Introduction

Differential Forms

The building blocks of differential forms on a variety of independent variables $\{x^m\}$ are scalar functions, $f(x^m)$, and 1-forms, $\omega = A_k(x^m)dx^k$. The 1-forms are to be thought of as integrands of a line integral. Now consider the behavior of these objects under mappings. In particular let the $\{x^m\}$ represent the independent variables of a range (the final state) which is the result of a map from a domain $\{\xi^n\}$.(the initial state). Note that the number of independent variables on the domain and range are not necessarily the same. Assume the map is at least C1 differentiable. The importance of differential forms is that they are **invariants** of such C1 functional substitutions. For consider the maps:

$$\phi : \{\xi^n\} \Rightarrow \{x^m\} = \phi^m(\xi^n)$$

$$d\phi : \{d\xi^n\} \Rightarrow \{dx^k\} = d\phi^k(\xi^n) = [\partial\phi^k(\xi^n)/\partial\xi^l]d\xi^l = [\mathbb{J}_l^k(\xi^n)]|d\xi^l\rangle$$

The (non-square) matrix $[\mathbb{J}_l^m(\xi^n)]$ represents the Jacobian matrix of the mapping. Now by functional substitution into the forms $f(x^m)$ and $\omega = A_k(x^m)dx^k \equiv \langle A_k(x^m)| \circ |dx^k\rangle$ on the final state, well defined pre-images are functionally obtained on the initial state.

$$f^*(\xi^n) \Leftarrow f(\phi^m(\xi^n)) \Leftarrow f(x^m)$$

$$A_l^*(\xi^n) \circ |d\xi^l\rangle \Leftarrow \langle A_k(\phi^m(\xi^n))| \circ [\mathbb{J}_l^k(\xi^n)] \circ |d\xi^l\rangle \Leftarrow A_k(x^m)dx^k$$

This operation has been called the ”pullback” in the literature. It is the essence of the invariance properties of differential forms. Given functional data on the final state, well defined functional data on the initial state is automatically obtained by the process of functional substitution.

Note that the functional components of the 1-form on the initial state are related to the functional components of the 1-form on the final state by means of the transpose of the Jacobian matrix.

$$\langle A_l^*| \Leftarrow \langle A_k| \circ [\mathbb{J}_l^m] \quad \equiv \quad [\mathbb{J}_l^k(\xi)]^{transpose} \circ |A_k(x)\rangle \Rightarrow |A_l^*(\xi)\rangle$$

This result of objects being functionally well defined is not necessarily obtained for the forward direction. That is, the ”pushforward” of objects on the initial state does not lead to functionally well defined objects on the final state unless the map and its Jacobian matrix are invertible. For maps between spaces of different dimension the ”pushforward” is impossible, but the ”pullback” always works. It is this feature of differential forms that permits differential forms to transcend ordinary tensor calculus, and to carry information about topological change. Note that the maps of interest may be immersions from spaces of lower dimension to spaces of higher dimension, or submersions (projections) from spaces of higher dimension to spaces of lower dimension.

Contravariant Vectors:

Consider an initial state as a manifold of independent variables $\{u^\sigma\} \equiv \{u, v, w\}$. Consider a final state as a manifold of independent variables $\{x^k\} \equiv \{x, y, z\}$. A point in the initial state can be considered as a position vector from the origin $\{0, 0, 0\}$ to the set $\{u, v, w\}$. It is conventional to consider the position vector, as a column vector. Similarly on the final state the position vector can be written as

$$\mathbf{R} = \left| \begin{matrix} x \\ y \\ z \end{matrix} \right\rangle$$

Presume a differentiable map ϕ exists between the initial and final state.

$$\phi : \{u^\sigma\} \Rightarrow \{x^k\} = \{f^k(u^\sigma)\}.$$

Example:

$$x = u$$

$$y = u + v^2$$

$$z = v + uw^3$$

The differentiability of the non-linear map leads to a linear relationship between the differentials:

$$d\phi : \{du^\sigma\} \Rightarrow \{dx^k\} = \{[\partial f^k(u^\rho)/\partial u^\sigma]du^\sigma\}.$$

Example:

$$dx = du$$

$$dy = du + 2v\,dv$$

$$dz = dv + w^3\,du + u\,3w^2 dw$$

The matrix

$$[\partial f^k(u^\rho)/\partial u^\sigma] = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2v & 0 \\ w^3 & 1 & 3uw^2 \end{bmatrix}$$

is defined as the Jacobian matrix of the mapping. Except at the origin on the initial state, this Jacobian map is well defined in the sense that it is invertible at all points of the initial domain except the origin. From the implicit function theorem there are restricted domains for which an inverse map exists, defining local diffeomorphisms.

