms must vanish. The bracketed factor is defined as the vector of Cartan Torsion 2-forms

$$\textbf{Cartan vector of Torsion 2-forms:}\quad \{d\left|\sigma^k\right\rangle - [\mathbb{C}_{left}]\,\hat{}\,\left|\sigma^k\right\rangle\} \Rightarrow 0. \tag{2.30}$$

This formula is not equivalent to the vector of Affine Torsion 2-forms. For a Cartan space, C_n(Cartan), the Cartan vector of Torsion 2-forms must vanish. This statement is Cartan's first equation of structure. In a C_n(Cartan) space, both equations of structure must vanish. However, the vector of Affine Torsion 2-forms does not vanish, necessarily. These concepts are free from metric considerations.

It is also true that in a C_n(Cartan) space, the Cartan left connection satisfies a matrix equations of 2-forms

$$\text{In a Cartan C}_n\text{ space:}\quad [\Theta_{left}] = [\mathbb{C}_{left}]\,\hat{}\,[\mathbb{C}_{left}] + d\,[\mathbb{C}_{left}] \Rightarrow 0. \tag{2.31}$$

In fact, the left Cartan connection is the similarity transform of the right Cartan connection:

$$[\mathbb{C}_{left}] = [\mathbb{F}] \circ [\mathbb{C}_{right}] \circ [\mathbb{G}]. \tag{2.32}$$

2.2. A C_n(Cartan) space with metric

The next step is to impose a metric on the domain space $[g]$, which, as a symmetric (covariant) invertible matrix of C2 functions admits a matrix of connection 1-forms $[\mathbf{\Gamma}_{christ}]$ in terms of the Christoffel formulas:

$$\begin{aligned} \text{Christoffel Coefficients} &: \mathbf{\Gamma}^b_{ac}(y^m) && (2.33)\\ &= g^{be}\{\partial g_{ce}/\partial y^a + \partial g_{ea}/\partial y^c - \partial g_{ac}/\partial y^e\}, && (2.34)\\ \text{Christoffel matrix } [\mathbf{\Gamma}_{christ}] &= [\mathbf{\Gamma}^b_{ac}(y^m) dy^c]. && (2.35) \end{aligned}$$

The Christoffel connection is built on a symmetric form $[g]$ which is a quadratic multiplicative congruence of functions,

$$\text{Symmetric congruence:}\quad [g] = [\mathbb{F}]^T \circ [\eta] \circ [\mathbb{F}]. \tag{2.36}$$

In this article the matrix $[\eta]$ will be defined as either a Minkowski matrix (signature -1) or Euclidean (signature +1). The Cartan connection is built on a non symmetric form $[\mathbb{F}]$ that can have symmetric parts composed by linearly additive methods, as well as anti-symmetric parts:

$$\text{Linearly symmetric} \quad : \quad \{[\mathbb{F}]^T + [\mathbb{F}]\}/2, \tag{2.37}$$

$$\text{Linearly anti-symmetric} : \quad \{[\mathbb{F}]^T - [\mathbb{F}]\}/2. \tag{2.38}$$

The symmetric congruence implies that a metric form might be included in the functional format of the Cartan connection. The Cartan Connection matrix of 1-forms $[\mathbb{C}]$ can then be decomposed into a symmetric Christoffel connection component as a matrix of 1-forms $[\mathbf{\Gamma}]$ (computed from the derivatives of the quadratic congruence as described by Levi-Civita) and a Residue component of matrix 1-forms, $[\mathbb{T}]$. The residue connection, $[\mathbb{T}]$, is defined by the equation,

$$\text{Connection decomposition } [\mathbb{C}_{right}] = [\mathbf{\Gamma}_{christ}] + [\mathbb{T}_{residue}]\,. \tag{2.39}$$

The Cartan connection is independent from metric, but the components of the decomposition law are metric dependent. Different Sylvester signatures produce different decompositions. The Christoffel component, by construction, is symmetric in the lower pair of indices, and does not produce any "affine torsion" effects. The anti-symmetric structure in the Cartan Connection, $[\mathbb{C}]$, that produces the vector of affine torsion 2-forms, must also appear in the Residue matrix, $[\mathbb{T}]$. It follows that, for a Jacobian Basis Frame, or for Basis Frames with a symmetric connection, the generalized vector of affine torsion 2-forms is zero: $[\mathbb{C}]\,\hat{}\,|dy\rangle \Rightarrow 0$.

