IMMERSIONS

Consider an initial state as a manifold of independent variables $\{u^\sigma\} \equiv \{u,v\}$. Consider a final state as a manifold of independent variables $\{x^k\} \equiv \{x,y,z\}$. (Note that the restriction is to a final state of 1 dimension more that the initial state in that which follows. A problem where the final state is of two dimensions more than the initial state will be considered in another section) A point in the final state can be considered as a position vector from the origin $\{0,0,0\}$ to the set $\{x,y,z\}$. It is conventional to consider the position vector, as a column vector.

$$\mathbf{R} = \left| \begin{matrix} x \\ y \\ z \end{matrix} \right\rangle$$

Presume a differentiable map ϕ exists between the initial and final state.

$$\phi : \{u^\sigma\} \Rightarrow \{x^k\} = \{f^k(u^\sigma)\}.$$

Example: The Monge Surface (z is a graph of u,v)

$$x = u$$

$$y = v$$

$$z = g(u,v)$$

The differentiability of the non-linear map leads to a linear relationship between the differentials:

$$d\phi : \{du^\sigma\} \Rightarrow \{dx^k\} = \{[\partial f^k(u^\rho)/\partial u^\sigma]du^\sigma\}.$$

Example:

$$dx = du$$

$$dy = dv$$

$$dz = (\partial g(u,v)/\partial u)du + (\partial g(u,v)/\partial v)dv$$

The Jacobian matrix becomes two columns of ”tangent” vectors”

$$\mathbb{J} = [\partial f^k(u^\rho)/\partial u^\sigma] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \partial g(u,v)/\partial u & \partial g(u,v)/\partial v \end{bmatrix};$$

The non-square matrix has a zero determinant, and therefore is not invertible. However, as the matrix is missing only one column from being square, it is possible to construct a third column of linear independent functions algebraically by means of the matrix adjoint process. This third linearly independent column vector is given to within a factor by the partial derivative components of the Monge function. The result is to construct at any point $\mathbf{R}(u,v)$ a basis frame of the form

$$\mathbb{F}(u,v) = \begin{bmatrix} 1 & 0 & -\rho\partial g(u,v)/\partial u \\ 0 & 1 & -\rho\partial g(u,v)/\partial u \\ \partial g(u,v)/\partial u & \partial g(u,v)/\partial v & +\rho \end{bmatrix}$$

This basis frame at a point p is an example of Cartan's Repere Mobile. The matrix elements in this case have been determined in terms of the two variables that make up a Monge surface and form an element of a two parameter group. It is possible to construct such a basis matrix from the definition of a curve in {x,y,z} which has been parametrized in terms of its arc length. The Basis Frame (an element of a single parameter group) is the called the Frenet-Serret basis frame.

It is to be noted that the matrix elements of this basis frame on the final state are defined in terms of the independent variables on the initial state. The determinant is never zero (for non-zero ρ):

$$det(\mathbb{F}) = \rho\{1 + (\partial g(u,v)/\partial u)^2 + (\partial g(u,v)/\partial v)^2\}.$$

As the determinant is nowhere zero the basis frame has a global inverse. The existence of the inverse matrix, of course, guarantees that the differential of every basis vector is decomposable into a linear combination of the original elements of the basis set, independent of the partition. The matrix of coefficients of this linear expansion defines the (right) Cartan matrix of connection 1-forms, $\mathbb{C}_r$,

$$d\mathbb{F} = \mathbb{F}\circ\mathbb{C}_r = \mathbb{F}\circ\{d\mathbb{F}\circ\mathbb{F}^{-1}\} = \mathbb{F}\circ\{-d\mathbb{F}^{-1}\circ\mathbb{F}\}$$

over the domain of support.

