Falaco Solitons

– Cosmic strings in a swimming pool.

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Abstract: A new dynamical bifurcation mechanism finally explains the formation of topological defects experimentally observed and defined in 1986 as Falaco Solitons. The Falaco Solitons are topologically universal phenomena created experimentally by a macroscopic rotational dynamics in a continuous media with a discontinuity surface, such as that found in a swimming pool. The topologically coherent structure of Falaco Solitons replicates certain features found at all physical scales, from spiral arm galaxies and cosmic strings to microscopic hadrons. The easy to replicate experiment indicates the creation of "stationary" thermodynamic states (or solitons) far from equilibrium, which are locally unstable but are globally stabilized. Several exact solutions to the Navier-Stokes equations are demonstrated to admit bifurcations to Falaco Solitons. It is conjectured that the universal coherent topological features of the Falaco Solitons can appear as cosmological realizations of Wheeler's wormholes, could represent spin pairing mechanisms in the microscopic Fermi surface, and exhibit the confinement problem of sub microscopic quarks on the end of a string, or are perhaps realizations of sub-submicroscopic strings connecting branes.

Keywords: Falaco Solitons, Hopf Breathers, Cosmic Strings, Global Stability

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1 The Falaco Soliton - A Topological Defect in a swimming pool.

1.1 Preface

During March of 1986, while visiting an old friend in Rio de Janeiro, Brazil, the present author became aware of a significant topological event involving solitons that can be replicated experimentally by almost everyone with access to a swimming pool. Study the photo which was taken by David Radabaugh, in the late afternoon, Houston, TX 1986.

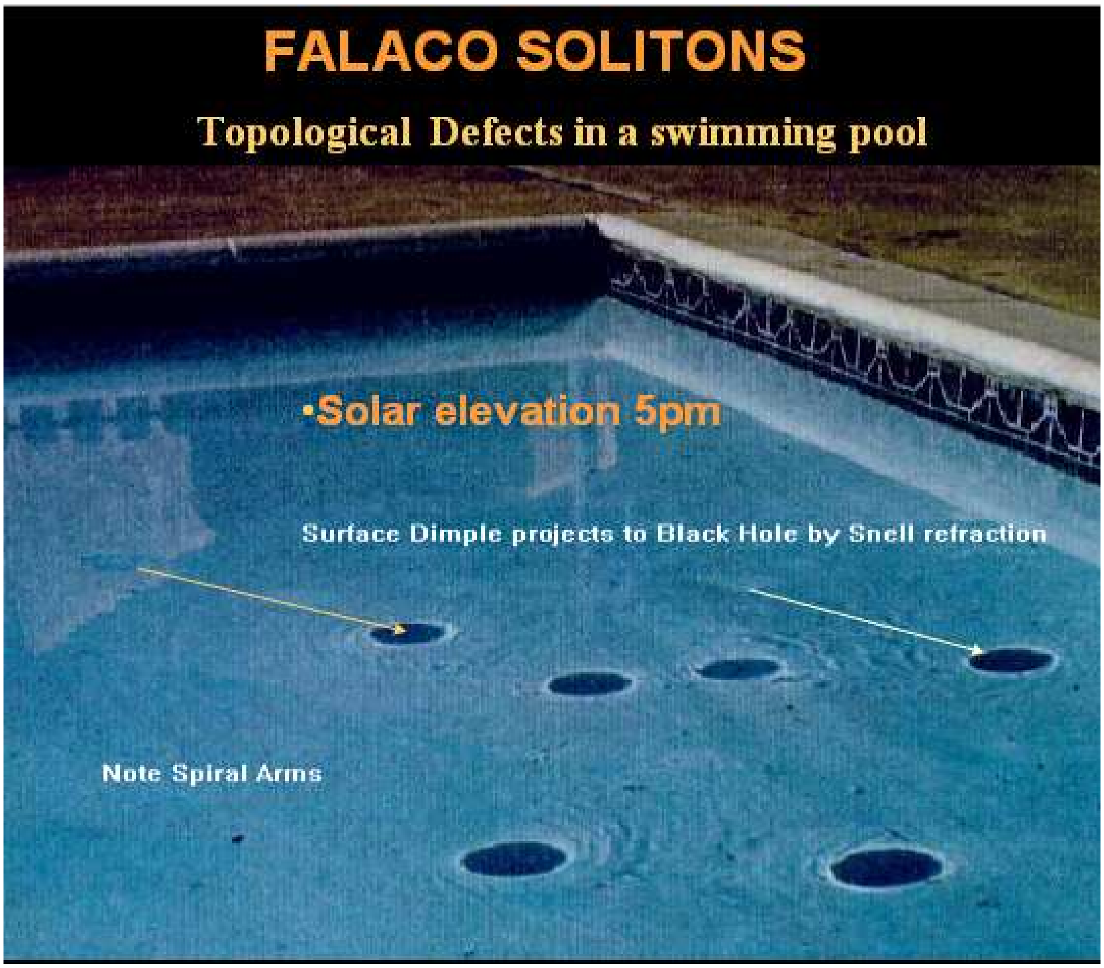

Figure 1. Three pairs of FALACO SOLITONS, a few minutes after creation. The kinetic energy and the angular momentum of a pair of Rankine vortices created in the free surface of water quickly decay into dimpled, locally unstable, singular surfaces that have an extraordinary lifetime of many minutes

in a still pool. These singular surfaces are connected by means of a stabilizing invisible singular thread, or string, which if abruptly severed will cause the end-caps to disappear in a rapid non-diffusive manner. The black discs are formed on the bottom of the pool by Snell refraction of a rotationally induced dimpled surface. Careful examination of contrast in the photo will indicate the region of the dimpled surface as deformed artifacts to the left of each black spot at a distance about equal to the separation distance of the top right pair and elevated above the horizon by about 25 degrees. The photo was taken in late afternoon. The fact that the projections are circular and not ellipses indicates that the dimpled surface is a minimal surface. (Photo by David Radabaugh, Schlumberger, Houston)

The extraordinary photo is an image of the 3 pairs of what are now called Falaco Solitons. The Falaco Soliton consists of a pair of globally stabilized rotational indentations in the free water-air surface of the swimming pool, and an (unseen in the photograph) interconnecting thread from the vertex of one dimple to the vertex of the other dimple that forms the rotational pair. The fluid motion is a local (non-rigid body) rotation motion about the interconnecting thread. In the photo the actual indentations of the free surface are of a few millimeters at most. The lighting and contrast optics enables the dimpled surface structures to be seen (although highly distorted) above and to the left of the black spots on the bottom of the pool. The experimental details of creating these objects are described below. From a mathematical point of view, the Falaco Soliton is a connected pair of two dimensional topological defects connected by a one dimensional topological defect or thread. The Falaco soliton is easily observed in terms of the black spots associated with the surface indentations. The black circular discs on the bottom of the pool are created by Snell refraction of sunlight on the dimpled surfaces of negative Gauss curvature. Also the vestiges of mushroom spirals in the surface structures around each pair can be seen. The surface spiral arms can be visually enhanced by spreading chalk dust on the free surface of the pool.

The photo demonstrates the existence of Falaco Solitons, a few minutes after creation, by a mechanism to be described below. The kinetic energy and the angular momentum initially given to a pair of Rankine vortices (of positive Gauss curvature) created in the free surface of water, quickly decay into dimpled, locally unstable, singular surfaces (of negative Gauss curvature) that have an extraordinary lifetime of more than 15 minutes in a still pool. These "solitons" are extraordinary for they are examples of solitons that can be created easily in a macroscopic environment. Very few examples of such long-lived topological structures can be so easily created in dynamical systems.

The surface defects of the Falaco Soliton are observed dramatically due the formation of circular black discs on the bottom of the swimming pool. The very dark black discs are emphasized in contrast by a bright ring or halo of focused light surrounding the black disc. All of these visual effects can be explained by means of the unique optics of Snell refraction from a surface of negative Gauss curvature. (This explanation was reached on the day, and about 30 minutes after, the present author became aware of the Falaco effect, while standing under a brilliant Brazilian sun and in the white marble swimming pool of his friend in Rio de Janeiro. An anecdotal history of the discovery is described below.) The dimpled surface created appears to be (almost) a minimal surface with negative Gauss curvature and mean curvature $Xm = 0$. This conclusion is justified by the fact that the Snell projection to the floor of the pool is almost conformal, preserving the circular appearance of the black disc, independent from the angle of solar incidence. (Notice that the black spots on the bottom of the pool in the photo are circular and not distorted ellipses, even though the solar elevation is less than 30 degrees.) The conformal projection property is a property of normal projection from minimal surfaces [Struik 1961].

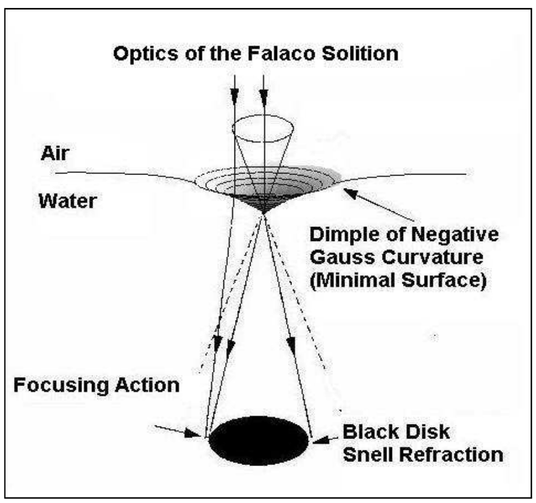

Figure 1. Optics of the Falaco Soliton

The effect is easily observed, for in strong sunlight the convex hyperbolic indentation will cause an intensely black circular disk (or absence of light) to be imaged on the bottom of the pool. In addition a bright ring of focused light will surround the black disk, emphasizing the contrast. During the initial few seconds of decay to the metastable soliton state, the large black disk is decorated with spiral arm caustics, remindful of spiral arm galaxies. The spiral arm caustics contract around the large black disk during the stabilization process, and ultimately disappear when the soliton state is achieved. It should be noted that if chalk dust is sprinkled on the surface of the pool during the formative stages of the Falaco soliton, then the topological signature of the familiar Mushroom Spiral pattern is exposed. The black disk optics are completely described by Snell refraction from a surface of revolution that has negative Gauss curvature.