The idea that a quadratic congruence can be used as a metric, producing a Christoffel connection, which may or may not have any Curvature relative to the Christoffel connection, is a typical result in the field of cohomology. In hydrodynamics, the analog would be that the Christoffel connection has two parts, one which is closed (equal to zero with respect to the exterior differentiation), and another part that is not closed. The closed part, if not exact, leads to circulation chiral effects, while non closed part leads to vorticity effects.

The metrical structure $[g]$ can be extended by assuming that the symmetric matrix, $[\eta]$, is equal to a set of constants, and not just elements in a diagonal structure equal to ± 1. These modifications are not investigated herein. Note that for a Basis Frame constructed from a symmetric congruence, the difference between $[g]$ and the identity matrix $[\mathbb{I}]$ is equal to twice the classic definition of the Strain matrix in elasticity theory.

When the Basis Frame is a Jacobian matrix, then the Cartan right connection is equal to the Christoffel connection computed from the quadratic symmetric congruence, $[\mathbb{J}]^T \circ [\mathbb{J}]$ generated by the Coordinate mapping.

$$\begin{aligned} If \;\; \left[\mathbb{F}^k_a\right] &= \left[\partial\phi^k(y^b)/\partial y^a\right] = \left[\mathbb{J}^k_a\right], &(2.40)\\ \text{then } [\mathbb{C}_{right}(y, dy))] &= [\mathbf{\Gamma}_{christ}(g)] &(2.41)\\ \text{where } \;\; [g] &= [\mathbb{J}]^T \circ [\mathbb{J}]\,. &(2.42)\end{aligned}$$

In general, the basis frame $\left[\mathbb{F}^k_a\right]$ need not be associated with an integrable mapping. The coefficient functions of the (right) Cartan matrix, symbolized by $[\mathbf{C}_{right}]$, are not necessarily symmetric in the lower indices, and the symmetric parts are not necessarily generated by the Christoffel formulas. It is of some interest to decompose the Cartan right connection into its Christoffel part and a Residue part. As the Christoffel connection does not support affine torsion, all of the Cartan affine torsion terms (if any) must be contained in the Residue connection matrix of 1-forms.

$$\begin{aligned} [\mathbb{C}_{right}]\,\hat{}\,|dy^a\rangle &= [\mathbf{\Gamma}_{christ}]\hat{}\,|dy^a\rangle + [\mathbb{T}_{residue}]\,\hat{}\,|dy^a\rangle &(2.43)\\ &= |0\rangle + [\mathbb{T}_{residue}]\,\hat{}\,|dy^a\rangle &(2.44)\end{aligned}$$

All of these concepts do not depend upon a diffeomorphic choice of coordinates. (Compare Shipov (5.77)).

3. The Strong Principle of Equivalence

The Cartan matrix of curvature 2-forms can be evaluated by substituting $[\mathbf{\Gamma}_{christ}]+[\mathbb{T}]$ for $[\mathbb{C}_{right}]$, to yield

$$\begin{aligned} \text{Christoffel Curvature 2-forms} &= [\Theta_{christ}] = [\mathbf{\Gamma}_{christ}]\,\hat{}\,[\mathbf{\Gamma}_{christ}] + d\,[\mathbf{\Gamma}_{christ}] &(3.1)\\ +\text{Interaction 2-forms} &= [\mathbf{\Gamma}_{christ}]\,\hat{}\,[\mathbb{T}] + [\mathbb{T}]\,\hat{}\,[\mathbf{\Gamma}_{christ}] &(3.2)\\ +\text{Residue curvature 2-forms} &= [\Theta_{residue}] = [\mathbb{T}]\,\hat{}\,[\mathbb{T}] + d\,[\mathbb{T}]\,. &(3.3)\end{aligned}$$

The sum of these 3 terms must vanish, as the sum equals the Cartan matrix of curvature 2-forms (which vanishes).

