FLUCTUATIONS, PFAFFIAN SYSTEMS AND COHERENT STRUCTURES

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Abstract. The fact that structures are recognized by their boundaries and the topological coherence of their defects focuses attention on topological rather than geometrical features of complex pattern forming systems. By identifying fluctuations as the compliment of a Pfaffian system, and by using Cartan's method of exterior differential (Pfaffian) forms to develop a theory of topological evolution, methods of differential topology may be used to find limit sets and boundaries of structures whose very existence in dynamical systems is generated by the fluctuations. This approach can be related to a variational principle subjected to non-integrable Pfaffian constraints, or fluctuations. By restricting the analysis to self-similar homogeneous integrands, mod fluctuations, the robust limit sets that define boundaries of structures, and which may contain a chaotic, fluctuating or turbulent interior dynamics, can be determined using the methods of Chern and Finsler.

1. Introduction

The recognition that a turbulent, irreversible, dissipative fluid can generate large scale structures has been the theme of several recent scientific conferences. [Kiehn 1991a]. The existence of such structures was not anticipated intuitively. At first glance, the result seems paradoxical, for according to "conventional dogma", a random system should not self-organize. Prigogine's thesis [Prigogine,1982] pointed a way to an alternate conclusion for non-linear synergetic systems. However, the details of how such structures are formed with finite robust lifetimes is not fully understood.

Recently it has been noted [Kiehn, 1992b] that the creation of persistent wake instability patterns in dissipative hydrodynamic flows can be put into correspondence with the creation of a class of topological limit sets. Rather than asking the deterministic question, "Given initial data, what is the UNIQUE outcome?", the alternate question can be asked: "Upon what domain do the equations of evolution permit non-unique or discontinuous solutions"? This method led to the discovery of closed form solutions [Kiehn 1992b,c] for those limits sets that form the classic Kelvin-Helmholtz and Rayleigh-Taylor instability patterns. The patterns, or structures, so formed are interpreted as the limit sets of instabilities induced by fluctuations.

Figure 1a. The pattern of the Kelvin-Helmholtz instability. $Q(s) = 1/\cos^2(s)$.

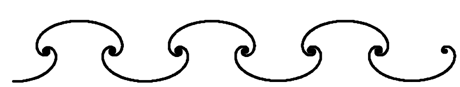

Figure 1b. The pattern of the Rayleigh-Taylor instability. $Q(s) = \tan(s)/\cos(s)$.

Formally, these plane curves can be generated by the equation of a complex unit tangent vector, **t**,

$$d\mathbf{t}/ds = \exp(iQ(s)) .$$

The choice of a phase factor, $Q(s) = 1/\cos^2(s)$, leads to the Kelvin-Helmholtz instability pattern of Figure 1a. Similarly, the Rayleigh-Taylor instability of Figure 1b is generated by the choice, $Q(s) = \tan(s)/\cos(s)$.

The argument was made [Kiehn 1992c] that these ever present patterns in hydrodynamic wakes, perhaps deformed, but always recognizable, are limit sets generated by surfaces of characteristics for a system of partial differential equations of evolution. In particular, these wake patterns, as limit sets, define domains of tangential discontinuities. However it is known that all such surfaces of tangential discontinuities [Landau, 1964] are locally unstable (as are all characteristic surfaces of negative Gaussian curvature). When subjected to the smallest of fluctuations, according to conventional wisdom, these surfaces should not persist. However, the conventional conclusion must be tempered, for there exists a globally stabilized subset of surfaces of negative Gaussian curvature that have finite robust lifetimes. The representatives of such robust surfaces are the minimal surfaces created by soap films built on finite boundaries. Recall that the theory of minimal surfaces historically was one of the first examples of a physical "field" theory based upon a variational principle [Nitsche,1990].

The concept to be studied in this article is the idea that, in locally unstable (or fluctuating) dynamical systems, the existence of chaos and fluctuations can drive the system to globally stabilized, persistently robust structures. The structures are observationally recognized by their boundaries of limit sets which can confine (like a Julia set) an interior with a chaotic, fluctuating, or even turbulent, dynamics. These topological boundaries in hydrodynamic wakes are to be considered as the equivalent of minimal surface soap films. This idea that a chaotic fluctuating system can be related to a variational principle is given further credence by the recognition that a minimal surface is generated by every holomorphic function in four dimensions [Nitsche, 1990]. Hence a sequence of functional iterates of holomorphic functions generates a sequence of minimal surfaces. However this is precisely the procedure used to generate the Julia set [Peitgen, 1988], which acts as a fractal boundary repellor in a dynamical system, separating those trajectories which wander off to an exterior infinity from those trajectories confined to an interior domain. Visually, the fractal boundary (consider a cloud) is the epitome of a "fluctuating" system. As extraordinary as it may seem, the conclusion is reached that fractal fluctuating boundaries, minimal surfaces, and coherent patterns in noisy fluctuating systems are all related to a variational principle with constraints.

The constraints of a variational principle are often written as a null set of Pfaffian forms, usually defined as a Pfaffian system. However, such a null set is typically non-integrable. Hence, if solutions are available to the Pfaffian system, they are not necessarily unique. The compliment of the null set of a Pfaffian system (the non-null set) can be used to define geometrically what intuitively are called "fluctuations". For example, consider the usual hypotheses of kinematics in which it is assumed that there exists a 1-parameter group of functions, $\mathbf{v}(\mathbf{x},t)$, that satisfy the constraints:

$$\Delta\mathbf{x} = d\mathbf{x} - \mathbf{v}\, dt = 0.$$

This is a classic example of a Pfaffian system. The compliment of this Pfaffian system is the set such that

$$\Delta\mathbf{x} = d\mathbf{x} - \mathbf{v}\, dt \neq 0,$$

and indeed this is intuitively close to what is meant by a fluctuation. However there is an even more subtle situation, whereby in the neighborhood $\Delta\mathbf{x} = 0$, the question can be asked, does the system admit unique solubility, or is there a possibility that more than one solution exists such that the evolution would have the appearance of a discontinuous bifurcation between the various solutions. This latter concept of a fluctuation will be defined as a second order fluctuation, where the first species, $\Delta\mathbf{x} \neq 0$, will be called a first order fluctuation. The usual kinematic assumption is that "particle" dynamics is governed by such Pfaffian systems without fluctuations, of the first or second order. This kinematic assumption need not be satisfied by the solutions to a set of partial differential equations that describe the evolution of synergetic field systems.