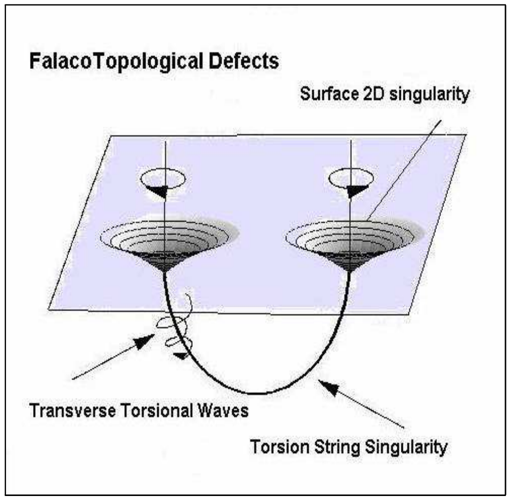

Figure 2. Falaco Topological Defects.

Dye injection near an axis of rotation during the formative stages of the Falaco Soliton indicates that there is a unseen thread, or 1-dimensional string singularity, in the form of a circular arc that connects the two 2-dimensional surface singularities or dimples. Transverse Torsional waves of dye streaks can be observed to propagate, back and forth, from one dimple vertex to the other dimple vertex, guided by the "string" singularity. The effect is remindful of the whistler propagation of electrons along the guiding center of the earth's magnetic field lines.

A feature of the Falaco Soliton [RMK 1986] that is not immediately obvious is that it consists of a pair of two dimensional topological defects, in a surface of fluid discontinuity, which are *connected* by means of a topological singular thread. It is conjectured that the tension in the singular connecting thread provides the force that maintains the global stability of the pair of locally unstable, dimpled surface structures. The equilibrium mode for the free surface requires that the surface be flat, of zero Gauss curvature, without dimples. If dye drops are injected into the water near the rotational axis, and during formative stages of the Falaco Soliton, the dye particles will execute a *torsional* wave motion that oscillates up and down, back and forth, until the dye maps out the thread singularity (a circular arc!) that connects the two vertices of the Falaco Soliton. The singular thread acts as a guiding center for the torsion waves. If the thread is severed, the endcap singularities disappear almost immediately, and not diffusively.

However, as a soliton, the topological system retains its coherence for remarkably long time - more than 15 minutes in a still pool. The long lifetime of the Falaco Soliton is due to this *global stabilization* of the connecting string singularity, even though the surface of negative Gauss curvature is locally unstable. The long life of the soliton state in the presence of a viscous media indicates that the flow vector field describing the dynamics is probably harmonic. This result is in agreement with the assumption that the fluid can be represented by a Navier-Stokes equation with a dissipation that is represented by the product of a viscosity and the vector Laplacian of the velocity field. If the velocity field is harmonic, the vector Laplacian vanishes, and the dissipation goes to zero no matter what the magnitude is of the viscosity term. Hence a palatable argument is offered for the long lifetime. More over it is known that minimal surfaces are generated by harmonic vector fields, hence the minimal surface endcaps give further credence to the idea of a harmonic velocity field.

The bottom line is that it is possible to produce, hydrodynamically, in a viscous fluid with a surface of discontinuity, a long lived coherent structure that consists of a set of macroscopic topological defects. The Falaco Solitons are representative of non-equilibrium long lived structures, or "stationary states", far from equilibrium. These observation were first reported at the 1987 Dynamics Days conference in Austin, Texas [RMK 1986], [RMK 1987 b], and subsequently in many other places, mostly in the hydrodynamic literature [RMK 1990], [RMK 1991 a], [RMK 1995], [RMK 1992 d], [RMK 1999 a], as well as several APS meetings.

These, long-lived topologically coherent objects, dubbed the Falaco Solitons (fore reasons explained below), have several features equivalent to those reported for models of the sub-microscopic hadron. String theorists take note, for the structure consists of a pair of topological 2-dimensional locally unstable rotational defects in a surface of discontinuity, globally connected and stabilized globally in the fluid by a 1 dimensional topological defect or string with tension. (The surface defects are of negative Gauss curvature, and are therefor locally unstable.) As mentioned above the experimental equilibrium state is a surface of zero Gauss curvature. However, the local instability is overcome globally by a string whose tension globally stabilizes the locally unstable endcaps. These observational conjectures have now been explained theoretically in terms of a bifurcation process, which is explained in detail below. Note that like hadrons, the endcaps represent the quarks which suffer a confinement problem, for when the confining string is severed, the hadrons (endcaps) disappear.

The reader must remember that the Falaco Soliton is a topological object that can and will appear at all scales, from the microscopic, to the macroscopic, from the sub-submicroscopic world of strings connection branes, to the cosmological level of spiral arm galaxies connected by threads. At the microscopic level, the method offers a view of forming spin pairs that is different from Cooper pairs and could offer insight into Superconductivity. At the level of Cosmology, the concept of Falaco Solitons could lead to explanations of the formation of flat spiral arm galaxies. At the submicroscopic level, the Falaco Solitons mimic quarks on a string. At the macroscopic level, the topological features of the Falaco Solitons can be found in solutions to the Navier-Stokes equations in a rotating frame of reference. Under deformation of the discontinuity surface to a flattened ball, the visual correspondence to hurricane structures between the earth surface and the tropopause is remarkable. In short, the concept of Falaco Solitons is a universal phenomena.

1.2 The Experiment

The Falaco Soliton phenomena is easily reproduced by placing a large circular disc, such as dinner plate, vertically into the swimming pool until the plate is half submerged and it oblate axis resides in the water-air free surface. Then move the plate slowly in the direction of its oblate axis. At the end of the stroke, smoothly extract the plate (with reasonable speed) from the water, imparting kinetic energy and distributed angular momentum to the fluid. Initially, the dynamical motion of the edges of the plate will create a pair of Rankine Vortices in the free surface of the water (a density discontinuity which can also be mimicked by salt concentrations). These Rankine vortices of opposite rotation will cause the initially flat surface of discontinuity to form a pair of parabolic concave indentations of positive Gauss curvature, indicative of the "rigid body" rotation of a pair of contra-rotating vortex cores of uniform vorticity. In a few seconds the concave Rankine depressions will decay into a pair of convex dimples of negative Gauss curvature. Associated with the evolution is a visible set of spiral arm caustics, As the convex dimples form, the surface effects can be observed in bright sunlight via their Snell projections as large black spots on the bottom of the pool. In a few tries you will become an expert experimentalist, for the drifting spots are easily created and, surprisingly, will persist for many minutes in a still pool. The dimpled depressions are typically of the order of a few millimeters in depth, but the zone of circulation typically extends over a disc of some 10 to 30 centimeters or more, depending on the plate diameter. This configuration, or coherent topological defect structure, has been defined as the Falaco Soliton. For purposes of illustration , the vertical depression has been greatly exaggerated in Figures 2 and 3.

If a thin broom handle or a rod is placed vertically in the pool, and the Falaco soliton pair is directed in its translation motion to intercept the rod symmetrically, as the soliton pair comes within range of the scattering center, or rod, (the range is approximately the separation distance of the two rotation centers) the large black spots at first shimmer and then disappear. Then a short time later, after the soliton has passed beyond the interaction range of the scattering center, the large black spots coherently reappear, mimicking the numerical simulations of soliton coherent scattering. for hydrodynamics, this observation firmly cements the idea that these objects are truly coherent "Soliton" structures. This is the only (known to this author) macroscopic visual experiment that demonstrates these coherence features of soliton scattering.

If the string connecting the two endcaps is sharply "severed", the confined, two dimensional endcap singularities do not diffuse away, but instead disappear almost explosively. It is this observation that leads to the statement that the Falaco soliton is the macroscopic topological equivalent of the illusive hadron in elementary particle theory. The two 2-dimensional surface defects (the quarks) are bound together by a string of confinement, and cannot be isolated. The dynamics of such a coherent structure is extraordinary, for it is a system that is globally stabilized by the presence of the connecting 1-dimensional string.

2 Bifurcation Process and the Production of Topological Defects

2.1 Lessons from the bifurcation to Hopf Solitons

2.1.1 Local Stability

Consider a dynamical system that can be encoded (to within a factor, $1/\lambda$) on the variety of independent variables $\{x, y, z, t\}$ in terms of a 1-form of Action:

$$A = \{A_k(x, y, z, t)dx^k - \phi(x, y, z, t)dt\}/\lambda(x, y, z, t). \tag{1}$$

Then construct the Jacobian matrix of the (covariant) coefficient functions:

$$[\mathbb{J}_{jk}(A)] = \left[\partial(A_j/\lambda)/\partial x^k\right]. \tag{2}$$

This Jacobian matrix can be interpreted as a projective correlation mapping of "points" (contravariant vectors) into "hyperplanes" (covariant vectors). The correlation mapping is the dual of a collineation mapping, $\left[\mathbb{J}(\mathbf{V}^k)\right]$, which takes points into points. Linear (local) stability occurs at points where the (possibly complex) eigenvalues of the Jacobian matrix are such that the real parts are not positive. The eigenvalues, ξ_k, are determined by solutions to the Cayley-Hamilton characteristic polynomial of the Jacobian matrix, $[\mathbb{J}(A)]$:

$$\Theta(x,y,z,t;\ \xi) = \xi^4 - X_M\xi^3 + Y_G\xi^2 - Z_A\xi + T_K \Rightarrow 0. \tag{3}$$

The Cayley-Hamilton polynomial equation defines a family of implicit functions in the space of variables, $X_M(x,y,z,t)$, $Y_G(x,y,z,t)$, $Z_A(x,y,z,t)$, $T_K(x,y,z,t)$. The functions X_M, Y_G, Z_A, T_K are defined as the similarity invariants of the Jacobian matrix. If the eigenvalues, ξ_k, are distinct, then the similarity invariants are given by the expressions:

$$\begin{aligned}
X_M &= \xi_1+\xi_2+\xi_3+\xi_4 = Trace\,[\mathbb{J}_{jk}]\,, && (4)\\
Y_G &= \xi_1\xi_2+\xi_2\xi_3+\xi_3\xi_1+\xi_4\xi_1+\xi_4\xi_2+\xi_4\xi_3, && (5)\\
Z_A &= \xi_1\xi_2\xi_3+\xi_4\xi_1\xi_2+\xi_4\xi_2\xi_3+\xi_4\xi_3\xi_1, && (6)\\
T_K &= \xi_1\xi_2\xi_3\xi_4 = \det\,[\mathbb{J}_{jk}]\,. && (7)
\end{aligned}$$

In the differential geometry of 3-dimensional space, $\{x,y,z\}$, when the scaling coefficient is chosen to be the quadratic isotropic Holder norm of index 1 (the Gauss map), then the determinant of the 3x3 Jacobian matrix vanishes, and the resulting similarity invariants become related to the mean curvature and the Gauss curvature of the Shape matrix.