It is convenient to partition the basis frame $\mathbb{F}$ in terms of the *associated* (other terms are horizontal, interior, coordinate or transversal) column vectors, $\mathbf{e}_k$, and the *adjoint* (other terms are normal, exterior, parametric or vertical) field, $\mathbf{n}_p$,

$$\mathbb{F}(u,v) = [\mathbf{e}_k, \mathbf{n}] = [\mathbf{e}_1, \mathbf{e}_2, \mathbf{n}].$$

The corresponding Cartan matrix has the partition,

$$d\mathbb{F} = d\begin{bmatrix} e_1^1 & e_2^1 & ... & n^1 \\ e_1^2 & e_2^2 & ... & n^2 \\ ... & ... & ... & ... \\ e_1^n & e_2^n & ... & n^n \end{bmatrix} = \mathbb{F}\circ\mathbb{C} = \mathbb{F}\circ\begin{bmatrix} \Gamma_1^1 & \Gamma_2^1 & ... & \gamma^1 \\ \Gamma_1^2 & \Gamma_2^2 & ... & \gamma^2 \\ ... & ... & ... & ... \\ h_1 & h_2 & ... & \Omega \end{bmatrix}$$

The corresponding vector equations written in longhand would read

$$d\mathbf{e}_k = \mathbf{e}_1\Gamma_k^1 + \mathbf{e}_2\Gamma_k^2 + \mathbf{n}h_k$$

$$d\mathbf{n} = \mathbf{e}_1\gamma^1 + \mathbf{e}_2\gamma^2 + \mathbf{n}\Omega.$$

It should be realized that the vectors are column elements of three functions which, as functions, premultiply the differential forms that make up the matrix elements of the Cartan matrix. It is for this reason that the right Cartan matrix is to be preferred over the left Cartan matrix of 1-forms. The expanded form of the matrix elements of the Cartan matrix are:

$$\Gamma^1_k \equiv \Gamma^1_{11}du + \Gamma^1_{12}dv$$

$$h_k \equiv h_{k1}du + h_{k2}dv$$

$$\gamma^k \equiv \gamma^k{}_1du + \gamma^k{}_2dv$$

and

$$\Omega \equiv \Omega_1du + \Omega_2dv$$

(In the appendix below, references to PDF files are given where all of the details, and the matrix elements, have been constructed by using MAPLE.)

The Cartan matrix, $\mathbb{C}$, is a matrix of differential 1-forms which can be evaluated explicitly from the functions that make up the basis frame if they admit first partial derivatives. Moreover, the differential of the position vector can be expanded in terms of the same basis frame and a set of Pfaffian 1-forms:

$$d\mathbf{R} = \mathbb{I} \circ \left| \begin{matrix} dx \\ dy \\ dz \end{matrix} \right\rangle = \mathbb{F} \circ \mathbb{F}^{-1} \circ \left| \begin{matrix} dx \\ dy \\ dz \end{matrix} \right\rangle = \mathbb{F} \circ \left| \begin{matrix} \boldsymbol{\sigma} \\ \omega \end{matrix} \right\rangle,$$

where the vector $\left| \begin{matrix} \boldsymbol{\sigma} \\ \omega \end{matrix} \right\rangle$ is a vector of 1-forms, in the same sense that the Cartan matrix is a matrix of 1-forms. In many exposes, this formula for $d\mathbf{R}$ is pulled out of the air, and written as:

$$d\mathbf{R} = \mathbf{e}_1\sigma^1 + \mathbf{e}_2\sigma^2 + \mathbf{n}\omega.$$

Remember that when written in detail the 1-forms become

$$\sigma^1 \equiv \sigma^1_1du + \sigma^1_2dv, \qquad \sigma^2 \equiv \sigma^2_1du + \sigma^2_2dv.$$

For parametrizations, $\omega = 0$.