The problem of second order fluctuations can be put into correspondence with the classic problem of when is a Pfaffian system completely integrable? If the system is integrable, then there are no fluctuations of the second order. A rather important result of Cartan is that the closure of a Pfaffian system is always integrable [Slebodzinsky, 1964]. A simple application of this idea is to the special case where the Pfaffian system, Σ, is closed, $d\Sigma = 0$, a case which according to Cartan's theorem is integrable. The implication of integrability leads to the conjecture that a frame of reference can be found in which the fluctuations are null. Cartan's idea of prolongation is to increase the dimensionality of the domain to include more and more variables, until ultimately the Pfaffian system is integrable. Then by choosing the "right" set of coordinates, the fluctuations or discontinuities, so apparent in the lower dimensional space, disappear. This idea will not be proved formally in this article, but the cases of closed, integrable and non-integrable Pfaffian systems will be studied and related to a topological theory of fluctuations.

The Homogeneous Variational Principle

This article is motivated by the fact that Cartan's theory of Action transforms the mechanics of an integral variational principle into an exterior differential procedure operating on sets of exterior differential p-forms. Indeed, if A is the integrand of a variational integral, then the search for those vector fields that annihilate the exterior derivative of A (e.g., those vector fields which are *extremal* and satisfy the equation $i(V)dA = 0$) is equivalent to finding solutions to the necessary equations for an extremum generated by variation of a Lagrange integral. *Associated* vector fields satisfy the "orthogonality" condition, $i(V)A = 0$. Vector fields that satisfy both conditions generate the characteristics of any given problem. AS will be developed below, extremal vector fields preserve the even dimensional topological features of the system, but not the odd dimensional topological features. Characteristics preserve both odd and even dimensional topological features, hence are homeomorphisms.

Perhaps surprisingly to some, the theory of differential forms, which at first glance might be conceived as only a local linear theory without global qualities, is in fact a mathematical vehicle that not only carries topological properties, but can also be used to assess topological change. In the standard integral form of the variational calculus, when a Riemannian quadratic form such as the line element, ds^2, is used as a variational integrand, then the concept of measure and geometrical length enters naturally. However, the choice of the integrand as a quadratic form is in itself a constraint on the allowed topologies. It is the thesis of this article, that such topological constraints based on quadratic invariants are too strong to describe all of nature's processes of evolution. Instead, the variational process will be applied to linear forms, or their duals, and their exterior derivatives, which makeup the Cartan exterior differential system.

This procedure is motivated by the known facts that when a projective geometry -- in which transformations necessarily do not preserve either size, or shape, (or parallelism) -- is constrained by an invariant quadratic form, the constrained geometry leads to a class of the euclidean similarities preserving geometric shape, and in the extreme, geometric size. On the other hand, when the projective geometry is constrained, not by a *quadratic* form, but by a contravaraint *linear* form, a deformable, (affine) space is produced, with an associated group of transformations which preserve parallelism but not necessarily the euclidean properties of size and shape. A price is to be paid for such a constraint, for the concept of a rational net of harmonically generated points [Meserve, 1978], always permissible in a projective space, may not be globally preserved by the group of affine transformations. Recall that a harmonic net of rationality is the foundation of what physically is called a lattice. A defect, in crystalline physics called a dislocation, may appear in a quantized, discontinuous fashion. This lattice defect is a topological obstruction that destroys the global simple connectivity of a lattice without defects, such as a tiled euclidean domain. The dislocation defect is often recognized by the appearance of a line that terminates within the interior of the domain in an anomalous manner, like a zebra stripe. Creation and destruction of such topological defects will lead to an appearance of fluctuations.

However, though little used in the applied sciences, it should be noted that there is both a covariant and a contravariant affine space, each of which is associated with a *linear* form or its *dual*. More will be said about the dual to the affine invariant linear form, later, but at this point note that the constraint of a latent contravariant invariant preserves the concept of parallelism with its associated defects of translational shears, called dislocations. On the other hand, the covariant linear form is to be associated with the concept of rotational defects, called disclinations. The two species of defects point out that there can be two species of fluctuations (amplitude and phase), and these concepts may be distinct an unrelated on non-compact or non-orientable domains, which are subjected to irreversible transformations.

The important point at this stage is to realize that these concepts of defects and defect fluctuations are not dependent upon a metric. They are not geometrical ideas built on sizes and shapes. The concepts can be related to the cohomology of the domain. This topological idea is based on the fact that a system of exterior differential forms can contain a certain number of *harmonic* p-forms. For each defect (or hole) there will exist one distinct harmonic p-form. These defects are topological invariants in that they are invariants of homeomorphisms, but they need not be invariants of continuous transformations which are irreversible. As will become apparent, it is the *harmonic* components of the p-forms which will attract the most attention, for like a hole, they are hard to miss, and they are rational. There can not be such a thing as 0.00034591 of a hole. The hole might have a metric size that is minute, but it is still one (1) hole.

It is remarkable, that a course topology (called herein the Cartan Topology) for any given domain may be determined merely by the specification of a global 1-form of action, A, its exterior derivatives and exterior products. This course topology can be refined by adjoining to the 1-form of Action a system of non-integrable Pfaffian constraints. The totality of forms and their exterior derivatives forms an exterior differential system, Σ. A fundamental theoretical axiom for this article is that a system of exterior differential forms and differential constraints may be used to define a topology on a domain. The fundamental intuitive idea is that the space-time domain of measurable science, with its defects and discontinuities, may be embedded in a space of higher dimensions in which the discontinuities disappear. When the smooth results of the embedding space are pulled back to the base space, the result can take on the appearance of a set with discontinuities that may have the appearance of Brownian fluctuations.