If (following Einstein's 1915 theory) the curvature 2-forms generated by the metric and the associated Christoffel connection are defined as the Gravitational stress energy,

$$[\Theta_{christ}] = [\text{"Gravitational stress energy"}], \tag{3.4}$$

and if the 2-forms which depend upon the interactions and curvatures of the Residue connection are defined as Inertial stress energy,

$$[\text{"Inertial stress energy"}] = -\{[\Theta_{residue}] + [\mathbf{\Gamma}_{christ}]\,\hat{}\,[\mathbb{T}] + [\mathbb{T}]\,\hat{}\,[\mathbf{\Gamma}_{christ}]\}, \tag{3.5}$$

then the fact that Cartan Curvature 2-forms must vanish leads to the result

$$[\text{"Gravitational stress energy"}] = [\text{"Inertial stress energy"}]. \tag{3.6}$$

The equation represents a "Strong Principle of Equivalence", built on the balance between gravitational stress energy and inertial stress energy. The acceleration does not compare accelerations explicitly, for the sum of $[\mathbf{\Gamma}_{christ}] + [\mathbb{T}]$ is not zero, but is equal to $[\mathbb{C}_{right}]$, which is not zero.

This result is universally true for any connection based upon a Basis Frame of C2 functions, and a metric field, [g].

4. Maple Examples

4.1. The Particle Affine Connection

As a first example, a particle affine Basis Frame in 4D will be examined. For simplicity, the space-space portions of the Basis Frame will be assumed to be the 3x3 Identity matrix, essentially eliminating spatial deformations and spatial rigid body motions. The 4th (space-time) column will consist of components that can be identified with a velocity field. The bottom row will have three zeros, and the 44 component will be described in terms of a function $\psi(x, y, z, t)$.

$$\left[\mathbb{F}_{Particle\_affine}\right] = \begin{bmatrix} 1 & 0 & 0 & -V^x \\ 0 & 1 & 0 & -V^y \\ 0 & 0 & 1 & -V^z \\ 0 & 0 & 0 & \psi \end{bmatrix}. \tag{4.1}$$

The projected 1-forms become

$$[\mathbb{F}] \circ |dy^a\rangle = \left|\sigma^k\right\rangle \tag{4.2}$$

$$\begin{bmatrix} 1 & 0 & 0 & -V^x \\ 0 & 1 & 0 & -V^y \\ 0 & 0 & 1 & -V^z \\ 0 & 0 & 0 & \psi \end{bmatrix} \circ \left| \begin{array}{c} dx \\ dy \\ dz \\ dt \end{array} \right\rangle \Rightarrow \left| \begin{array}{c} \sigma^x \\ \sigma^y \\ \sigma^z \\ \omega \end{array} \right\rangle = \left| \begin{array}{c} dx - V^x dt \\ dy - V^y dt \\ dz - V^z dt \\ \psi dt \end{array} \right\rangle \tag{4.3}$$

The Cartan right Connection matrix of 1-forms is given by the expression

$$[\mathbb{C}_{Particle\_affine\_right}] = \begin{bmatrix} 0 & 0 & 0 & -dV^x + V^x d(\ln \psi) \\ 0 & 0 & 0 & -dV^y + V^y d(\ln \psi) \\ 0 & 0 & 0 & -dV^z + V^z d(\ln \psi) \\ 0 & 0 & 0 & d(\ln \psi) \end{bmatrix} \tag{4.4}$$