By the Poincare lemma, it follows that

$$dd\mathbf{R} = d\mathbb{F}\wedge \left| \begin{matrix} \boldsymbol{\sigma} \\ \omega \end{matrix} \right\rangle + \mathbb{F} \circ \left| \begin{matrix} d\boldsymbol{\sigma} \\ d\omega \end{matrix} \right\rangle = \mathbb{F} \circ \{\mathbb{C}\wedge \left| \begin{matrix} \boldsymbol{\sigma} \\ \omega \end{matrix} \right\rangle + \left| \begin{matrix} d\boldsymbol{\sigma} \\ d\omega \end{matrix} \right\rangle\} = 0$$

and

$$d\mathbb{F} = d\mathbb{F}\wedge\mathbb{C} + \mathbb{F}\wedge d\mathbb{C} = \mathbb{F} \circ \{\mathbb{C}\wedge\mathbb{C} + d\mathbb{C}\} = 0.$$

As the Frame matrix and the Cartan matrix are partitioned relative to the tangent (or interior) vectors **e** and the normal (or exterior) vectors, **n**, the Poincare lemma breaks up into linearly independent factors, each of which must vanish. The results are:

$$dd\mathbf{R} = \mathbf{e}\{d|\boldsymbol{\sigma}\rangle + [\boldsymbol{\Gamma}]\wedge|\boldsymbol{\sigma}\rangle - \omega\wedge|\boldsymbol{\gamma}\rangle\} + \mathbf{n}\{d\omega + \Omega\wedge\omega + \langle\mathbf{h}|\wedge|\boldsymbol{\sigma}\rangle\} = 0$$

$$dd\mathbf{e} = \mathbf{e}\{d[\mathbf{\Gamma}] + [\mathbf{\Gamma}]\hat{}[\mathbf{\Gamma}] + |\boldsymbol{\gamma}\rangle\hat{}\langle\mathbf{h}|\} + \mathbf{n}\{d\langle\mathbf{h}| + \Omega\hat{}\langle\mathbf{h}| + \langle\mathbf{h}|\hat{}[\mathbf{\Gamma}]\} = 0$$

$$dd\mathbf{n} = \mathbf{e}\{d|\boldsymbol{\gamma}\rangle + [\mathbf{\Gamma}]\hat{}|\boldsymbol{\gamma}\rangle - \Omega\hat{}|\boldsymbol{\gamma}\rangle\} + \mathbf{n}\{d\Omega + \Omega\hat{}\Omega + \langle\mathbf{h}|\hat{}|\boldsymbol{\gamma}\rangle\} = 0$$

By reasons of linear independence, each of the curly bracket factors must vanish, leading to the results on the interior domain (coefficients of **e**):

$$d|\boldsymbol{\sigma}\rangle + [\mathbf{\Gamma}]\hat{}|\boldsymbol{\sigma}\rangle = \omega\hat{}|\boldsymbol{\gamma}\rangle$$

$$\equiv |\mathbf{\Sigma}\rangle = \left|\begin{matrix} \omega\hat{}\gamma^1 \\ \omega\hat{}\gamma^2 \end{matrix}\right\rangle \text{ the interior torsion vector of dislocation 2-forms.}$$

$$d[\mathbf{\Gamma}] + [\mathbf{\Gamma}]\hat{}[\mathbf{\Gamma}] = -|\boldsymbol{\gamma}\rangle\hat{}\langle\mathbf{h}|$$

$$\equiv [\mathbf{\Theta}] = \begin{bmatrix} \gamma^1\hat{}h_1 & \gamma^1\hat{}h_2 \\ \gamma^2\hat{}h_1 & \gamma^2\hat{}h_2 \end{bmatrix} \text{the matrix of interior curvature 2-forms}$$

$$\{d|\boldsymbol{\gamma}\rangle + [\mathbf{\Gamma}]\hat{}|\boldsymbol{\gamma}\rangle = \Omega\hat{}|\boldsymbol{\gamma}\rangle$$

$$\equiv |\mathbf{\Psi}\rangle = \left|\begin{matrix} \Omega\hat{}\gamma^1 \\ \Omega\hat{}\gamma^2 \end{matrix}\right\rangle \text{the exterior torsion vector of disclination 2-forms.}$$

The first two equations are precisely Cartan's equations of structure (on an affine domain).