For simplicity, the exterior differential system considered in this article will be a single 1-form of action, A, constructed in part from a LaGrange function, $L(\mathbf{x},\mathbf{v},t)$, and subject to certain non-integrable Pfaffian constraints. The complete 1-form will be created by use of a procedure classically called the method of LaGrange multipliers. The resultant exterior differential system does *not* generate, necessarily, a Riemannian geometry. A point of departure in the method is to recognize that the usual kinematic statement, $\Delta\mathbf{x} = d\mathbf{x} - \mathbf{v}dt = 0$, when interpreted as a Pfaffian system of constraints, is indeed a *topological* constraint on a domain, that need not be true for the evolution of deformable media. The non-zero values of $\Delta\mathbf{x}$ will be defined as *deformation-fluctuations* of position. Higher order fluctuations also may be defined. The base space for the system to be studied in this article will be a space time variety of dimension 4. This system will be prolonged to higher dimensions to accommodate the possible fluctuations. Then, following Cartan's method, which is equivalent to a variational principle, certain topological constraints, not equivalent to the null fluctuation constraints of rigid body dynamics, can be placed on this higher dimensional space. Finally, these results can be pulled back to space-time by functional substitution. An example of this technique will be used to demonstrate that the Navier-Stokes equations follow from such a procedure. For certain classes of solutions, the resultant geometry is non-Riemannian, but contains many of the features developed by Finsler [Chern 1948] and Cartan for spaces that support torsion, and non-integrable vector fields.

The Cartan Topology

Suppose that such a 1-form, A, is given. Then, the exterior derivative of A produces a 2-form, F = dA, with components,

$$F = dA = F_{\mu\nu}dx^{\mu}dx^{\nu}$$

The 1-form, A, and the 2-form, F=dA, form a "closed" exterior differential system, {A, F}. Using these two elements, a sequence of higher order sets can be constructed algebraically by forming all possible exterior products of A and F: The resulting system of forms is defined as the Pfaff sequence: {A, F=dA, H=A^dA, K=dA^dA...}. The union of all elements of the Pfaff sequence and their closures forms the elements of the Cartan topological base: $\{ A, A \cup F, H, H \cup K \}$.

On space-time of 4 dimensions, there are only 4 possibilities for the Pfaff sequence, and these sets are defined as:

TOPOLOGICAL ACTION

$$A = A_{\mu}dx^{\mu}$$

TOPOLOGICAL TORSION

$$F = dA = F_{\mu\nu}dx^{\mu}dx^{\nu}$$

TOPOLOGICAL TORSION

$$H = A\hat{}dA = H_{\mu\nu\rho}dx^{\mu}dx^{\nu}dx^{\rho}$$

TOPOLOGICAL PARITY

$$K = dA\hat{}dA= K_{\mu\nu\rho\sigma}dx^{\mu}dx^{\nu}dx^{\rho}dx^{\sigma}.$$

The largest non-null element of the Pfaff sequence defines the Pfaff dimension of the domain. Certain flows have Pfaff dimension 1; others, Pfaff dimension 2, 3, and 4. Examples are given in Kiehn (1990).

The domain of support of each element of the Pfaff sequence may be considered as a "point" of what will be called the Cartan topology. In this sense the "point", A, and its closure, $A\cup F$ may be used as base elements to define "open sets" of the Cartan topology. If the domain is such that H and K are null, then that is the end of the construction, and the Cartan topology is a connected topology. However, suppose that there exists a domain of support for the exterior product of A and F, such that H is not empty. Then construct the closure of H as the union, $H\cup K$, and use these 4 "points", $\{A, A\cup F, H, H\cup K\}$, as a basis of open sets for the Cartan topology. The resulting topology is NOT connected! [Baldwin, 1991] This choice for a topology is extraordinary in that the Cartan exterior derivative may be interpreted as a "limit point operator" relative to the Cartan topology. Given any set of the topology, if the exterior derivative vanishes, then the set has no limit points. For currents constructed in terms of contravariant vector fields (or N-1 forms), then the vanishing of the exterior derivative implies that no limit points exist within the domain of support. If true, then the associated axial vector current satisfies a conservation law, equivalent to the vanishing of an N dimensional divergence. The "lines" associated with this vector field do not stop or start within the domain.

In this article, the technique will be to define a 1-form of action, A, similar to that variational integrand proposed by Finsler [Chern 1948] and Cartan in their studies of non-Riemannian geometries that admit torsion. The Finsler methods motivated this author to treat the general problem of topological evolution as an extremal problem on a variety of higher dimensions. In effect, the classic kinematic constraints of rigid body motion are considered to be "overly severe" topological constraints on fluid motion, and are relaxed such that the new topology admits "fluctuations". The fluctuations can be interpreted in many ways; in particular they can represent deviations from the classic kinematic constraints, $d\mathbf{x}-\mathbf{v}dt = 0$. These deviations may be interpreted as variations of the initial conditions, or they may be interpreted as an uncertainty of the origin, but for this author they do not imply that a *statistical* analysis is necessary. A refinement, or specialization, of this "fluctuation" topology leads to the set of necessary partial differential equations which for the given example are recognized to be the Navier-Stokes equations for a compressible, viscous flow. Of particular importance is the recognition that the dynamics of a deformable system are to be associated with a field of axial "current", whose 4 components form the 3-form of Topological Torsion on (x,y,z,t). This axial-vector current is a completely anti-symmetric third rank tensor field, and is evanescent in rigid body systems that satisfy the topological constraints of a perfect kinematics. The divergence of this axial current may or may not vanish for deformable systems, and it may or may not be an evolutionary invariant. When the divergence is anomalous, the Torsion Current can stop or start in the interior, thereby generating a topological defect in the domain. Almost no attention has been given to this Torsion Current by researchers in hydrodynamics, but the relationship of this covariant tensor field to the theory of *non*-integrability will demonstrate its importance to systems with defects and fluctuations.