Bifurcation and singularity theory involves the zero sets of the similarity invariants, and the algebraic intersections of the implicit hypersurfaces so generated by these zero sets. Recall that the theory of linear (local) stability requires that the eigenvalues of the Jacobian matrix have real parts which are not greater than zero. For a 4th order polynomial, either all 4 eigenvalues are real; or, two eigenvalues are real, and two eigenvalues are complex conjugate pairs; or there are two distinct complex conjugate pairs. Local stability therefor requires:

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$$\begin{aligned}
\text{Odd}\ \ X_M &\leq 0, && (8)\\
\text{Odd}\ \ Z_A &\leq 0, && (9)\\
\text{Even}\ \ Y_G &\geq 0, && (10)\\
\text{Even}\ \ T_K &\geq 0. && (11)
\end{aligned}$$

2.1.2 The Hopf Map

The Hopf map is a rather remarkable projective map from 4 to 3 (real or complex) dimensions that has interesting and useful topological properties related to links and braids and other forms of entanglement. As will be demonstrated, the Hopf map satisfies the criteria of Local

Stability, and yet is not an integrable system, and admits irreversible dissipation. The map can be written as $\{x, y, z, s = ct\} \Rightarrow \{x1, x2, x3\}$

$$\textbf{Hopf Map} \quad \mathbf{H}1 = [x1, x2, x3] = [2(xz + ys), 2(xs - yz), (x^2 + y^2) - (z^2 + s^2)]. \tag{12}$$

A remarkable feature of this map is that

$$\mathbf{H}1 \cdot \mathbf{H}1 = (x1)^2 + (x2)^2 + (x3)^2 = (x^2 + y^2 + z^2 + s^2)^2. \tag{13}$$

Hence a real (and imaginary) 4 dimensional sphere maps to a real 3 dimensional sphere. If the functions $[x1, x2, x3]$ are defined as $[x/ct, y/ct, z/ct]$, then the 4D sphere $(X^2+Y^2+Z^2+S^2)^2 = 1$, implies that the Hopf map formulas are equivalent to the 4D light cone. The Hopf map can also be represented in terms of complex functions by a map from C2 to R3, as given by the formulas:

$$\mathbf{H}1 = [x1, x2, x3] = [\alpha \cdot \beta^* + \beta \cdot \alpha^*, i(\alpha \cdot \beta^* - \beta \cdot \alpha^*), \alpha \cdot \alpha^* - \beta \cdot \beta^*. \tag{14}$$

For **H**1, the 4 independent 1 forms are given by the expressions (where $\Lambda(x, y, z, s)$ is an arbitrary scaling function):

$$d(x1) = 2zd(x) + 2sd(y) + 2xd(z) + 2yd(s) \tag{15}$$

$$d(x2) = 2sd(x) - 2zd(y) - 2yd(z) + 2xd(s) \tag{16}$$

$$d(x3) = 2xd(x) + 2yd(y) - 2zd(z) - sd(s) \tag{17}$$

$$A = \{-yd(x) + xd(y) - sd(z) + zd(s)\}/\Lambda. \tag{18}$$

The formula for the, A, 1-form can be generalized to include constant coefficients of chirality and expansion, to read

$$A_{Hopf} = \{a(-yd(x) + xd(y)) + b(-sd(z) + zd(s))\}/\Lambda. \tag{19}$$

It is some interest to examine the properties of the 1-form, A_{Hopf},defined hereafter as the canonical Hopf 1-form. The Jacobian matrix (for $\Lambda = 1$) becomes

$$JAC_{Hopf} := \begin{bmatrix} 0 & a & 0 & 0 \\ -a & 0 & 0 & 0 \\ 0 & 0 & 0 & -b \\ 0 & 0 & b & 0 \end{bmatrix} \tag{20}$$

with eigenvalues $e1 = ia$, $e2 = -ia$, $e3 = ib$, $e4 = -ib$, and with similarity invariants,

$$X_M = 0. \tag{21}$$
$$Y_G = a^2 + b^2 \geq 0 \tag{22}$$
$$Z_A = 0. \tag{23}$$
$$T_K = a^2 b^2 \geq 0. \tag{24}$$

Hence the canonical Hopf 1-form, A_{Hopf}, is locally stable. It the 1-form is scaled by the factor, $1/\sqrt{(x^2 + y^2 + z^2 + s^2)}$, then the similarity invariants and in all cases represents an imaginary minimal surface. The curvatures are pure imaginary, but the Gauss curvature is positive! For the simple case where $b = 0$, the Hopf map describes an minimal surface with imaginary individual curvatures. The classic real minimal surface has a Gauss curvature Y_G which is negative, and for which the individual curvatures are real.

For $\Lambda = 1$, it follows that the Hopf 1-form is of Pfaff dimension 4, and has a topological torsion 4-vector proportional to the ray vector from the origin to a point in the space,

$$\mathbf{T}_4 = -2ab[x, y, z, t] \ . \tag{25}$$

Any process that evolves with a component in the direction of $\mathbf{T}_4$ is thermodynamically *irreversible*, as

$$L(\mathbf{T}_4)A = -8ab\ A = Q, \tag{26}$$
$$\text{and } Q\hat{\ }dQ \neq 0. \tag{27}$$

Consider the Falaco Solitons to be represented by a dynamical system topologically equivalent to an exterior differential system of 1-forms,

$$\omega^x = dx - \mathbf{V}^x(x, y, z, t)dt \Rightarrow 0, \tag{28}$$
$$\omega^y = dy - \mathbf{V}^y(x, y, z, t)dt \Rightarrow 0, \tag{29}$$
$$\omega^z = dz - \mathbf{V}^z(x, y, z, t)dt \Rightarrow 0. \tag{30}$$

When all three 1-forms vanish, imposing the existence of a topological limit structure on the base manifold of 4 dimensions, $\{x, y, z, t\}$, the result is equivalent to a 1D solution manifold defined as a kinematic system. The solution manifold to the dynamical system is in effect a parametrization of the parameter t to the space curve $C_{parametric}$ in 4D space, where for kinematic perfection, $[\mathbf{V}^k, 1]$ is a tangent vector to the curve $C_{parametric}$. Off the kinematic solution submanifold, the non-zero values for the 1-forms, ω^k, can be interpreted as topological fluctuations from "kinematic perfection".

If "kinematic perfection" is not exact, then the three 1-forms ω^k are not precisely zero, and have a finite triple exterior product that defines a N-1=3 form in the 4 D space. From the theory of exterior differential forms it is the intersection of the zero sets of these three hypersurfaces ω^k that creates an implicit curve $C_{implicit}$ in 4D space.

$$\begin{aligned} C_{implicit} &= \omega^x\hat{}\omega^y\hat{}\omega^z && (31)\\ &= dx\hat{}dy\hat{}dz - \mathbf{V}^x dy\hat{}dz\hat{}dt + \mathbf{V}^y dx\hat{}dz\hat{}dt - \mathbf{V}^z dx\hat{}dy\hat{}dt && (32)\\ &= -i([\mathbf{V},1])\Omega_4. && (33)\end{aligned}$$

The discussion brings to mind the dualism between points (rays) and hypersurfaces (hyperplanes) in projective geometry.

If a ray (a point in a the projective 3 space of 4 dimensions) is specified by the 4 components of a the 4D vector $[\mathbf{V},1]$ multiplied by any non-zero factor, κ, (such that $[\mathbf{V},1] \approx \kappa[\mathbf{V},1]$), then the equation of a dual projective hyperplane is given by the expression $[\mathbf{A},-\phi]$ such that

$$\langle\gamma[\mathbf{A},-\phi]| \circ |\kappa[\mathbf{V},1]\rangle = 0. \quad (34)$$

The principle of projective duality [Pedoe (1988)] implies that (independent from the factors γ and κ)

$$\phi = \mathbf{A}\circ\mathbf{V}. \quad (35)$$

A particularly easy choice is to assume that (to within a factor)

$$\begin{aligned} \mathbf{A}_k &= \mathbf{V}^k, \text{ and } \quad \phi = \mathbf{V}\circ\mathbf{V}, && (36)\\ A &= V_k dx^k - V_k V^k dt. && (37)\\ V_k(x,y,z,t) &\equiv V^k(x,y,z,t), \text{ the 3 functions of the dynamical system.} && (38)\end{aligned}$$

It should be remembered that not all dynamic features are captured by the similarity invariants of a dynamic system . The antisymmetric features of the dynamics is better encoded in terms of Cartan's magic formula. Cartan's formula expresses the evolution of a 1-form of Action, A, in terms of the Lie differential with respect to a vector field, V, acting on the 1-form that encodes the properties of the physical system. For example, consider the 1-form of Action (the canonical form of a Hopf system) given by the equation

$$A_{Hopf} = \alpha(ydx - xdy) + \beta(tdz - zdt). \tag{39}$$

The Jacobian matrix of this Action 1-form has eigenvalues which are solutions of the characteristic equation,

$$\Theta(x, y, z, t;\ \xi)_{Hopf} = (\xi^2 + \alpha)(\xi^2 + \beta) \Rightarrow 0. \tag{40}$$

The eigenvalues are two conjugate pairs of pure imaginary numbers, $\{i\alpha, -i\alpha, i\beta, -i\beta\}$ and are interpreted as "oscillation" frequencies. The similarity invariants are $X_M = 0$, $Y_G = \alpha^2 + \beta^2 > 0$, $Z_A = 0$, $T_K = \alpha^2\beta^2 > 0$. The Hopf eigenvalues have no real parts that are positive, and so the Jacobian matrix is locally stable. The criteria for a double Hopf oscillation frequency requires that the algebraically odd similarity invariants vanish and the algebraically even similarity invariants are positive definite. The stability critical point of the Hopf bifurcation occurs when all similarity invariants vanish. In such a case the oscillation frequencies are zero. This Hopf critical point is NOT necessarily the same as the thermodynamic critical point, as exhibited by a van der Waals gas. The oscillation frequencies have led the Hopf solution to be described as a "breather". The Hopf system is a locally stable system in four dimensions. Each of the pure imaginary frequencies can be associated with a "minimal" hypersurface. .