The Connection coefficients can be computed and exhibit components of non zero "Affine Torsion ". The vector of "Affine Torsion" 2-forms is

$$\left|\Sigma_{affine\_torsion}\right\rangle = \mathbb{C}\hat{}\,|dy^m\rangle = \left| \begin{array}{c} -d(V^x)\hat{}d(t) + V^x d(\ln \psi)\hat{}d(t)) \\ -d(V^y)\hat{}d(t) + V^y d(\ln \psi)\hat{}d(t)) \\ -d(V^z)\hat{}d(t) + V^z d(\ln \psi)\hat{}d(t)) \\ d(\ln \psi)\hat{}d(t) \end{array} \right\rangle \tag{4.5}$$

$$\neq \left|\Sigma_{Cartan\_torsion}\right\rangle \tag{4.6}$$

If the velocity field is a function of time only, then total differential of the velocity field leads to the classic kinematic concept of accelerations, and the "affine" torsion 2-forms depend only on the potential ψ. If the potential function is such that its total differential is zero, or a function of time, then all of the affine torsion coefficients vanish (for the example).

4.2. The Wave Affine Connection

The next example coincides the "wave affine" Basis Frame which will have zeros for the first 3 elements of the rightmost column. For simplicity, the space-space portions of the Basis Frame will be assumed to be the 3x3 Identity matrix, essentially eliminating spatial deformations and spatial rigid body motions. The 4th (space-time) column will have three zeros, and the 44 component will be described in terms of a function $\phi(x, y, z, t)$.

$$[\mathbb{F}_{wave\_affine}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ Ax & Ay & Az & -\psi \end{bmatrix}. \tag{4.7}$$

The projected 1-forms become

$$[\mathbb{F}_{wave\_affine}] \circ |dy^a\rangle = |\sigma^k\rangle \tag{4.8}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ A_x & A_y & A_z & -\phi \end{bmatrix} \circ \left| \begin{array}{c} dx \\ dy \\ dz \\ dt \end{array} \right\rangle \Rightarrow \left| \begin{array}{c} \sigma^x \\ \sigma^y \\ \sigma^z \\ Action \end{array} \right\rangle = \left| \begin{array}{c} dx \\ dy \\ dz \\ A_x dx + A_y dy + A_z dz - \phi dt \end{array} \right\rangle \tag{4.9}$$

Note that the Action 1-form produced by the wave affine Basis Frame is precisely the format of the 1-form of Action used to construct the Electromagnetic field intensities. It will be evident from that which is displayed below, that the 2-form $F = dA$ can be identified with the coefficients of "Affine Torsion". This result does not occur in P-affine group, but appears naturally in the W-Affine group.

The Cartan right Connection matrix of 1-forms is given by the expression

$$[\mathbb{C}_{wave\_affine\_right}] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -d(A_x)/\phi & -d(A_y)/\phi & -d(A_z)/\phi & d(\ln \phi) \end{bmatrix} \tag{4.10}$$

The Connection coefficients can be computed and exhibit components of non zero "Affine Torsion ". The vector of "Affine Torsion" 2-forms is given by the expression:

$$\begin{aligned} |\Sigma_{affine\_torsion}\rangle &= [\mathbb{C}_{wave\_affine\_right}] \,\hat{}\, |dy^m\rangle & (4.11) \\ &= \left| \begin{array}{c} 0 \\ 0 \\ 0 \\ F/\phi \end{array} \right\rangle = \left| \begin{array}{c} 0 \\ 0 \\ 0 \\ -d(Action)/\phi \end{array} \right\rangle & (4.12) \\ &\neq |\Sigma_{Car\tan\_torsion}\rangle & (4.13) \end{aligned}$$

It is remarkable that, for the wave affine group of Basis Frames, the coefficients of "Affine torsion" of the Right Cartan matrix of connection 1-forms are directly related to the structural format of the electromagnetic field 2-form, $F = dA$.

The **wave affine group** of Basis Frames has a connection with Affine Torsion 2-forms that are abstractly related to the forces per unit charge of electromagnetic theory. The **particle affine** group of Basis Frames has a connection with Affine Torsion 2-forms which are directly related to Accelerations.