In this article a specific class of variational principles with non-integrable constraints will be studied. The class associated with integrands which are homogeneous of degree one is singled out as special Not only is this the class that generates non-Riemannian or Finsler spaces, but also it is the class that generates minimal surfaces, and, perhaps not too surprisingly, the special theory of relativity. This special class of homogeneous integrands of degree one is directly related to the theory of projective geometries, where a degree of self-similarity (a pattern generator) is evident immediately, for in projective geometries, vectors $\mathbf{v}$ and $\lambda\mathbf{v}$ are considered to be equivalent. "Rays" - not magnitudes - are of importance to projective geometries. It also is of interest to note that in projective geometries, the existence of duality implies that there are two species of fundamental objects. In projective 2-space, these dual sets are rays (lines) and surfaces, in projective 3-space, the rays physically can be associated with currents, while their duals are "hyper-surfaces" of Action. More explicitly, the rays (or currents) are exterior 3-forms, J, and the dual hyper-surfaces are 1-forms, A.

2. The Cartan Action as a Fluctuation 1-form

Global, or topological, features of limit sets and boundaries are the important attributes of patterns, hence in this article the emphasis is on the topological perspective of dynamical synergetic systems. It is known that a exterior differential system, Σ, and its closure formed by adjoining to the original system its exterior derivatives, $d\Sigma$, defines a topology on the domain. Herein, the topology of interest will be that defined by a single 1-form, the Cartan1-form of Action.

The format of the Cartan 1-form, A, studied in this article will be taken to be that of the Cartan-Hilbert invariant integrand,

$$A = L(\mathbf{x},t;\mathbf{v})dt + \mathbf{p} \bullet (d\mathbf{x} - \mathbf{v}dt). \tag{1}$$

The objective is to study the topology induced by such a form, without imposing a priori constraints that correspond to "no fluctuations". Note that the original space-time, $\{\mathbf{x}, t\}$ has been extended or prolonged to a 10 dimensional space of functions, $\{\mathbf{x}, t; \mathbf{v}, \mathbf{p}\}$. Differential and functional constraints will be imposed on this 10 dimension space thereby refining the topology. Such constraints will add physical significance to the functions, $\{\mathbf{x}, t; \mathbf{v}, \mathbf{p}\}$, which at these point are arbitrary variables. .Further note that formally the Cartan-Hilbert action given by (1) involves a classic Lagrange function, $L(\mathbf{x},t;\mathbf{v})$, and a linear combination of non-zero position "fluctuation or deformation" 1-forms, $\Delta\mathbf{x}$, defined as:

$$\Delta\mathbf{x} = d\mathbf{x} - \mathbf{v}dt \neq 0 . \tag{2}$$

The usual hypothesis of classical physics is that of kinematic perfection. That is, the "velocity" functions, $\mathbf{v}$, admit a 1 parameter group of integrable solutions such that

$$\Delta\mathbf{x} = d\mathbf{x} - \mathbf{v}dt = 0.$$

It is not generally appreciated that this kinematic assumption of point particle mechanics is a topological constraint on the domain that need not be true for the evolution of synergetic systems, such as a turbulent fluid. It is possible that there exist vector field solutions to, say, the Navier-Stokes equations, that do not admit a 1 parameter group.

The covariant array, $\mathbf{p}$, of coefficients of the fluctuation 1-forms in (1) may be described as set of Lagrange multipliers. It will be demonstrated that this covariant field, $\mathbf{p}$, dual to the contravariant velocity field, $\mathbf{v}$, plays the role of the canonical momentum, when the system is subjected to those additional, but classic, constraints that are equivalent to the constraint of zero temperature. It follows that the concept of temperature can be given meaning in terms of the non-null fluctuations associated with the difference between, the Lagrange multipliers, $\mathbf{p}$, and the canonical momentum, $\partial L/\partial\mathbf{v}$.

The 1-forms, $\Delta\mathbf{x}$, are defined as fluctuation-deformation 1-forms for they represent deviations from the pure kinematic point of view associated with a rigid body dynamics or the evolution of a point particle in terms of a single parameter group of transformations. Although not always true, these deviations are often small corrections to the kinematic constraints, $\Delta\mathbf{x} = 0$, and have the appearance of "fluctuations" about the "kinematic" lines that act as guiding centers for the evolution. The idea is that $\Delta\mathbf{x} \approx 0$, but $d(\Delta\mathbf{x}) \neq 0$.

The fluctuation 1-forms given by (2) are not necessarily zero even for classical deformable media. For example consider a flowing fluid in the Lagrangian representation. Assumed that the "points", **x**, evolve in terms of a map, f, from a set of initial conditions, **y**, such that the map

$$x^k = f^k(\mathbf{y},t),$$

describes the evolution of the fluid. Then, it is apparent that the fluctuation, $\Delta\mathbf{x}$, is not zero, and can be explicitly written as:

$$\Delta x^k = dx^k - v^k dt = (\partial f_k/\partial y^i) dy^i \neq 0 \ . \tag{2b}$$

If the parameters, **y**, which could be interpreted either as initial conditions or as the coordinates of the origin, are not constants, then the RHS of (2b) is not zero, and the system does not evolve according to the kinematic rules associated with a single parameter group. The basic idea is that the statement, d**x** - **v**dt = 0, must be interpreted as a topological constraint, just as the statement xdx+ydy+zdz = 0 is a topological constraint on Euclidean 3-space that produces the topology of a spherical surface.

In classical hydrodynamics, the non-zero "fluctuations" given by (2) are usually constrained by topological conditions such that even though the individual values of Δx^k are not zero identically, their associated 3-form exterior product admits an integrating factor, $\rho = \rho(x,y,z,t)$. In other words, the non-zero 3-form,

$$\Omega = \rho(dx-v^x dt)\wedge(dy-v^y dt)\wedge(dz-v^z dt) \ , \tag{3}$$

is presumed to have a vanishing exterior derivative,

$$d\Omega = \{ \operatorname{div} \rho\mathbf{v} + \partial\rho/\partial t \} \, dx\wedge dy\wedge dz\wedge dt = 0, \tag{4}$$

and is therefore integrable. This topological constraint represented by the vanishing of the partial differential equation in the brackets is usually called the "equation of continuity" for deformable media. Such a constraint makes Ω an absolute invariant of the evolution. If the flow lines are retraceable, implying that the Jacobian determinant of the assumed mapping is of rank 3, then the topological constraint may be interpreted as the "conservation of mass". However, it is not apparent that nature always insists on the such a topological constraint among the fluctuation-deformation 1-forms. Such a constraint is a matter for test, especially in the case of a turbulent, irreversible, evolutionary process.