Suppose that $\beta = 0$. Then the resulting characteristic equation represents a "minimal surface" as $X_M = 0$, but with a Gauss curvature which is positive definite, $Y_G = \alpha^2 > 0$. The curvatures of the implicit surface are imaginary. In differential geometry, where the eigenfunctions can be put into correspondence with curvatures, the Hopf condition, $X_M = 0$, for a single Hopf frequency would be interpreted as "strange" minimal surface (attractor). The surface would be strange for the condition $Y_{G(hopf)} = \alpha^2 > 0$ implies that the Gauss curvature for such a minimal surface is positive. A real minimal surface has curvatures which are real and opposite in sign, such that the Gauss curvature is negative.

As a real minimal surface has eigenvalues with one positive and one negative real number, the criteria for local stability is not satisfied for real minimal surfaces. Yet experience indicates that soap films can occur as "stationary states". The implication is that soap films can be globally stabilized, even though they are locally unstable.

As developed in the next section, the Falaco critical point and the Hopf critical point are the same: all similarity invariants vanish. For the autonomous examples it is possible to find an implicit surface, $Y_{G(hopf)} = Y_{G(falaco)} = 0$, in terms of the variables $\{x, y, z;\ A, B, C...\}$ where $A, B, C...$ are the parameters of the dynamical system.

Recall that the classic (real) minimal surface has real curvatures with a sum equal to zero, but with a Gauss curvature which is negative ($X_M = 0, Y_G < 0$). Such a system is not locally stable, for there exist eigenvalues of the Jacobian matrix with positive real parts. Yet persistent soap films exist under such conditions and are apparently stable macroscopically (globally). This experimental evidence can be interpreted as an example of global stability overcoming local instability.

2.2 The bifurcation to Falaco Solitons

Similar to and guided by experience with the Hopf bifurcation, the bifurcation that leads to Falaco Solitons must agree with the experimental observation that the endcaps have negative Gauss curvature, and are in rotation. The stability of the Falaco Soliton is global, experimentally, for if the singular thread connecting the vertices is cut, the system decays non-diffusively. Hence the bifurcation to the Falaco Soliton can not imply local stability. This experimental result is related to the theoretical confinement problem in the theory of quarks. To analyze the problem consider the case where the T_K term in the Cayley-Hamilton polynomial vanishes (implying that one eigenvalue of the 4D Jacobian matrix is zero). Experience with the Hopf bifurcation suggests that Falaco Soliton may be related to another form of the characteristic polynomial, where $X_M = 0,\ Z_A = 0,\ Y_G < 0$. This bifurcation is not equivalent to the Hopf bifurcation, but has the same critical point, in the sense that all similarity invariants vanish at the critical point. Similar to the Hopf bifurcation this new bifurcation scheme can be of Pfaff topological dimension 4, which implies that the abstract thermodynamic system generated by the 1-form (which is the projective dual to the dynamical system) is an open, non-equilibrium thermodynamic system. The odd similarity invariants of the 4D Jacobian matrix must vanish. However there are substantial differences between the bifurcation that lead to Hopf solitons (breathers) and Falaco solitons. Experimentally, the Falaco soliton appears to have a projective cusp at the critical point (the vertex of the dimple) and that differs from the Hopf bifurcation which would be expected to have a projective parabola at the critical point.

When $T_K = 0$, the resulting cubic factor of the characteristic polynomial will have 1 real eigenvalue, b, one eigen value equal to zero, and possibly 1 pair of complex conjugate eigenvalues, $(\sigma + i\Omega), (\sigma - i\Omega)$. To be stable globally it is presumed that

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$$\text{Odd } X_M = b + 2\sigma \leq 0, \tag{41}$$

$$\text{Odd } Z_A = b(\sigma^2 + \Omega^2) \leq 0, \tag{42}$$

$$\text{Even } Y_G = \sigma^2 + \Omega^2 + 2b\sigma \quad \text{undetermined} \tag{43}$$

$$\text{Even } T_K = 0 \tag{44}$$

If all real coefficients are negative then $Y_G > 0$, and the system is locally stable. Such is the situation for the Hopf bifurcation. However, the Falaco Soliton experimentally requires that $Y_G < 0$.

By choosing $b \leq 0$, in order to satisfy $Z_A \leq 0$, leads to the constraint that $\sigma = -b/2 > 0$, such that the real part of the complex solution is positive, and represents an expansion, not a contraction. Substitution into the formula for Y_G leads to the condition for generation of a Falaco Soliton:

$$Y_{G(falaco)} = \Omega^2 - 3b^2/4 < 0. \tag{45}$$

It is apparent that local stability is lost for the complex eigenvalues of the Jacobian matrix can have positive real parts, $\sigma > 0$. Furthermore it follows that $Y_G < 0$ (leading to negative Gauss curvature) if the square of the rotation speed, Ω, is smaller than the 3/4 of the square of the real (negative) eigen value, b. This result implies that the "forces" of tension overcomes the inertial forces of rotation. In such a situation, a real minimal surface is produced (as visually required by the Falaco soliton). The result is extraordinary for it demonstrates a global stabilization is possible for a system with one contracting direction, and two expanding directions coupled with rotation. The contracting coefficient b (similar to a spring constant) is related to the surface tension in the "string" that connects the two global endcaps of negative Gaussian curvature. The critical point occurs when $\Omega^2 = 3b^2/4$.

It is conjectured that if the coefficient b is in some sense a measure of of a reciprocal length (such that b ˜ $1/R$, a curvature), then there are three interesting formulas comparing angular velocity (orbital period) and length (orbital radius).

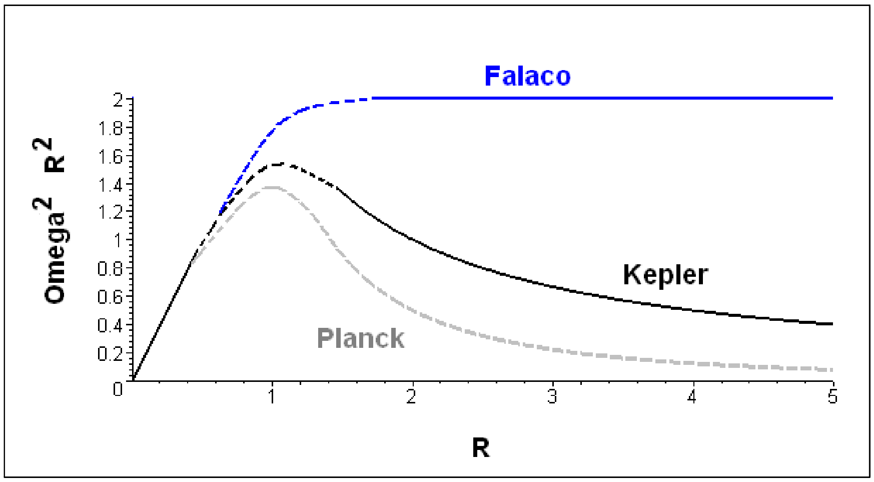
$$\text{Falaco}: \qquad \Omega^2 R^2 = \text{constant} \tag{46}$$

$$\text{Kepler}: \qquad \Omega^2 R^3 = \text{constant} \tag{47}$$

$$\text{Planck}: \qquad \Omega^2 R^4 = \text{constant}. \tag{48}$$

The bifurcations to Hopf Solitons suggest oscillations of expansions and contractions of imaginary minimal surfaces (or Soliton concentration breathers) and have been exhibited in the certain chemical reactions such as the Besalouv-Zhabotinski system. On the other hand, the bifurcations to Falaco Solitons suggest the creation of spiral concentrations, or density waves, on real rotating minimal surfaces. The molal density distributions (or order parameters) are complex. The visual bifurcation structures of the Falaco Solitons in the swimming pool would appear to offer an explanation as to the origin of (˜flat) spiral arm galaxies at a cosmological level, and would suggest that the spiral arm galaxies come in pairs connected by a topological string. Moreover, the kinetic energy of the stars far from the galactic center would not vary as the radius of the "orbit" became very large. This result is counter to the Keplerian result that the kinetic energy of the stars should decrease as 1/R.

If is assumed that the density distribution of star mass is more or less constant over the central region of the spiral arm flat disc-like structures, then over this region, the Newtonian gravitation force would lead to a "rigid body" result, $\Omega^2 R^2 = R$. If it is assumed that the density distribution then decreases dramatically in the outer regions of the spiral arms, then it has been assumed that Keplerian formula holds. The following Figure demonstrates the various options:



3 Falaco Solitons in exact solutions to the Navier-Stokes equations.

The idea that multiple parameter Dynamical Systems can produce tertiary bifurcations was studied by Langford [Langford 1983]. His developments were organized about certain non-linear equations in polar coordinates, with multiple parameters $(r, \theta, z, t; A, B, , ..)$.

$$dr/dt = rg(r, z, A, G, C) \tag{49}$$

$$d\theta/dt = 1 \tag{50}$$

$$dz/dt = f(r, z, A, B, D) \tag{51}$$

It is remarkable that these tertiary bifurcations can be demonstrated to be solutions of the Navier-Stokes equations in a rotating frame of reference [RMK 1991 e]. Langford was interested in how these "normal" forms of dynamical systems could cause bifurcations to Hopf breather-solitons. Herein, it is also of interest to determine how and where these dynamical systems can cause bifurcations to Falaco rotational solitons.