4.3. Perturbations of Coordinate (diffeomorphic) maps

The formulas described above will be evaluated for a number of cases using Maple to relieve the algebraic burden. The starting point will be based upon a true Affine map of (13 parameters) mapping the coordinates, $\{y^a\} = \{r, \theta, \phi, \tau\}$ into the 4F space $\{x^k\} = \{x, y, z, t\}$. The mapping will produce a Jacobian matrix which will be used as the candidate for the Basis Frame, $[\mathbb{F}]$, of the domain space $\{\xi^a\}$. The Basis Frame on the range space, initially formulated as the identity matrix, will be perturbed in several ways in order to examine situations where the Basis Frame maps the domain differentials $|dy^a\rangle$ into 1-forms $|\sigma\rangle$. The 1-forms will be constructed to have various Pfaff Topological dimensions, greater than 1.

$$[\mathbb{F}] \circ |dy^a\rangle = \left|\sigma^k\right\rangle \neq \left|dx^k\right\rangle \tag{4.14}$$

The cases to be studied are simplified such that $\sigma^1 = dx$, $\sigma^2 = dy$, $\sigma^3 = variable, \sigma^4 = dt$. One of the specialized cases corresponds to :

$$\begin{array}{lcc} \text{Pfaff Dimension 1} & \sigma^3 = dz & \text{exact} \\ \text{Pfaff Dimension 1} & \sigma^3 = dz + \Gamma(ydx - xdy)/(ax^2 + by^2) & \text{closed} \\ \text{Pfaff Dimension 2} & \sigma^3 = dz + cc \cdot zdx & \text{integrable} \\ \text{Pfaff Dimension 3} & \sigma^3 = dz + aa \cdot (ydx - xdy) & \text{non integrable} \end{array} \tag{4.15}$$

The perturbation of the Basis Frame consists of additions to the exact differential, dz. When the system is integrable, then there exists an integrating factor, β, such that $d(\beta\sigma^3) = 0$. The non-integrable case has a non-zero 3-from, such that $\sigma^3\hat{}d(\sigma^3) \neq 0$.

If each of the perturbations are added to the perfect differential, such that

$$\sigma^3 = dz + \Gamma(ydx - xdy)/(ax^2 + by^2) + cc \cdot zdx + aa \cdot (ydx - xdy) \tag{4.16}$$

then the Basis Frame on the range space becomes

$$[\mathbb{F}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ cc \cdot z + \Gamma y/(ax^2 + by^2) + aa \cdot y & -\Gamma x/(ax^2 + by^2) - aa \cdot x & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{4.17}$$

The right Cartan matrix of Connection 1-forms becomes

TO BE CONCLUDED SHORTLY with a number of Maple examples,

1. where a coordinate mapping is given ,
2. where a coordinate Jacobian matrix is perturbed
3. where a Quadratic congruence is perturbed
4. where both perturbations take place simultaneously

The 3 - Connection matrices of 1-forms will be evaluated.

The matrix of curvature 2-forms and the matrix of interaction 2-forms will be evaluated.

The vector of Affine curvature 2-forms will be evaluated.

5. Partitioned Basis Frames

In this section, a Basis Frame will partitioned such that the right Cartan connection will also be partitioned. I find it convenient and informative to partition the (arbitrary) basis frame $\mathbb{F}$ in terms of the n-1 *associated* (horizontal, interior, transversal) column vectors, $\mathbf{e}_k$, and the *adjoint* (normal, exterior, parametric or vertical) field, or column vector, $\mathbf{n}_p$,

$$\mathbb{F} = [\mathbf{e}_k, \mathbf{n}] = [\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{n}].$$

The corresponding Cartan matrix has the partition,

$$d\mathbb{F} = d\begin{bmatrix} e_1^1 & e_2^1 & ... & n^1 \\ e_1^2 & e_2^2 & ... & n^2 \\ ... & ... & ... & ... \\ e_1^n & e_2^n & ... & n^n \end{bmatrix} = \mathbb{F} \circ \mathbb{C} = \mathbb{F} \circ \begin{bmatrix} \Gamma_1^1 & \Gamma_2^1 & ... & \gamma^1 \\ \Gamma_1^2 & \Gamma_2^2 & ... & \gamma^2 \\ ... & ... & ... & ... \\ h_1 & h_2 & ... & \Omega \end{bmatrix}$$