The Cartan 1-form will be used not only to generate the Cartan topology, but also to generate, by means of a procedure equivalent to a variational principle, a set of partial differential equations of evolution with solution vector fields, **V**. These vector fields, **V**, will have smooth curves tangent to the **V** in the higher dimensional or prolonged geometry necessary to admit fluctuations. When these smooth curves are pulled back, or projected to the lower dimensional geometry, they may have a discontinuous appearance of fluctuations. The ordinary kinematic differential equations based on **v**, not **V**, yield solutions that act as "guiding centers" for the "fluctuation" fields, **V**, in the limit that the fluctuations are small. The projections of the continuous curves in the geometry of the higher dimensional space may have gaps and tangential discontinuities on space-time. The discontinuities would be interpreted as defects or fluctuations in an otherwise homogeneous and continuous system. These ideas may be compared to the concept of Poincare sections in the theory of non-linear dynamics. The Cartan method permits the concepts of discontinuous fluctuations to be put on a continuous basis in a space of higher dimension. This topological idea is similar to the geometric idea where a curved space may be embedded in a higher dimensional euclidean flat space.

Physicists often recognize the Cartan Action in the format,

$$A = \mathbf{p} \bullet d\mathbf{x} - (\mathbf{p} \bullet \mathbf{v} - L(\mathbf{x},t;\mathbf{v}))dt = \mathbf{p} \bullet d\mathbf{x} - H(\mathbf{x},t;\mathbf{p},\mathbf{v})dt \ , \tag{5}$$

but do not seem to appreciate that this composition may be interpreted in terms of a fluctuation geometry on a space of 10 dimensions, as given by (1). In current physical theories, it is often assumed that the function, $H(\mathbf{x},t;\mathbf{p},\mathbf{v})$, can be written entirely in terms of the variables $(\mathbf{x},t;\mathbf{p})$ alone. In such cases the function $H^*(\mathbf{x},t;\mathbf{p})$ becomes the Hamiltonian function of classical mechanics, and the Lagrange multipliers, **p**, would be identified as the canonical momentum. This assumption corresponds to a functional relationship or constraint between the variables **v** and **p** such that $\partial H/\partial \mathbf{v} = 0$. If the relationship is linear, then there would exist a constitutive or metrical relationship between the dual fields, **v** and **p**. Such assumptions are NOT made *a priori* in this article.

Consider first a Cartan 1-form of action where the fluctuations, $\Delta\mathbf{x}$, are assumed to be identically zero over the domain. Then by taking the exterior derivative of (1), the 2-form of limit points becomes $F = dA = dL\hat{}dt$. It follows that $H = A\hat{}dA = 0$, and $K = 0$. The Pfaff dimension of such systems is 2 at most, and the system is completely integrable in the sense of Frobenious. Such systems in a fluid sense can have vorticity but are without helicity, or Topological Torsion. Examples of systems that do support Topological Torsion are presented in reference [Kiehn 1991a]. In this article, systems of non-zero H and non-zero K are of interest. From this point of view, both Topological Torsion and Topological Parity are to be associated with the concept of non-null kinematic fluctuations which are not transversal to the system momentum, $\mathbf{p} \bullet \Delta\mathbf{x} \neq 0$.

When fluctuations are permitted, $\Delta\mathbf{x} \neq 0$, then the exterior derivative of the Cartan action on the 10 dimensional space becomes explicitly,

$$dA = (\partial L/\partial \mathbf{v} - \mathbf{p}) \bullet d\mathbf{v}\hat{}dt + d\mathbf{p} \bullet (d\mathbf{x} - \mathbf{v}dt) + \partial L/\partial \mathbf{x} \bullet (d\mathbf{x}\hat{}dt)$$

$$= (\partial L/\partial \mathbf{v} - \mathbf{p}) \bullet \Delta\mathbf{v}\hat{}dt + \Delta\mathbf{p} \wedge \Delta\mathbf{x} \tag{6}$$

Continuing with the objective to study the Cartan topology, without a priori constraints, note that the term, $\Delta\mathbf{v}$, represents the non-zero 1-forms of velocity fluctuations, defined as,

$$\Delta\mathbf{v} = d\mathbf{v} - \mathbf{a}dt \neq 0 , \tag{7}$$

and, $\Delta\mathbf{p}$, represents the non-zero 1-form of Lagrange multiplier fluctuations,

$$\Delta\mathbf{p} = d\mathbf{p} - (\partial L/\partial\mathbf{x})\, dt \neq 0. \tag{8}$$

The functions, **a,** are defined to be to the contravariant acceleration vector field (with velocity fluctuations) in the same extremal sense that **v** is defined as the contravariant velocity vector field (with position fluctuations).

The notation, $(\Delta\mathbf{p} \wedge \Delta\mathbf{x})$, stands for the sum of 2-forms,

$$(\Delta\mathbf{p} \wedge \Delta\mathbf{x}) = \sum_k (dp_k - \{\partial L / \partial x^k\}dt)\wedge(dx^k - v^k dt) \tag{9}$$

which is similar to the dot product of two vectors, but here the combinatorial action is through the exterior product, ^. Although closely related to an expectation value generated by an inner product, or to the integrand of a cross-correlation integral, no statistical or ensemble averaging of (9) is assumed in this article. The beauty of the Cartan analysis is that it is retrodictively deterministic and well defined in a pullback sense, even when unique, deterministic prediction is impossible [Kiehn,1976b].

The bracket factor, $(\partial L/\partial\mathbf{v} - \mathbf{p}) = - \partial H/\partial\mathbf{v}$ will be defined as the scaled covariant vector field, $\mathbf{k}/S$. The topological constraint $\mathbf{k} = 0$ permits the Lagrange multipliers to be uniquely determined as the canonical momenta of classical mechanics, $\mathbf{p} = \partial L/\partial\mathbf{v}$.