It is of some pedagogical utility to transform the Langford equations to $\{x, y, z\}$ coordinates with parameters, $A, B, ...$ In polar coordinates, a map between the variables $\{x, y, z\} \Rightarrow \{r, \theta, z\}$ leads to the following expressions:

$$r = \sqrt{x^2 + y^2}, \qquad dr = (xdx + ydy)/\sqrt{x^2 + y^2}, \tag{52}$$
$$\theta = \tan(y/x), \qquad d\theta = \pm(ydx - xdy)/(x^2 + y^2) \tag{53}$$
$$z = z, \qquad dz = dz \tag{54}$$

Substitution of the differentiable map into the polar equations yields the system of 1-forms

$$\omega^1 = dr - rg(r, z, A, B...)dt = (xdx + ydy)/\sqrt{x^2 + y^2} - rg(r, z, A, B...)dt \tag{55}$$
$$\omega^2 = d\theta - \Omega dt = \pm(ydx - xdy)/(x^2 + y^2) - \omega dt \tag{56}$$
$$\omega^3 = dz - f(r, z, A, B)dt. \tag{57}$$

The 3-form C composed from the three 1-forms becomes to within an arbitrary factor,

$$\omega^1\hat{}\omega^2\hat{}\omega^3 = i(\rho\mathbf{V}_4)\Omega_4 = i(\rho\mathbf{V}_4)dx\hat{}dy\hat{}dz\hat{}dt, \tag{58}$$
$$= -\{\mathbf{V}^x dy\hat{}dz\hat{}dt - \mathbf{V}^y dx\hat{}dz\hat{}dt + \mathbf{V}^z dx\hat{}dy\hat{}dt - dx\hat{}dy\hat{}dt\}, \tag{59}$$
$$\text{where } \mathbf{V}_4 = [\mathbf{V}_3, 1]. \tag{60}$$

with

$$\mathbf{V}^x = \{\mp\Omega y + (xg(r, z, A, B...\} \tag{61}$$
$$\mathbf{V}^y = \{\pm\Omega x + (yg(r, z, A, B...)\} \tag{62}$$
$$\mathbf{V}^z = f(r, z, \lambda, \alpha) \tag{63}$$

The rotation speed (angular velocity) is represented by Ω. The Langford examples are specializations of the functions $f(r, z, A, B...)$ and $g(r, z, A, B...)$. The following examples yield solutions to the similarity invariants for three of the Langford examples that he described as the Saddle Node Hopf bifurcation, the Hysteresis Hopf bifurcation, and the Transcritical Hopf bifurcation.

The similarity invariants are computed for the projective dual 1-form,

$$\text{Projective dual 1-form } A = V_k dx^k - V_k V^k dt. \tag{64}$$

It can be shown that the Pfaff topological dimension of the projective dual 1-form of each of the examples below is 4. This fact implies that the abstract thermodynamic system is an open system far from thermodynamic equilibrium [RMK2005]. Thermodynamic systems of

Pfaff dimension 4 are inherently dissipative, and admit processes which are thermodynamically irreversible. Such irreversible processes are easily computable, and are proportional to the 3-form of Topological Torsion, $A\hat{}dA$. If

$$\text{for } i(\mathbf{T}_4)\Omega_4 = A\hat{}dA, \quad (65)$$

$$\text{such that } \langle \mathbf{T}_4| \circ |\mathbf{V}_4\rangle = 0, \quad (66)$$

then the dynamical system is *not* representative of an irreversible process. For a physical system represented by the projective dual 1-form, and a process defined by the direction field of the dynamical system, the process is irreversible only if it has a component in the direction of the Topological Torsion vector. Direct computation indicates that for the projective dual 1-form composed of the components of an autonomous dynamical system, then a necessary condition for reversibility in an abstract thermodynamic sense is that the helicity density must vanish. Hence a sufficient condition for irreversibility is

Theorem 1 *Autonomous processes* $\mathbf{V}$ *(such that* $\partial\mathbf{V}/\partial t = 0$*) are irreversible if* $\mathbf{V}\circ curl\ \mathbf{V} \neq 0$

3.1 Saddle-Node: Hopf and Falaco bifurcations

The dynamical system

$$f = A + Bz^2 + D(x^2 + y^2) \quad (67)$$

$$g = (G + Cz) \quad (68)$$

$$dx/dt = \mathbf{V}^x = x(G + Cz) \mp \Omega y \quad (69)$$

$$dy/dt = \mathbf{V}^y = y(G + Cz) \pm \Omega x \quad (70)$$

$$dz/dt = \mathbf{V}^z = A + Bz^2 + D(x^2 + y^2) \quad (71)$$

Similarity Invariants for $A = V_k dx^k - V^k V_k dt$

$$X_M = 2(G + Cz + Bz) \quad (72)$$

$$Y_G = +\Omega^2 - 2CD(x^2 + y^2) + G^2 + 2G(C + 2B)z + C^2(1 + 4B)z^2 \quad (73)$$

$$Z_A = +\{2Bz\}\Omega^2 + 2(G + Cz)(GBz + CBz^2 - DC(x^2 + y^2)$$

$$T_K = 0 \quad (74)$$

The similarity invariants are also chiral invariants with respect to the sign of the rotation parameter, Ω. The criteria for Hopf oscillations requires that $X_M = 0$, and $Z_A = 0$. When these constraints are inserted into the formula for Y_G they yield $Y_{G(hopf)}$. The criteria for oscillations is that $Y_{G(hopf)} > 0$.

$$\text{Hopf Constraint } \; Y_{G(hopf)} = +3\Omega^2 - B^2 z^2 > 0, \tag{75}$$

$$\text{Oscillation frequencies } \; : \; \omega = \pm\sqrt{Y_{G(hopf)}} \tag{76}$$

Note that $Y_{G(hopf)}$ is a quadratic form in terms of the rotation parameter. When $Y_{G(hopf)} < 0$, it is defined as $Y_{G(falaco)}$. It is therefor easy to identify the tension parameter, b, for the Falaco Soliton by evaluating the Falaco formula

$$Y_{G(falaco)} = \Omega^2 - 3b^2/4 < 0. \tag{77}$$

$$\text{Falaco tension } \; b^2 = 4B^2 z^2/9. \tag{78}$$

The coefficient b can be interpreted as the Hooke's law force (tension) associated with a linear spring extended in the z direction, with a spring constant equal to $2/3B$. Indeed, computer solutions to the Saddle node Hopf system indicate the trajectories can be confined internally to a sphere, and that Falaco surfaces of negative Gauss curvature are formed at the North and South poles by the solution trajectories.

$$\text{Helicity} = \mathbf{V} \circ curl\ \mathbf{V}$$

$$Helicity = -(C(x^2 + y^2) + 2A + 2Bz^2)\Omega \tag{79}$$

If the process described by the dynamical system is to be reversible in a thermodynamic sense, then the Helicity must vanish. This constraint fixes the value of the rotation frequency Ω in the autonomous system for reversible bifurcations.

The Hopf-Falaco critical point in similarity coordinates can be mapped to an implicit surface in xyz coordinates, eliminating the rotation parameter, Ω.

$$Y_{G(hopf-critical)} = Y_{G(falaco-critical)} = -(3DC(x^2 + y^2) + 4B^2 z^2) \Rightarrow 0. \tag{80}$$

Depending on the values assigned to the parameters, and especially the signs of C and D, the critical surface is either open or closed. When the critical surface function is positive, the Hopf-Falaco bifurcation leads to Hopf Solitons (breathers), and if the critical surface function is negative, the bifurcation leads to Falaco Solitons.

3.2 Hysteresis-Hopf and Falaco bifurcations

$$f = A + Bz + Ez^3 + D(x^2 + y^2) \quad (81)$$
$$g = (-G + Cz) \quad (82)$$
$$dx/dt = \mathbf{V}^x = x(-G + Cz) \mp \Omega y \quad (83)$$
$$dy/dt = \mathbf{V}^y = y(-G + Cz) \pm \Omega x \quad (84)$$
$$dz/dt = \mathbf{V}^z = A + Bz + Ez^3 + D(x^2 + y^2) \quad (85)$$

Similarity Invariants for $A = V_k dx^k - V^k V_k dt$

$$X_M = 2(Cz + G) + (B + 3Ez^2) \quad (86)$$
$$Yg = \Omega^2 + \{G^2 - 2GB - 2DC(x^2 + y^2)\} + \{2G(B - C)\}z \quad (87)$$
$$+\{C^2 - 6GE\}z^2 + \{6CE\}z^3 \quad (88)$$
$$Z_A = \{B + 3Ez^2\}\Omega^2 + \{2GCD(x^2 + y^2) + G^2B\} + \quad (89)$$
$$\{-2C^2D(x^2 + y^2) - 2GCB\}z + \{3G^2E + C^2B\}z^2 \quad (90)$$
$$+\{-6GCE\}z^3 + \{3C^2E\}z^4 \quad (91)$$
$$T_K = 0 \quad (92)$$

The criteria for Hopf oscillations requires that $X_M = 0$, and $Z_A = 0$. When these constraints are inserted into the formula for Y_G they yield $Y_{G(hopf)}$. The criteria for oscillations is that $Y_{G(hopf)} > 0$.

$$\text{Hopf Constraint :} \quad Y_{G(hopf)} = 3\Omega^2 - 9/4E^2z^4 - 3/2BEz^2 - 1/4B^2 > 0, \quad (93)$$
$$\text{Oscillation frequencies} \; : \; \omega = \pm\sqrt{Y_{G(hopf)}}. \quad (94)$$

Note that (like the Saddle Node Hopf case) $Y_{G(hopf)}$ is a quadratic form in terms of the rotation parameter. It is therefor easy to identify the tension parameter for the Falaco Soliton by evaluating the Falaco formula

$$Y_{G(falaco)} = \Omega^2 - 3b^2/4 < 0, \quad (95)$$
$$\text{Falaco tension } b^2 = (9E^2z^4 + 6BEz^2 + B^2)/9. \quad (96)$$

In this case the tension is not that of a linear spring, but instead can be interpreted as a non-linear spring constant for extensions in the z direction. Indeed, computer solutions to the Hysteresis - Hopf - Falaco system indicate the trajectories can be confined internally to a sphere-like surface, and that Falaco minimal surfaces are visually formed at the North and South poles [Langford 1983].

$$\text{Helicity} = \mathbf{V} \circ curl\ \mathbf{V}$$

$$Helicity = -\{C(x^2 + y^2) + 2A + 2z(B + Ez^2)\}\Omega.$$

If the process described by the dynamical system is to be reversible in a thermodynamic sense, then the Helicity must vanish. This constraint fixes the value of the rotation frequency Ω in the autonomous system for reversible bifurcations.