The Cartan matrix, $\mathbb{C}$, is a matrix of differential 1-forms which can be evaluated explicitly from the functions that make up the basis frame, if they admit first partial derivatives. Moreover, the differential of the position vector can be expanded in terms of the same basis frame and a set of Pfaffian 1-forms:

$$d\mathbf{R} = \mathbb{I} \circ \left| \begin{array}{c} dy^a \\ d\tau \end{array} \right\rangle = \mathbb{F} \circ \mathbb{F}^{-1} \circ \left| \begin{array}{c} dy^a \\ d\tau \end{array} \right\rangle = \mathbb{F} \circ \left| \begin{array}{c} \boldsymbol{\varpi} \\ \omega \end{array} \right\rangle,$$

where the vector $\left| \begin{array}{c} \varpi \\ \omega \end{array} \right\rangle$ is a vector of 1-forms that I can compute explicitly. Note that these 1-forms are constructed from the projections by the inverse Basis Frame $\mathbb{F}^{-1} \circ \left| \begin{array}{c} dy^a \\ d\tau \end{array} \right\rangle$, and not as described above in terms of the formula, $\mathbb{F} \circ \left| \begin{array}{c} dy^a \\ d\tau \end{array} \right\rangle$.

By the Poincare lemma, it follows that

$$dd\mathbf{R} = d\mathbb{F}\hat{} \left| \begin{array}{c} \boldsymbol{\varpi} \\ \omega \end{array} \right\rangle + \mathbb{F} \circ \left| \begin{array}{c} \boldsymbol{\varpi} \\ \omega \end{array} \right\rangle = \mathbb{F} \circ \{\mathbb{C}\hat{} \left| \begin{array}{c} \boldsymbol{\varpi} \\ \omega \end{array} \right\rangle + \left| \begin{array}{c} d\boldsymbol{\varpi} \\ d\omega \end{array} \right\rangle\} = 0$$

and

$$d\mathbb{F} = d\mathbb{F}\hat{}\mathbb{C} + \mathbb{F}\hat{}d\mathbb{C} = \mathbb{F} \circ \{\mathbb{C}\hat{}\mathbb{C} + d\mathbb{C}\} = 0.$$

As the Frame matrix and the Cartan matrix are partitioned relative to the tangent (or interior) vectors $\mathbf{e}$ and the normal (or exterior) vectors, $\mathbf{n}$, the Poincare lemma breaks up into linearly independent factors, each of which must vanish. The results are:

$$dd\mathbf{R} = \mathbf{e}\{d\left|\boldsymbol{\varpi}\right\rangle + [\boldsymbol{\Gamma}]\hat{}\left|\boldsymbol{\varpi}\right\rangle - \omega\hat{}\left|\boldsymbol{\gamma}\right\rangle\} + \mathbf{n}\{d\omega + \Omega\hat{}\omega + \left\langle\mathbf{h}\right|\hat{}\left|\boldsymbol{\varpi}\right\rangle\} = 0$$

$$dd\mathbf{e} = \mathbf{e}\{d[\boldsymbol{\Gamma}] + [\boldsymbol{\Gamma}]\hat{}[\boldsymbol{\Gamma}] + \left|\boldsymbol{\gamma}\right\rangle \hat{} \left\langle\mathbf{h}\right|\} + \mathbf{n}\{d\left\langle\mathbf{h}\right| + \Omega\hat{}\left\langle\mathbf{h}\right| + \left\langle\mathbf{h}\right|\hat{}[\boldsymbol{\Gamma}]\} = 0$$

$$dd\mathbf{n} = \mathbf{e}\{d\left|\boldsymbol{\gamma}\right\rangle + [\boldsymbol{\Gamma}]\hat{}\left|\boldsymbol{\gamma}\right\rangle - \Omega\hat{}\left|\boldsymbol{\gamma}\right\rangle\} + \mathbf{n}\{d\Omega + \Omega\hat{}\Omega + \left\langle\mathbf{h}\right|\hat{}\left|\boldsymbol{\gamma}\right\rangle\} = 0$$