A direct computation of the Topological Torsion, H, on the 10 dimensional space yields,

$$H = A\wedge dA = L\, dt\wedge (\Delta\mathbf{p} \wedge \Delta\mathbf{x}) + (\mathbf{k}/S \bullet \Delta\mathbf{v})\wedge(\mathbf{p} \wedge \Delta\mathbf{x})\wedge dt. \tag{10}$$

which may be evaluated in principle on 4 dimensional space time by functional substitution. A similar direct computation in the higher dimensional geometry of variables {x, t; v, p} of the exterior derivative, K = dH, produces a 4-form that also can be pulled back to {x,y,z,t} by functional substitution. The Topological Parity 4-form becomes,

$$K = dA\wedge dA = 2 \{(\partial L/\partial\mathbf{v} - \mathbf{p}) \bullet \Delta\mathbf{v} \}\wedge (\Delta\mathbf{p} \wedge \Delta\mathbf{x}) + 2 (\Delta\mathbf{p} \wedge \Delta\mathbf{x})\wedge (\Delta\mathbf{p} \wedge \Delta\mathbf{x}). \tag{11}$$

On a space-time variety, this 4-form becomes Chern's top Pfaffian whose integral gives information concerning the Euler characteristic of space-time. It is apparent that K depends on the triple exterior product of the fluctuations of position, Lagrange multipliers, and velocity, as well as the bracket factor, $(\partial L/\partial\mathbf{v} - \mathbf{p}) = - \partial H/\partial\mathbf{v}$, involving the Lagrange multipliers, and dt.

A first constraint on the Cartan system will be to consider the classic first variation of the Action integral, $\int A$, as an extremal principle in the sense of Finsler. The geometries so constructed are not necessarily Riemannian. According to Chern, the Finsler variation is equivalent to setting dA of equation (6) equal to zero, mod $\Delta\mathbf{x}$. In addition, for Finsler geometries, the Lagrange function is presumed to be homogeneous of degree 1 in the functions, **v**, and this constraint is used by Chern to construct what he calls a "projectivized" tangent bundle. The homogeneity condition implies that the variable, t, can be reparametrized, and the vector **v** forms the elements of a projective geometry. If it is further assumed that the velocity fluctuations do not vanish, $\Delta\mathbf{v} \neq 0$, then the only way for the Finsler variation to be valid is for the factor, $(\partial L/\partial\mathbf{v} - \mathbf{p})$, to vanish.

In classical field theory, this Finsler constraint is often imposed arbitrarily:

$$\mathbf{k}/S \approx (\partial L/\partial\mathbf{v} - \mathbf{p}) = - \partial H/\partial\mathbf{v} = 0. \tag{12}$$

As mentioned above, such a constraint uniquely defines the Lagrange multipliers, **p**, as the components of the canonical momentum. The Topological Parity 4-form (11) is then dependent on the exterior product of "fluctuations" in position and momentum only, and has the same physical dimensions as Planck's constant. The Euler characteristic of the constrained topology does not depend upon the velocity fluctuations. From a qualitative point of view, fluctuations in velocity correspond to the property of temperature. In this sense, the fluctuations of temperature are distinct from the fluctuations of spin.

4. Equations of Topological Evolution

From the dynamical point of view, the variation procedures that lead to integrability of the Action depend upon the choice of paths. Note that for a 1-form of action, A, to be closed, such that dA = 0, and hence A is integrable, makes no statement about paths. In fact, if dA = 0, then the system, {A, dA} is equivalent to {A} and as {A, dA} is completely integrable, then so is {A}, for all paths. This is the basis of exact systems. But a somewhat different idea is that dA may be zero when projected along certain paths. Then integrability is path dependent. The criteria will be given by expressions of the form i(**V**)dA = 0, where the objective, given the A, is to find the **V** and thereby the paths, for which the Extremal condition is true. The basic constraint i(**V**)dA = 0 leads to systems of partial differential equations of evolution.

In the Cartan method, an evolutionary process relative to a vector field, **V**, is described by the action of the Lie derivative on the p-forms of interest [Slebodzinsky, 1970]. The action of the Lie derivative on 1-forms is equivalent to the "convective" derivative in Cartesian hydrodynamics, but the Lie derivative is defined without the geometric constraints of a metric or a connection. In the present article, the p-forms of evolutionary interest are those p-forms that make up the Pfaff sequence. If any p-form is invariant with respect to the evolutionary process, **V**, then its Lie derivative vanishes: L(**V**)A = 0. If all p-forms that make up the topological base of the Cartan topology are invariant, then the topology is invariant, and the process generated by **V** is a homeomorphism: (L(**V**) A = 0 and L(**V**)dA =0). Such processes are both continuous and reversible, and are to be ignored in this article.

A more general set of evolutionary processes that can admit topological evolution is described by the topological constraint that the limit sets, dA=F, are evolutionary invariants, but no such constraint is placed on A. Such processes are described by the statements:

$$L(\mathbf{V})dA = 0, \quad L(\mathbf{V})A = Q \neq 0. \tag{13}$$

It may be demonstrated that such concepts are equivalent to Helmholtz theorem of the conservation of vorticity [Kiehn 1975, Tur 1991]. In terms of the Cartan topology , such processes are uniformly continuous. In engineering terms, such results correspond to the master equation for the "perfect plasma" of magneto-hydrodynamics, $\partial\mathbf{B}/\partial t = \mathrm{curl}(\mathbf{v} \times \mathbf{B})$. Direct computation on C^2 functions indicates that the statement is equivalent to the constraint that 1-form, $W = i(\mathbf{V})dA$, is closed (and therefore has no limit points with respect to the Cartan topology):

$$d(i(\mathbf{V})dA) = 0. \tag{14}$$

This constraint of uniform continuity is satisfied by the more stringent sufficient condition,

$$i(\mathbf{V})dA = 0 . \tag{15}$$

For a given Lagrange Action, A, Cartan has demonstrated that the first variation of the Action integral is equivalent to the search for those vector fields, **V**, that satisfy the equation (15) given above. Such vector fields are called extremal vector fields [Klein 1962, Kiehn 1975]. The resultant equations deduced from (15) are a set of partial differential equations that represent extremal evolution. Note that the extremal conditions are insensitive to any renormalization of the vectors, **V**. That is, if **V** satisfies the equations, then $\rho\mathbf{V}$ also satisfies the equations. Such a result is commonplace in the projective geometry of lines, and does not require the Riemannian or euclidean concept of an inner product or a metric [Meserve 1983]. Cartan [1958] has shown that this projective extremal condition is necessary and sufficient for the dynamical system, **V**, relative to the action, A, to be Hamiltonian [Kiehn 1974]. Such Hamiltonian systems are not dissipative, and guarantee the existence of a single parameter group for the velocity field generated by the solutions of the necessary partial differential equations that describe the topological constraint.