The Hopf-Falaco critical point in similarity coordinates can be mapped to an implicit surface in xyz coordinates, eliminating the rotation parameter, Ω.

$$Y_{G(hopf-critical)} = Y_{G(falaco-critical)} = -(3DC(x^2 + y^2) + (3EZ^2 + B)^2) \Rightarrow 0. \tag{97}$$

Depending on the values assigned to the parameters, and especially the signs of C and D, the critical surface is either open or closed. When the critical surface function is positive, the Hopf-Falaco bifurcation leads to Hopf Solitons (breathers), and if the critical surface function is negative, the bifurcation leads to Falaco Solitons. Note that if $E = 0$, $DC < 0$, then there is a circular limit cycle in the x,y plane. Direct integration of the differential equations demonstrates the decay to this attractor.

3.3 Transcritical Hopf and Falaco Bifurcations

The dynamical system

$$f = Az + Bz^2 + D(x^2 + y^2)) \tag{98}$$

$$g = A - G + Cz \tag{99}$$

$$dx/dt = \mathbf{V}^x = x(A - G + Cz) \mp \Omega y \tag{100}$$

$$dy/dt = \mathbf{V}^y = y(A - G + Cz) \pm \Omega x \tag{101}$$

$$dz/dt = \mathbf{V}^z = Az + Bz^2 + D(x^2 + y^2) \tag{102}$$

Similarity Invariants for $A = V_k dx^k - V^k V_k dt$

$$\begin{aligned} X_M &= 3A - 2G + 2(C+B)z && (103) \\ Yg &= +\Omega^2 - 2CD(x^2+y^2) + (4CB+C^2)z^2 + 2z(2B(A-G) && (104) \\ &\quad +C(2C-G) + (G^2+3A^2-4GA) \\ Z_A &= +\{A+2Bz\}\Omega^2 + A^3 + 2A^2Bz - 2GA^2 - 4AGBz + 2CzA^2 && (105) \\ &\quad +4ACz^2B - 2ACy^2D + G^2A + 2G^2Bz - 2GCzA - 4GCz^2B \\ &\quad +2GCy^2D + C^2z^2A + 2C^2z^3B - 2C^2zy^2D + 2Dx^2C(G-A-Cz) \\ T_K &= 0 && (106) \end{aligned}$$

The similarity invariants are chiral invariants relative to the rotation parameter Ω. The criteria for Hopf oscillations requires that $X_M = 0$, and $Z_A = 0$. When these constraints are inserted into the formula for Y_G they yield $Y_{G(hopf)}$. The criteria for (breather) oscillations is that $Y_{G(hopf)} > 0$.

$$\text{Hopf Constraint :} \quad Y_{G(hopf)} = 3\Omega^2 ABz - B^2z^2 - 1/4A^2 > 0 \qquad (107)$$

$$\text{Oscillation frequencies} \quad : \quad \omega = \pm\sqrt{-Y_{G(hopf)}} \qquad (108)$$

Note that (again) $Y_{G(hopf)}$ is a quadratic form in terms of the rotation parameter. It is therefor easy to identify the tension parameter for the Falaco Soliton by evaluating the Falaco formula

$$\begin{aligned} Y_{G(falaco)} &= \Omega^2 - 3b^2/4. && (109) \\ \text{Falaco tension } b^2 &= (4B^2z^2 + A^2)/9ABz). && (110) \end{aligned}$$

In this case the tension is again to be associated with a non-linear spring with extensions in the z direction.

$$\text{Helicity} = \mathbf{V} \circ curl\ \mathbf{V}$$

$$H_{bifurcation} = -\{C(x^2+y^2) + 2z(A+Bz)\}\Omega.$$

If the process described by the dynamical system is to be reversible in a thermodynamic sense, then the Helicity must vanish. This constraint fixes the value of the rotation frequency Ω in the autonomous system for reversible bifurcations.

The Hopf-Falaco critical point in similarity coordinates can be mapped to an implicit surface in xyz coordinates, eliminating the rotation parameter, Ω.

$$Y_{G(hopf-critical)} = Y_{G(falaco-critical)} = -(3DC(x^2+y^2) + (2Bz+A)^2) \Rightarrow 0. \qquad (111)$$

Depending on the values assigned to the parameters, and especially the signs of C and D, the critical surface is either open or closed. When the critical surface function is positive, the Hopf-Falaco bifurcation leads to Hopf Solitons (breathers), and if the critical surface function is negative, the bifurcation leads to Falaco Solitons. Note that if $B = 0$, $DC < 0$, then there is a circular limit cycle in the x,y plane. Direct integration of the differential equations demonstrates the decay to this attractor.

3.4 Minimal Surface Hopf and Falaco Bifurcations

The utility of Maple becomes evident when generalizations of the Langford systems can be studied.

The dynamical system

$$f = A + Bz + Fz^2 + Ez^3 + D(x^2 + y^2)) \tag{112}$$

$$g = G + Cz \tag{113}$$

$$dx/dt = \mathbf{V}^x = x(G + Cz) \mp \Omega y \tag{114}$$

$$dy/dt = \mathbf{V}^y = y(G + Cz) \pm \Omega x \tag{115}$$

$$dz/dt = \mathbf{V}^z = A + Bz + Fz^2 + Ez^3 + D(x^2 + y^2)) \tag{116}$$

can be studied with about as much ease as all of the preceding examples. An especially interesting case is given by the system

$$f = A + P\sinh(\alpha z) + D(x^2 + y^2)) \tag{117}$$

$$g = G + Cz \tag{118}$$

$$dx/dt = \mathbf{V}^x = x(G + Cz) \mp \Omega y \tag{119}$$

$$dy/dt = \mathbf{V}^y = y(G + Cz) \pm \Omega x \tag{120}$$

$$dz/dt = \mathbf{V}^z = A + P\sinh(\alpha z) + D(x^2 + y^2)) \tag{121}$$

Similarity Invariants for $A = V_k dx^k - V^k V_k dt$

$$X_M = 2(G + Cz) + \alpha P\cosh(\alpha z) \tag{122}$$

$$Yg = +\Omega^2 - 2CD(x^2 + y^2) + (G + Cz)^2 + 2(G + Cz)P\alpha\cosh(\alpha z)$$

$$Z_A = (+\Omega^2 + (G + Cz)^2)P\alpha\cosh(\alpha z) - 2CD(G + Cz)(x^2 + y^2)$$

$$T_K = 0 \tag{123}$$

The similarity invariants are chiral invariants relative to the rotation parameter Ω. The criteria for Hopf oscillations requires that $X_M = 0$, and $Z_A = 0$. When these constraints are inserted into the formula for Y_G they yield $Y_{G(hopf)}$. The criteria for (breather) oscillations is that $Y_{G(hopf)} > 0$.

$$\text{Hopf Constraint :} \quad Y_{G(hopf)} = 3\Omega^2 - 1/4\alpha^2 P^2(\cosh(\alpha z))^2 > 0 \tag{124}$$

$$\text{Oscillation frequencies} \;:\; \omega = \pm\sqrt{-Y_{G(hopf)}} \tag{125}$$

Note that (again) $Y_{G(hopf)}$ is a quadratic form in terms of the rotation parameter. It is therefor easy to identify the tension parameter for the Falaco Soliton by evaluating the Falaco formula

$$Y_{G(falaco)} = \Omega^2 - 3b^2/4. \tag{126}$$

$$\text{Falaco tension } b^2 = (\alpha^2 P^2(\cosh(\alpha z))^2)/3). \tag{127}$$

In this case the tension is again to be associated with a non-linear spring with extensions in the z direction.

$$\text{Helicity} = \mathbf{V} \circ curl\ \mathbf{V}$$

$$H_{bifurcation} = -\{C(x^2 + y^2) + 2(A + P\sinh(\alpha z))\}\Omega.$$

If the process described by the dynamical system is to be reversible in a thermodynamic sense, then the Helicity must vanish. This constraint fixes the value of the rotation frequency Ω in the autonomous system for reversible bifurcations.