Note that the linear mapping of the differentials can be written in matrix form

$$\left| \begin{matrix} dx \\ dy \\ dz \end{matrix} \right\rangle = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2v & 0 \\ w^3 & 1 & 3uw^2 \end{bmatrix} \left| \begin{matrix} du \\ dv \\ dw \end{matrix} \right\rangle,$$

an equation that takes a row vector on the initial state and creates a row vector on the final state. **This linear mapping (the push forward) is the epitome of a contravariant tensor mapping**. That is, those ordered arrays of functions of arguments on the initial state that are linearly mapped into ordered arrays of values on the final state by means of the Jacobian matrix are **defined** as *contravariant* tensors of rank 1, or a *contravariant vector*.

Example: Consider a dynamical system on the initial state as a velocity vector field **U** of functions on the initial state.

$$\mathbf{U} = \left| \begin{matrix} \mathbf{U}^u \\ \mathbf{U}^v \\ \mathbf{U}^w \end{matrix} \right\rangle = \left| \begin{matrix} du/dt \\ dv/dt \\ dw/dt \end{matrix} \right\rangle \doteq \left| \begin{matrix} 1+uv \\ v^3 \\ uw \end{matrix} \right\rangle$$

This vector on the initial state can be transformed into a vector on the final state by the Jacobian matrix rule

$$\left| \begin{matrix} \mathbf{V}^x \\ \mathbf{V}^y \\ \mathbf{V}^z \end{matrix} \right\rangle = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2v & 0 \\ w^3 & 1 & 3uw^2 \end{bmatrix} \left| \begin{matrix} \mathbf{U}^u \\ \mathbf{U}^v \\ \mathbf{U}^w \end{matrix} \right\rangle$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2v & 0 \\ w^3 & 1 & 3uw^2 \end{bmatrix} \left| \begin{matrix} 1+uv \\ v^3 \\ uw \end{matrix} \right\rangle = \left| \begin{matrix} 1+uv \\ 1+uv+2v^4 \\ w^3+w^3uv+v^3+3u^2w^3 \end{matrix} \right\rangle$$

Note that the ordered array of functions on the final state has arguments on the initial state! In order to obtain an ordered array of functions with arguments on the final state it is necessary to know the inverse mapping such that the $\{u,v,w\}$ can be expressed in terms of the $\{x,y,z\}$. Given a point on the initial state, a value of the contra-variant vector on the final state is readily computed. However, the functional form of the components of the contra-variant vector on the final state with arguments over the base $\{x,y,z\}$ is impossible without knowledge of the inverse mapping.

Co-variant Vectors:

Next consider a function defined on the final state, $\Theta(x,y,z)$. Compute the differential of this function using the chain rule:

$$d\Theta(x,y,z) = \{\partial\Theta(x,y,z)/\partial x\}dx + \{\partial\Theta(x,y,z)/\partial y\}dy + \{\partial\Theta(x,y,z)/\partial z\}dz.$$

Note that this construction can be written as the linear-row column product:

$$d\Theta(x,y,z) = \left\langle \; \partial\Theta(x,y,z)/\partial x \quad \partial\Theta(x,y,z)/\partial y \quad \partial\Theta(x,y,z)/\partial z \; \right| \left| \begin{matrix} dx \\ dy \\ dz \end{matrix} \right\rangle.$$

Now use the mapping to rewrite the expression on terms of the variables of the initial state, $\{u,v,w\}$.

$$d\Theta(x,y,z)$$

$$= \left\langle \partial\Theta(x,y,z)/\partial x \quad \partial\Theta(x,y,z)/\partial y \quad \partial\Theta(x,y,z)/\partial z \;\middle|\; \begin{bmatrix} \partial x/\partial u & \partial x/\partial v & \partial x/\partial w \\ \partial y/\partial u & \partial y/\partial v & \partial y/\partial w \\ \partial z/\partial u & \partial z/\partial v & \partial z/\partial w \end{bmatrix} \middle| \begin{matrix} du \\ dv \\ dw \end{matrix} \right\rangle$$

$$= \left\langle A_x(x,y,z) \quad A_y(x,y,z) \quad A_z(x,y,z) \;\middle|\; \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2v & 0 \\ w^3 & 1 & 3uw^2 \end{bmatrix} \middle| \begin{matrix} du \\ dv \\ dw \end{matrix} \right\rangle$$

$$= \left\langle A_u(x,y,z) \quad A_v(x,y,z) \quad A_w(x,y,z) \;\middle|\; \begin{matrix} du \\ dv \\ dw \end{matrix} \right\rangle$$

$$= \left\langle \overset{*}{A}_u\ (u,v,w) \quad \overset{*}{A}_v\ (u,v,w) \quad \overset{*}{A}_w\ (u,v,w) \;\middle|\; \begin{matrix} du \\ dv \\ dw \end{matrix} \right\rangle = d\ \overset{*}{\Theta}\ (u,v,w).$$