However, on the fluctuation space of 10 dimensions, $\{\mathbf{x}, t; \mathbf{v}, \mathbf{p}\}$, contraction of dA with the vector field, $\mathbf{V} = \{\mathbf{v}, 1; \mathbf{a}, \mathbf{f}\}$, yields explicitly,

$$i(\mathbf{V})dA = -\mathbf{F} \bullet (d\mathbf{x} - \mathbf{v}\,dt) - \mathbf{k}/S \bullet (d\mathbf{v} - \mathbf{a}\,dt) = -\mathbf{F} \bullet \Delta\mathbf{x} - \mathbf{k}/S \bullet \Delta\mathbf{v} , \tag{16}$$

where $\mathbf{F} = \mathbf{f} - \partial L/\partial\mathbf{x}$ represents the dissipative components of the "force". The RHS of (16) is not necessarily zero, so such evolutionary processes are not necessarily Hamiltonian. The RHS of (16) depends explicitly upon the fluctuations in position and velocity, and is explicitly independent from fluctuations in momentum. The two covariant vector fields, $\mathbf{F} = i(\mathbf{V})\Delta\mathbf{p}$, and $\mathbf{k}/S$, represent the irreversible dissipative mechanisms of friction and radiation in the system fluctuation dynamics. These dissipative terms are not included explicitly in the usual Lagrange theory, and represent fluctuation interactions with the environment. Note that the Finsler-Chern condition, $dA = 0 \mod \Delta\mathbf{x}$, implies that $\mathbf{k}/S = 0$, a constraint that on physical grounds implies that the system does not depend on velocity fluctuations (temperature). Further note that the Cartan-Hamilton constraint given by (15) is satisfied by eight distinct sets of sufficient conditions, depending on choices for the four factors in the RHS of (16).

5. The Navier Stokes equations

For the liquid state, guided by the physical idea that fluctuations in velocity are related to the concept of temperature, assume that the equations of topological evolution are refined by the constraints,

$$\mathbf{k} \neq 0\,,\ \ \mathbf{k} \bullet \Delta\mathbf{v} = S\, d(kT),\ \mathbf{F} = \upsilon \text{ curl curl } \mathbf{v}\,,\ \ \mathbf{F} \bullet \Delta\mathbf{x} \neq 0. \tag{17}$$

In this example, the constraint of canonical momentum ($\mathbf{k}/S = 0$) is relaxed to the alternate constraint that the deviation from canonical momentum, k/S, be transversal to the fluctuations in velocity, if the system is at constant temperature, then $d(kT) = 0$. The equations of motion (16) become

$$i(\mathbf{V})dA = -\ \upsilon(\text{ curl curl } \mathbf{v}) \bullet \Delta\mathbf{x} + d(kT) \tag{18}$$

For the case of a fluid, define the Cartan fluctuation 1-form, A, as,

$$A = \mathbf{v} \bullet d\mathbf{x} - H\, dt, \tag{19}$$

with the "barotropic Hamiltonian" function specified as,

$$H = \mathbf{v} \bullet \mathbf{v}/2 + \int dP/r - l\ div\mathbf{v} + kT \tag{20}$$

Substitution into (16) yields a necessary system of partial differential equations for the constrained topological evolution:

$$\partial\mathbf{v}/\partial t + \text{grad}(\mathbf{v} \bullet \mathbf{v}/2) - \mathbf{v} \times \text{curl}\mathbf{v}) = -\text{grad } P/\rho + \lambda \text{grad div } \mathbf{v} - \nu \text{ curl curl } \mathbf{v}\,, \tag{21}$$

Equations (21) are exactly the Navier-Stokes partial differential equations for the evolution of a compressible viscous irreversible flowing fluid. In other words, the Navier-Stokes equations of hydrodynamics have been deduced by imposing a set of topological constraints on the equations of topological evolution. However, the vector field which is a solution to the partial differential system may not be the generator of a single parameter group of transformations, and thereby may exhibit fluctuations and tangential discontinuities when projected to space time.

By direct computation, the 2-form $F = dA$ has components

$$F = dA = \boldsymbol{\omega}_z\, dx\wedge dy + \boldsymbol{\omega}_x\, dy\wedge dz + \boldsymbol{\omega}_y\, dz\wedge dx + (\mathbf{a} \bullet d\mathbf{r})\wedge dt$$

where

$$\boldsymbol{\omega} = \text{curl}\,\mathbf{v} \qquad \text{and} \qquad \mathbf{a} = -\partial\mathbf{v}/\partial t - \text{grad}\, H \tag{22}$$

The 3-form of Helicity or Topological Torsion, H, is constructed from the exterior product of A and dA as,

$$\begin{aligned} H = A\wedge dA &= H_{ijk}\, dx^i\wedge dx^j\wedge dx^k \\ &= \mathbf{T}_x\, dy\wedge dz\wedge dt + \mathbf{T}_y\, dz\wedge dx\wedge dt + \mathbf{T}_z\, dx\wedge dy\wedge dt + h\, dx\wedge dy\wedge dz, \end{aligned} \tag{23}$$

where **T** is the fluidic Torsion axial vector current,

$$\mathbf{T} = (\mathbf{a} \times \mathbf{v}) + H\,\boldsymbol{\omega} \quad , \tag{24}$$

and h is the torsion (helicity) density,

$$h = \mathbf{v} \bullet \boldsymbol{\omega}. \tag{25}$$

The torsion axial vector, **T**, consists of two parts. The first term represents the shear of translational accelerations, ($\mathbf{a} \times \mathbf{v}$), and the second part represents the shear of rotational accelerations, $H\,\boldsymbol{\omega}$. Tubular domains, or singular lines of **T**, in a hydrodynamic system can be interpreted as "defect" structures that can be put into correspondence with dislocation and disclination defects in continuous media, depending on whether or not the field is dominated by translational or rotational shears. The topological torsion tensor, H_{ijk} , is a third rank completely anti-symmetric covariant tensor field, with four components on the variety {x,y,z,t}.