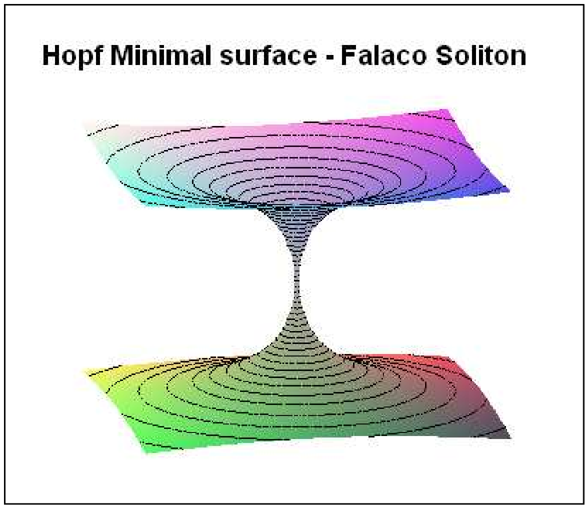
The Hopf-Falaco critical point in similarity coordinates can be mapped to an implicit surface in xyz coordinates, eliminating the rotation parameter, Ω.

$$Y_{G(hopf\_critical)} = Y_{G(falaco\_critical)} = -\{3DC(x^2 + y^2) + \alpha^2 P^2(\cosh(\alpha z))^2\} \Rightarrow 0. \tag{128}$$

When the parameters DC have a product which is negative, then the critical surface is the catenoid – *A Minimal Surface*. That is the Hopf critical surface is an implicit surface of given by the equation,

$$(x^2 + y^2) = \{(\alpha^2 P^2)/(3|DC|)\}(\cosh(\alpha z))^2 \tag{129}$$

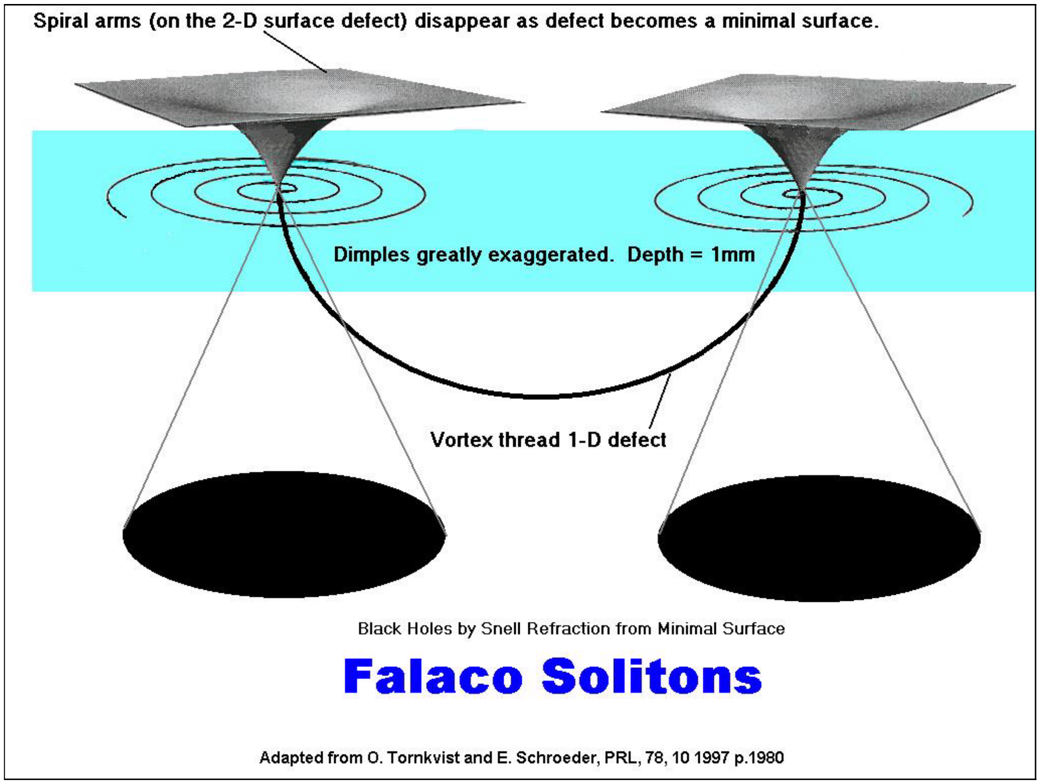
The throat diameter of the catenoid is proportional to the coefficient $\sqrt{(\alpha^2 P^2)/(3|DC|)}$.



4 Falaco Solitons as Landau Ginsburg structures in micro, macroscopic and cosmological systems

The Falaco experiments lead to the idea that such topological defects are available at all scales. The Falaco Solitons consist of spiral "vortex defect" structures (analogous to CGL theory) on a two dimensional minimal surface, one at each end of a 1-dimensional "vortex line" or thread (analogous to GPG theory). Remarkably the topological defect surface structure is locally unstable, as the surface is of negative Gauss curvature. Yet the pair of locally unstable 2-D surfaces is *globally* stabilized by the 1-D line defect attached to the "vertex" points of the minimal surfaces.

For some specific physical systems it can be demonstrated that period (circulation) integrals of the 1-form of Action potentials, A, lead to the concept of "vortex defect lines". The idea is extendable to "twisted vortex defect lines" in three dimensions. The "twisted vortex defects" become the spiral vortices of a Complex Ginsburg Landau (CGL) theory , while the "untwisted vortex lines" become the defects of Ginzburg-Pitaevskii-Gross (GPG) theory [Tornkvist 1997].



In the macroscopic domain, the experiments visually indicate "almost flat" spiral arm structures during the formative stages of the Falaco solitons. In the cosmological domain, it is suggested that these universal topological defects represent the ubiquitous "almost flat" spiral arm galaxies. Based on the experimental creation of Falaco Solitons in a swimming pool, it has been conjectured that M31 and the Milky Way galaxies could be connected by a topological defect thread [RMK 1986]. Only recently has photographic evidence appeared suggesting that galaxies may be connected by strings.

COSMIC STRINGS FROM HUBBLE

At the other extreme, the rotational minimal surfaces of negative Gauss curvature which form the two endcaps of the Falaco soliton, like quarks, apparently are confined by the string. If the string (whose "tension" induces global stability of the unstable endcaps) is severed, the endcaps (like unconfined quarks in the elementary particle domain) disappear (in a non-diffusive manner). In the microscopic electromagnetic domain, the Falaco soliton structure offers an alternate, topological, pairing mechanism on a Fermi surface, that could serve as an alternate to the Cooper pairing in superconductors.

It is extraordinary, but the Falaco Solitons appear to be macroscopic realizations of the Wheeler wormhole. This structure was presented early on by Wheeler (1955), but was considered to be unattainable in a practical sense. To quote Zeldovich p. 126 [Zeldovich 1996]

> "The throat or "wormhole" (in a Kruskal metric) as Wheeler calls it, connects regions of the same physical space which are extremely remote from each other. (Zeldovich then gives a sketch that topologically is equivalent to the Falaco Soliton). Such a topology implies the existence of 'truly geometrodynamic objects' which are unknown to physics. Wheeler suggests that such objects have a bearing on the nature of elementary particles and anti particles and the relationships

> between them. However, this idea has not yet borne fruit; and there are no macroscopic"geometrodynamic objects" in nature that we know of. Thus we shall not consider such a possibility."

This quotation dates back to the period 1967-1971. Now the experimental evidence justifies (again) Wheeler's intuition.

5 A Cosmological Conjecture

The objective of this section is to examine topological aspects and defects of thermodynamic physical systems and their possible continuous topological evolution, creation, and destruction on a cosmological scale. The creation and evolution of stars and galaxies will be interpreted herein in terms of the creation of topological defects and evolutionary phase changes in a very dilute turbulent, non-equilibrium, thermodynamic system of maximal Pfaff topological dimension 4. The cosmology so constructed is opposite in viewpoint to those efforts which hope to describe the universe in terms of properties inherent in the quantum world of Bose-Einstein condensates, super conductors, and superfluids [Volovik 2003]. Both approaches utilize the ideas of topological defects, but thermodynamically the approaches are opposite in the sense that the quantum method involves, essentially, equilibrium systems, while the approach presented herein is based upon non-equilibrium systems. Based upon the single assumption that the universe is a non-equilibrium thermodynamic system of Pfaff topological dimension 4 leads to a cosmology where the universe, at present, can be approximated in terms of the non-equilibrium states of a very dilute van der Waals gas near its critical point. The stars and the galaxies are the topological defects and coherent - but not equilibrium - structures of Pfaff topological dimension 3 in this non-equilibrium system of Pfaff topological dimension 4. The topological theory of the ubiquitous van der Waals gas leads to the concepts of negative pressure, string tension, and a Higgs potential as natural consequences of a topological point of view applied to thermodynamics. Perhaps of more importance is the fact that these concepts do not depend explicitly upon the geometric constraints of metric or connection, and yield a different perspective on the concept of gravity.

The original motivation for this conjecture is based on the classical theory of correlations of fluctuations presented in the Landau-Lifshitz volume on statistical mechanics [Landau 1958]. However, the methods used herein are not statistical, not quantum mechanical, and instead are based on Cartan's methods of exterior differential forms and their application to the topology of thermodynamic systems and their continuous topological evolution [RMK1991 b]. Landau and Lifshitz emphasized that real thermodynamic substances, near the thermodynamic critical point, exhibit extraordinary large fluctuations of density and entropy. In fact, these authors demonstrate that for an almost perfect gas near the critical point, the correlations of the fluctuations can be interpreted as a $1/\mathrm{r}$ potential giving a $1/\mathrm{r}^2$ force law of attraction. Hence, as a cosmological model, the almost perfect gas - such as a very dilute van der Waals gas - near the critical point yields a reason for both the granularity of the night sky and for the $1/\mathrm{r}^2$ force law ascribed to gravitational forces between for massive aggregates.

A topological (and non statistical) thermodynamic approach can be used to demonstrate how a four dimensional variety can support a turbulent, non-equilibrium, physical system with universal properties that are homeomorphic (deformable) to a van der Waals gas [RMK2004 b]. The method leads to the necessary conditions required for the existence, creation or destruction of topological defect structures in such a non-equilibrium system. For those physical systems that admit description in terms of an exterior differential 1-form of Action potentials of maximal rank, a Jacobian matrix can be generated in terms of the partial derivatives of the coefficient functions that define the 1-form of Action. When expressed in terms of intrinsic variables, known as the similarity invariants, the Cayley-Hamilton 4 dimensional characteristic polynomial of the Jacobian matrix generates a universal phase equation. Certain topological defect structures can be put into correspondence with constraints placed upon those (curvature) similarity invariants generated by the Cayley-Hamilton 4 dimensional characteristic polynomial. These constraints define equivalence classes of topological properties.