By reasons of linear independence, each of the curly bracket factors must vanish, leading to the results on the interior domain (coefficients of $\mathbf{e}$):

$$d\left|\boldsymbol{\varpi}\right\rangle + [\mathbf{\Gamma}]\hat{}\left|\boldsymbol{\varpi}\right\rangle = \omega\hat{}\left|\boldsymbol{\gamma}\right\rangle \equiv \left|\mathbf{\Sigma}\right\rangle = \left|\begin{array}{l} \omega\hat{}\gamma^1 \\ \omega\hat{}\gamma^2 \\ \omega\hat{}\gamma^3 \end{array}\right\rangle$$

the interior torsion vector of dislocation 2-forms.

$$d[\mathbf{\Gamma}] + [\mathbf{\Gamma}]\hat{}[\mathbf{\Gamma}] = -\left|\boldsymbol{\gamma}\right\rangle\ \hat{}\ \left\langle\mathbf{h}\right| \equiv [\mathbf{\Theta}] = \begin{bmatrix} \gamma^1\hat{}h_1 & \gamma^1\hat{}h_2 & \gamma^1\hat{}h_3 \\ \gamma^2\hat{}h_1 & \gamma^2\hat{}h_2 & \gamma^2\hat{}h_3 \\ \gamma^3\hat{}h_1 & \gamma^3\hat{}h_2 & \gamma^3\hat{}h_3 \end{bmatrix}$$

the matrix of interior curvature 2-forms

$$\{d\left|\boldsymbol{\gamma}\right\rangle + [\mathbf{\Gamma}]\hat{}\left|\boldsymbol{\gamma}\right\rangle = \Omega\hat{}\left|\boldsymbol{\gamma}\right\rangle \equiv \left|\mathbf{\Psi}\right\rangle = \left|\begin{array}{l} \Omega\hat{}\gamma^1 \\ \Omega\hat{}\gamma^2 \\ \Omega\hat{}\gamma^3 \end{array}\right\rangle$$

the exterior torsion vector of disclination 2-forms.

The first two equations are recognized as Cartan's equations of structure (on an affine domain).

The last equation appears to be a new equation of structure valid on a projective domain, when $\Omega \neq 0$.

$\left|\mathbf{\Psi}\right\rangle$ physically seems to represent a different kind of "torsion" which is perhaps associated with W-Affine Basis Frames. I intuit that this "torsion" is more to be associated with rotations and expansions about a fixed point. Perhaps this could be a better description of disclination defects. Recall that Kondo has developed the theory of dislocation defects based on P-Affine Basis Frames, $\left|\mathbf{\Sigma}\right\rangle$.

There are also three equations of structure on the exterior domain (coefficients of $\mathbf{n}$) which are given by the constructions:

$$d\omega + \Omega\hat{}\omega = -\left\langle\mathbf{h}\right|\hat{}\left|\boldsymbol{\varpi}\right\rangle$$

$$d\left\langle\mathbf{h}\right| + \Omega\hat{}\left\langle\mathbf{h}\right| = -\left\langle\mathbf{h}\right|\hat{}[\mathbf{\Gamma}]$$

$$d\Omega + \Omega\hat{}\Omega = \theta = -\left\langle\mathbf{h}\right|\hat{}\left|\boldsymbol{\gamma}\right\rangle \text{ the exterior curvature 2-forms}$$

A remarkable result (to me) of this construction is the fact that the matrix of interior curvature 2-forms, $[\mathbf{\Theta}]$, can be constructed in two ways. The classical method utilizes differential processes $\{d[\mathbf{\Gamma}] + [\mathbf{\Gamma}]\hat{}[\mathbf{\Gamma}]\}$, while the second method is purely algebraic $\{-|\boldsymbol{\gamma}\rangle\ \hat{}\ \langle\mathbf{h}|\}$. The order of partial derivatives contained in the algebraic (exterior) expression for the interior curvature $\{-|\boldsymbol{\gamma}\rangle\ \hat{}\ \langle\mathbf{h}|\}$ is one less than the classic expression built on the connection coefficients, $\{d[\mathbf{\Gamma}] + [\mathbf{\Gamma}]\hat{}[\mathbf{\Gamma}]\}$.