An ordered array of functions on the final state that behaves like the components of the gradient of a function (a row vector) producing an ordered array of functions as a row vector on the initial state is the epitome of what is defined classically as a co-variant vector. In the usual realm of tensor analysis, the gradient is defined on the initial state, and the Jacobian maps are presumed to be invertible. Then the covariant tensor field is defined as the ”pushforward” with respect to the *inverse* Jacobian matrix transposed. It is quite common in engineering applications to restrict the mappings such that the Jacobians are elements of the orthogonal group. In such restricted cases the inverse and the transpose are identical. It is much more illuminating to consider the covariants as defined on the final state and consider the functional substitution and the pullback as the proper definitions of covariant tensor fields. The reason is that the co-tensors then are well defined with respect to pullbacks whether or not the Jacobian is invertible.

Further note that the process is deduced by the behavior of the gradient, but the mapping and pullback concepts can start with the set

$$\left\langle A_x(x,y,z) \quad A_y(x,y,z) \quad A_z(x,y,z) \;\right|$$

whether or not it can be constructed from a gradient operation. In other words the differential form

$$A = \sum_{k=1}^{3} A_k(x,y,z)dx^k$$

need not be exact. Objects that behave as under the pullback transform as the Jacobian transpose are defined as covariant tensor fields.

Example:

Define

$$\Theta(x,y,z) = 1 + x^2 + yxz$$

Then

$$\left\langle \partial\Theta(x,y,z)/\partial x \quad \partial\Theta(x,y,z)/\partial y \quad \partial\Theta(x,y,z)/\partial z \right|$$

$$= \left\langle A_x(x,y,z) \quad A_y(x,y,z) \quad A_z(x,y,z) \right|$$

$$= \left\langle 2x + yz \quad xz \quad yx \right|$$

and

$$\left\langle \overset{*}{A}_u\ (u,v,w) \quad \overset{*}{A}_v\ (u,v,w) \quad \overset{*}{A}_w\ (u,v,w) \right|$$

$$= \left\langle A_x(x,y,z) \quad A_y(x,y,z) \quad A_z(x,y,z) \right| \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2v & 0 \\ w^3 & 1 & 3uw^2 \end{bmatrix}$$

$$= \left\langle 2x + yz + xz + yxw^3 \quad xz2v + yx \quad yx3uw^3 \right|$$

Finally the components of the co-vector field on the initial state can be expressed as three functions with arguments on the initial state, as the pull-back:

$$\overset{*}{A}_u\ (u,v,w) = 2u + (u+v^2)(v+uw^3) + uu(v+uw^3) + (u+v^2)uw^3$$
$$\overset{*}{A}_v\ (u,v,w) = u(v+uw^3)2v + (u+v^2)u$$
$$\overset{*}{A}_w\ (u,v,w) = (u+v^2)u3uw^3$$

Everything with respect to the pullback is well defined without need for a inverse mapping or for a inverse Jacobian mapping.

Contra-tensor densities

In addition to the pullback of 1-forms there is another construction that can pullback properly and yields contravariant tensor densities (not tensors). Consider the volume element on the final state, with perhaps some measure function, $\rho(x,y,z)$.

$$\Omega = \rho(x,y,z)dx\wedge dy\wedge dz$$

Contract the volume element with a contravariant vector on the final state to form the N-1 form density:

$$J = i(\mathbf{V}^k(x,y,z))\Omega = \rho(x,y,z)\{V^x(x,y,z)dy\wedge dz - V^y(x,y,z)dx\wedge dz + V^z(x,y,z)dx\wedge dy\}$$

Now substitute for each differential is expression in terms of differentials of the variables of the initial state, and collect terms. The ordered array is equivalent to multiplying the contravariant vector by the adjoint matrix (matrix of cofactors transposed) of the original Jacobian matrix. This adjoint matrix is well defined and exists even though the determinant of the Jacobian vanishes. Finally, substitute in the pulled back ordered array of functions the values of x,y,z in terms of u,v,w to create a contravariant tensor density field on the initial state.

In other words, there are **two species of differential forms** that have functionally well defined formats on the initial state given functional data on the final state, and without knowledge of an inverse mapping or and inverse Jacobian mapping. One species defines covariant tensors (Field intensities) and the other defines contravariant tensor densities (Field excitations) Not only can values of such objects be retrodicted to the initial state from the final data, but also the functional forms on the initial state are well define in terms of the functional data on the final state.

The duals of these concepts, contravariant tensors and co-variant capacities, are not well behaved with respect to non-invertible mappings. That is values may be predicted, but functional formats cannot.