The Navier-Stokes constraint given by (21) may be used to express the acceleration term, **a**, kinematically; i.e.,

$$\mathbf{a} = -\text{grad}\, H - \partial\mathbf{v}/\partial t = -\mathbf{v} \times \text{curl}\,\mathbf{v} + \upsilon\{\text{curl curl}\,\mathbf{v}\}. \tag{26}$$

By substituting (26) into (24), the torsion axial vector current becomes expressible in terms of the helicity density, h, the Lagrangian function, *L*, and the viscosity as:

$$\mathbf{T} = \{ h\,\mathbf{v} - L\,\text{curl}\,\mathbf{v} \} - \upsilon\,\{\mathbf{v} \times (\text{curl curl}\,\mathbf{v})\} \tag{27}$$

Note that the torsion axial vector current persists even for Euler flows, where $\upsilon = 0$. The measurement of the components of the Torsion vector have been completely ignored by experimentalists in hydrodynamics.

The Topological Parity 4-form can be evaluated by exterior differentiation as,

$$K = dH = dA\wedge dA = -2\,(\mathbf{a} \bullet \boldsymbol{\omega})\, dx\wedge dy\wedge dz\wedge dt. \tag{28}$$

The identity given by eq. (24) is in the form of a divergence when expressed on {x,y,z,t},

$$\mathrm{div}\,\mathbf{T} + \partial h/\partial t \;=\; -2(\,\mathbf{a} \bullet \boldsymbol{\omega}\,)\;, \tag{29}$$

and yields the helicity-torsion current conservation law if the anomaly, $-2(\,\mathbf{a} \bullet \boldsymbol{\omega}\,)$, on the RHS vanishes. It is to be observed that when K = 0, the Euler index is zero, and the integral of H over a boundary of support vanishes by Stokes theorem. This idea is the generalization of the conservation of the integral of helicity density in an Eulerian flow. Note the result is true for a *viscous* fluid, subject to the constraint of zero Euler index. However, if the topological parity 4-form, K, does not vanish, then the lines of the torsion current, T, do not satisfy a conservation law and can start or stop within the fluid interior. The parity 4-form, K, is the source for the production or destruction of topological defects in the evolutionary process.

By a similar substitution of (26) into (28), the topological parity pseudo-scalar becomes expressible in terms of engineering quantities as,

$$K = -2\,\upsilon\,\{\,\mathrm{curl}\,\mathbf{v} \bullet (\mathrm{curl\,curl}\,\mathbf{v})\} = -2\,\upsilon\,\{\,\boldsymbol{\omega} \bullet \mathrm{curl}\,\boldsymbol{\omega}\}. \tag{30}$$

The integral of K over {x,y,z,t} gives the Euler index of the flow. It is to be observed that the Topological Parity pseudo-scalar, K, is always zero for non-viscous Eulerian flows, and can be zero for viscous Navier-Stokes flows if the vorticity vector, $\boldsymbol{\omega}$, satisfies the Frobenius integrability condition, $\{\,\boldsymbol{\omega} \bullet \mathrm{curl}\,\boldsymbol{\omega}\} = 0$. From the result given by (30), it is apparent that K, and therefore, the Euler index of the domain of support, is not necessarily zero, unless the vorticity field, curl v, admits a foliation of codimension 1. See Arnold [1981]. If the flow is to be irreversible, the flowlines must have at least one point of intersection in space-time, and therefore the Euler index of the domain of support cannot be zero. It follows that K for the domain of support cannot be zero, and therefore the Pfaff dimension of the turbulent state must be 4. The production of defects is to be associated with the turbulent state.

If the RHS of (16) goes to zero, then the constraint becomes equivalent to the Euler equations of motion of a non-viscous fluid. The criteria that $i(\mathbf{V})dA = 0$ has been shown by Cartan [Cartan 1958] to be necessary and sufficient for the solution vector to have a Hamiltonian representation relative to the Action, A. This result corresponds to the well known Clebsch potential representation for an Euler flow [Lamb 1945]. Note that even in a viscous flow, if $\mathbf{F} = \upsilon$ curl curl $\mathbf{v} = 0$, the system has a Hamiltonian representation! If the kinematics are without fluctuations, then the conclusion is again reached that the system is Hamiltonian.

Analytically, if the RHS of (16) vanishes or is closed, then the solution vector fields are uniformly continuous, for the even dimensional elements of the Pfaff sequence, (the limit sets, F=dA and K = dH = F^F) are invariants of the flow. However, the odd dimensional intersections of the Pfaff sequence need not be invariant, implying the possibility of a continuous but changing topology. If topology changes, such flows can not be homeomorphisms, but are uniformly continuous; therefore, they must be irreversible, in the sense that a continuous inverse does not exist. Time reversal symmetry is broken.

The constraint of uniform continuity may be relaxed, for a flow is continuous if the limit points of the flow permute into the closure of the evolving topology [Hocking, 1984]. Such a flow is said to be continuous, but not necessarily uniformly continuous. It is possible to show that relative to the Cartan topology, all evolutionary processes generated by the Lie derivative with respect to C2 vector fields are continuous. The equations of continuous, but not uniformly continuous, evolution must satisfy transversal constraints of the form [Kiehn 1974],

$$i(\mathbf{V})dA = \Gamma A + \{\text{closed 1-forms to fit boundary conditions}\}$$

$$= \Gamma\{L(\mathbf{x},t;\,\mathbf{v})dt + \mathbf{p} \bullet (d\mathbf{x} - \mathbf{v}dt)\} + \{\text{closed 1-forms}\} \ . \qquad (31)$$

This result implies that the viscous forces, **F**, in (16) must be proportional to the canonical momentum, **p**, a result typical of empirical viscous friction assumptions. In other words, solutions to the Navier-Stokes equations (21), which are of the form (31), may be continuous but irreversible relative to the Cartan fluctuation topology.

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