The last equation appears to be a new equation of structure not usually seen in the literature. Parametric 2-surfaces in 3-D are usually limited by the two constraints of Gauss-Weingarten. The first constraint presumes that the position vector has no differential component in the direction of any tangent vector, $(\mathbf{n} \bullet d\mathbf{R} = 0)$. This constraint physically implies that motion is confined to the surface. The second G-W constraint presumes that the surface normal has no differential component in the direction of the normal field $(\mathbf{n} \bullet d\mathbf{n} = 0)$. The first constraint forces $\omega \Rightarrow 0$, and the second constraint forces $\Omega \Rightarrow 0$. For immersive maps of 2-surfaces into 3-D, the first G-W constraint is always satisfied, but it requires an embedding for $\Omega \Rightarrow 0$. The idea is that if the determinant of the basis frame is globally non-zero (embedding) then the adjoint-normal field can be "normalized" by choice of the scaling factor, $\rho = (\mathbf{n} \bullet \mathbf{n})^{1/2}$. For the Monge surface, the surface is an embedding as the normal field never goes to zero. It is then possible to choose the "scaling parameter" such that the disclinations "disappear". If the surface has self-intersections, this renormalization cannot be done globally. The line of self intersections of the surface becomes the "disclination defect". (Cross Cap??). The concept of self intersections does not show up in the Gauss curvature or the Mean Curvature expressions.

$|\mathbf{\Psi}\rangle$ physically seems to represent a different kind of "torsion". For parametric surfaces, it will be demonstrated that the dislocation torsion 2-forms, $|\mathbf{\Sigma}\rangle$, vanish and the disclination torsion 2-forms, $|\mathbf{\Psi}\rangle$, are proportional Gauss curvature (implying a "rotation").

There are also three equations of structure on the exterior domain (coefficients of **n**) which are given by the constructions:

$$d\omega + \Omega\hat{}\omega = -\langle\mathbf{h}|\hat{}|\boldsymbol{\sigma}\rangle$$

$$d\langle\mathbf{h}| + \Omega\hat{}\langle\mathbf{h}| = -\langle\mathbf{h}|\hat{}[\boldsymbol{\Gamma}]$$

$$d\Omega + \Omega\hat{}\Omega = \theta = -\langle\mathbf{h}|\hat{}|\boldsymbol{\gamma}\rangle \text{ the exterior curvature 2-forms}$$

For immersions, $\omega \Rightarrow 0,$ and Ω is an exact 1-form. The constraint forces $\langle\mathbf{h}|\hat{}|\boldsymbol{\gamma}\rangle \Rightarrow 0.$. Exterior differentiation of the matrix of interior curvature 2-forms yields:

$$d[\boldsymbol{\Theta}] = -d|\boldsymbol{\gamma}\rangle\hat{}\langle\mathbf{h}| = (-|d\boldsymbol{\gamma}\rangle\hat{}\langle\mathbf{h}|) + (|\boldsymbol{\gamma}\rangle\hat{}\langle d\mathbf{h}|) =$$

$$([\boldsymbol{\Gamma}]\hat{}|\boldsymbol{\gamma}\rangle\hat{}\langle\mathbf{h}|) - (\Omega\hat{}|\boldsymbol{\gamma}\rangle\hat{}\langle\mathbf{h}|) - (|\boldsymbol{\gamma}\rangle\hat{}\Omega\hat{}\langle\mathbf{h}|) - (|\boldsymbol{\gamma}\rangle\hat{}\langle\mathbf{h}|\hat{}[\boldsymbol{\Gamma}]) = 0$$

The fundamental result is that the matrix of 2-forms that forms the interior curvature matrix is closed!

MORE DETAIL HERE SOON about Mean and Gauss Curvature

Under construction