Exterior differentiation of the matrix of interior curvature 2-forms yields:

$$d[\mathbf{\Theta}] = -d\,|\boldsymbol{\gamma}\rangle\ \hat{}\ \langle\mathbf{h}| = (-\,|d\boldsymbol{\gamma}\rangle\ \hat{}\ \langle\mathbf{h}|) + (|\boldsymbol{\gamma}\rangle\ \hat{}\ \langle d\mathbf{h}|) =$$

$$([\mathbf{\Gamma}]\hat{}\,|\boldsymbol{\gamma}\rangle\ \hat{}\ \langle\mathbf{h}|) - (\Omega\hat{}\,|\boldsymbol{\gamma}\rangle\ \hat{}\ \langle\mathbf{h}|) - (|\boldsymbol{\gamma}\rangle\ \hat{}\,\Omega\hat{}\,\langle\mathbf{h}|) - (|\boldsymbol{\gamma}\rangle\ \hat{}\ \langle\mathbf{h}|\ \hat{}[\mathbf{\Gamma}]) = 0$$

The fundamental result is that the matrix of 2-forms that forms the interior curvature matrix is closed!

It is important to note that due to the partition, the exterior curvature is a closed (in this example a scalar valued) 2-form $\theta = -\,\langle\mathbf{h}|\ \hat{}\ |\boldsymbol{\gamma}\rangle$ with

$$\begin{aligned} d\theta &= -\,\langle d\mathbf{h}|\ \hat{}\ |\boldsymbol{\gamma}\rangle + \langle\mathbf{h}|\ \hat{}\ |d\boldsymbol{\gamma}\rangle \\ &= +\Omega\hat{}\,\langle\mathbf{h}|\ \hat{}\ |\boldsymbol{\gamma}\rangle + \langle\mathbf{h}|\ \hat{}[\mathbf{\Gamma}]\hat{}\,|\boldsymbol{\gamma}\rangle - \langle\mathbf{h}|\ \hat{}[\mathbf{\Gamma}]\hat{}\,|\boldsymbol{\gamma}\rangle + \langle\mathbf{h}|\ \hat{}\,\Omega\hat{}\,|\boldsymbol{\gamma}\rangle = 0. \end{aligned}$$

Both the exterior and the interior curvature 2-forms can be matrix valued depending upon the partition of the Frame. Each curvature matrix exhibits a set of similarity invariants deduced from the coefficients of the Cayley-Hamilton characteristic polynomial. It would appear therefore that their are two species of Chern characteristic classes that can be constructed from the Cayley-Hamilton polynomial similarity invariants.

If (in the example) the projective Cartan matrix is constrained to be euclidean, then $\Omega = 1$, and both $\mathbf{h} = 0$,and $\boldsymbol{\gamma} = 0$. Hence both the interior and the exterior curvature vanish. Indeed, then both types of torsion 2-forms vanish.

On the other hand, if the Cartan matrix is anti-symmetric (as it must be for an orthonormal frame matrix) then $\Omega = 0$, and $\boldsymbol{\gamma} = -\mathbf{h}$. Hence, the exterior curvature vanishes, and $|\mathbf{\Psi}\rangle = 0$, but the domain could support interior curvature and dislocation torsion 2-forms, $|\mathbf{\Sigma}\rangle \neq 0$. I believe this to be a serious flaw in the ubiquitous assumption that the Basis Frame should be orthonormal.

If the Cartan matrix is left affine, then $\mathbf{h} = 0,\ \Omega = 1$. The interior and exterior domains are flat, but the structure could admit both forms of torsion 2-forms.

I am very much interested in this new class of projective structural equations and would appreciate any direction or comments in this area.

6. References

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