The characteristic polynomial, or Phase function, can be viewed as representing a family of implicit hypersurfaces. The hypersurface has an envelope which, when constrained to a minimal hypersurface, is related to a swallowtail bifurcation set. The swallowtail defect structure is homeomorphic to the Gibbs surface of a van der Waals gas. Another possible defect structure corresponds to the implicit hypersurface surface defined by a zero determinant condition imposed upon the Jacobian matrix. On 4 dimensional variety (space-time) , this non-degenerate hypersurface constraint leads to a cubic polynomial that always can be put into correspondence with a set of non-equilibrium thermodynamic states whose kernel is a van der Waals gas. Hence this universal topological method for creating a low density turbulent non-equilibrium media leads to the setting examined statistically by Landau and Lifshitz in terms of classical fluctuations about the critical point.

The conjecture presented herein is that non-equilibrium topological defects in a non-equilibrium 4 dimensional medium represent the stars and galaxies, which are gravitationally attracted singularities (correlations of fluctuations of density fluctuations) of a real gas near its critical point. Note that the Cartan methods do not impose (*a priori.)* a constraint of a metric, connection, or gauge, but do utilize the topological properties associated with constraints placed on the similarity invariants of the universal phase function.

Based upon the single assumption that the universe is a non-equilibrium thermodynamic system of Pfaff topological dimension 4 leads to a cosmology where the universe, at present, can be approximated in terms of the non-equilibrium states of a very dilute van der Waals gas near its critical point. The stars and the galaxies are the topological defects and coherent (but not equilibrium) self-organizing structures of Pfaff topological dimension 3 formed by irreversible topological evolution in this non-equilibrium system of Pfaff topological dimension 4.

The turbulent non-equilibrium thermodynamic cosmology of a real gas near its critical point yields an explanation for:

1. The granularity of the night sky as exhibited by stars and galaxies.
2. The Newtonian law of gravitational attraction proportional to $1/\mathrm{r}^2$.
3. The expansion of the universe (4th order curvature effects).
4. The possibility of domains of negative pressure (explaining what has recently been called dark energy) due to a classical Higgs mechanism for aggregates below the critical temperature (3rd order curvature effects)
5. The possibility of domains where gravitational effects (2nd order Gauss curvature effects) appear to be related to entropy and temperature properties of the thermodynamic system.
6. The possibility of cohesion properties (explaining what has recently been called dark matter) due to string or surface tension (1st order Mean curvature effects)
7. Black Holes (generated by Petrov Type D solutions in gravitational theory [**?**]) are to be related to Minimal Surface solutions to the Universal thermodynamic 4th order Phase function.

6 Summary

As the Falaco phenomena appears to be the result of a topological defect, it follows that as a topological property of hydrodynamic evolution, it could appear in any density discontinuity, at any scale. This rotational pairing mechanism, as a topological phenomenon, is independent from size and shape, and could occur at both the microscopic and the cosmic scales. In fact, as mentioned above, during the formative stages of the Falaco Soliton pair, the decaying

Rankine vortices exhibit spiral arms easily visible as caustics emanating from the boundary of each vortex core. The observation is so striking that it leads to the conjecture: Can the nucleus of M31 be connected to the nucleus of our Milky way galaxy by a tubular cosmic thread? Can material be ejected from one galaxy to another along this comic thread? Can barred spirals be Spiral Arm galaxies at an early stage of formation - the bar being and exhibition of material circulating about the stabilizing thread? At smaller scales, the concept also permits the development of another mechanism for producing spin-pairing of electrons in the discontinuity of the Fermi surface, or in two dimensional charge distributions. Could this spin pairing mechanism, depending on transverse wave, not longitudinal wave, coupling be another mechanism for explaining superconductivity? As the defect is inherently 3-dimensional, it must be associated with a 3-form of Topological Torsion, A^dA, introduced by the author in 1976 [RMK1976] [RMK 1990] [RMK 1991 a] [RMK 1992 d], but now more commonly called a Chern Simons term. These ideas were exploited in an attempt to explain high TC superconductivity [RMK 1991 c]. To this author the importance of the Falaco Solitons is that they offer the first clean experimental evidence of topological defects taking place in a dynamical system. Moreover, the experiments are fascinating, easily replicated by anyone with access to a swimming pool, and stimulate thinking in almost everyone that observes them, no matter what his field of expertise. They certainly are among the most easily produced solitons.

7 Some History

Just after WW II, one of my first contacts at MIT was a Brazilian young man named Jose Haraldo Hiberu FALCAO. He was in metallurgy and I was in physics. We became close friends and roommates during the period 1946-1950. He spent much of his time chasing the girls and playing soccer for MIT. Now MIT is not known for its athletic achievements, and when one weekend Haraldo scored two goals - giving MIT one of its few wins (ever) - the sports section of one of the Boston papers, misspelled his name with the headline ~

"FALACO SCORES TWO GOALS - MIT WINS"

Frankly I do not remember the exact headline from more than 55 years ago, but one thing is sure: Haraldo was known as FALACO ever since.

Haraldo moved back to Brazil and our ways parted. I became interested in many things, the most pertinent to this story included topological defects and topological evolution in physical systems. In 1986 I thought it would be great fun to go to Rio to see my old college friend, and then go to Machu Pichu to watch Haley's comet go by. My son was an AA pilot, so as parents we got a free Airline Ticket ticket to Brazil. Haraldo had married into a very wealthy family and had constructed a superb house, that his wife had designed, hanging onto a cliff-side above Sao Coronado beach south of Rio. Haraldo had a white marble swimming pool next to the house fed by a pristine stream of clear water.

The morning after my wife and I arrived in Rio (Haraldo's chauffeur met us at the airport in a big limo) I got up, after sleeping a bit late, and went to the pool for a morning dip. Haraldo and his family had gone to work, and no one was about. I sat in the pool, wondering about the fortunes of life, and how Haraldo - who I had help tutor to get through MIT - was now so very wealthy, and here I was - just a poor university professor. I climbed out of the pool, and was met by two servants who had been waiting in the wings. One handed me a towel and a terry cloth robe, and the other poured coffee and set out some breakfast fruit, croissants, etc.

I put a lot of sugar into the strong Brazilian coffee, as Haraldo had taught me to do long ago, and was stirring the coffee when I turned toward the pool (about 5-10 minutes after climbing out of the pool). In the otherwise brilliant sunshine, two black disks (about 15 cm in diameter) with bright halo rings were slowing translating along the pool floor. The optics caught my attention. Is there something about the southern hemisphere that is different? Does the water go down the drain with a different rotation? What was the cause of these Black Discs?

I went over to the pool, jumped in to investigate what was going on, and Voila!!!, the black discs disappeared. I thought: Here was my first encounter of the third kind and I blew it.

I climbed out of the pool, again, and then noticed that a pair of Rankine vortices was formed as my hips left the water, and that these rotational surfaces of positive Gauss curvature, within a few seconds, decayed into a pair of rotational surfaces of negative Gauss curvature. Each of the ultimate rotational surfaces were as if someone had depressed slightly a rubber sheet with a pencil point forming a dimple. As the negative Gauss curvature surfaces stabilized, the optical black disks were formed on the bottom of the pool. The extraordinary thing was that the surface deformations, and the black spots, lasted for some 15 minutes !!!. They were obviously rotational solitons.

The rest is history, and is described on my website and several published articles in some detail. The first formal presentation was at the 1987 Austin Dynamic Days get together, where my presentation and photos cause quite a stir. The Black Discs were quickly determined to be just an artifact of Snell's law of refraction of the solar rays interacting with the dimpled surfaces of negative Gauss curvature. What was not at first apparent was that there is a circular "string" – a 1D topological defect – that connects the two 2D topological defects of negative Gauss curvature. The string extends from one dimple to the other, and is evident if you add a few drops of dye to the water near the rotation axis of one of the "dimples". Moreover, experimentation indicated that the long term soliton stability was due to the global effect of the "string" connecting the two dimpled rotational surfaces. If the arc is sharply severed, the dimples do not "ooze" away, as you would expect from a diffusive process; instead they disappear quite abruptly. It startled me to realize that the Falaco Solitons have the confinement properties (and problems) of two quarks on the end of a string.

I called the objects FALACO SOLITONS, for they came to my attention in Haraldo's pool in Rio. Haraldo will get his place in history. I knew that finally I had found a visual, easily reproduced, experiment that could be used to show people the importance and utility of Topological Defects in the physical sciences, and could be used to promote my ideas of Continuous Topological Evolution.

The observations were highly motivating. The experimental observation of the Falaco Solitons greatly stimulated me to continue research in applied topology, involving topological defects, and the topological evolution of such defects which can be associated with phase changes and thermodynamically irreversible and turbulent phenomena. When colleagues in the physical and engineering sciences would ask "What is a topological defect?" it was possible for me to point to something that they could replicate and understand visually at a macroscopic level.

The topological ideas have led ultimately to

1. A non-statistical method of describing processes that are thermodynamically irreversible.

2. Applications of Topological Spin and Topological Torsion in classical and quantum field theories.

3. Another way of forming Fermion pairs

4. A suggestion that spiral galaxies may be stabilized by a connecting "thread", and an explanation of the fact that stars in the far reaches of galactic spiral arms do not obey the Kepler formula.

5. A number of patentable ideas in fluids, electromagnetism, and chemistry.

More detail (with downloadable pdf files of almost all publications) may be found on the web site:

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The original observation was first described at a Dynamics Days conference in Austin, TX, [RMK 1986] and has been reported, as parts of other research, in various hydrodynamic publications, but it is apparent that these concepts have not penetrated into other areas of research. As the phenomena is a topological issue, and can happen at all scales, the Falaco Soliton should be a natural artifact of both the sub-atomic and the cosmological worlds. The reason d'etre for this short article is to bring the idea to the attention of other researchers who might find the concept interesting and stimulating to their own research

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