IMPLICIT SURFACES

In parametric surface theory, the fundamental concept is a position vector to a point on the ”surface” of dimension n from a (possibly) exterior point called the origin an a space of dimension n+1. A tangent space to the surface is defined by a *differential procedure* in which derivatives of the position vector with respect to the parameters are used to construct the tangent vectors on the final state. Note that the components of the tangent vectors in the space of dimension n+1 are functions of the parametric variables on the space of dimension n.

In implicit surface theory, the fundamental entity is the ”normal” vector in a space of dimension n+1, from a point on the ”surface” of dimension n, but orthogonal to the ”surface”. The tangent vectors to the surface are constructed by an *algebraic procedure* relative to the normal field. The n+1 components of the tangent vectors are functions of the n+1 variables. In the simplest case, consider a function $\Theta(x,y,z)$ defined on {x,y,z}. Use the chain rule to construct the total differential. Presume the coefficients of the total differential (the gradient field) defines a normal, **N**, to some ”surface”. Tangent vectors to the ”surface” are algebraically defined as the n vectors, $\mathbf{e}_k$, of n+1 components orthogonal to **N**.

$$i(\mathbf{e}_k)d\Theta(x,y,z) = 0.$$

The submersive map, $\theta,$ form $\{x,y,z\}$ to c defined by

$$\theta : \{x,y,z\} \Rightarrow c = \Theta(x,y,z)$$

implies that $d\theta$ is not identically zero (the gradient does not vanish identically)

$$d\theta : dc = d\Theta(x,y,z) = (\partial\Theta/\partial x)dx + (\partial\Theta/\partial y)dy + (\partial\Theta/\partial z)dz \neq 0$$

Vectors that satisfy $i(\mathbf{e}_k)d\Theta(x,y,z) = 0$ are defined as ”associated” vectors. Note that motion in the direction of the associated vectors leaves the function Θ invariant. The Normal field is then

$$\mathbf{N} = grad\,\Theta.$$

Often the ”surface” is defined by the zero set of $\Theta,$ but what is important to the definition of the surface is the construction of the tangent field. There are many ways to construct this tangent field, and the collection of all such n vectors is often defined as a ”distribution”.

However, the implicit process can be generalized. Instead of specifying a function on the domain, it is possible to specify a 1-form on the domain, and then search for the ”distribution” of associated or tangent vectors to the ”surface” which are ”orthogonal” to the 1-form, ω.

The functional coefficients of the 1-form may be viewed as the ”normal” field,

$$\omega = N_x(x,y,z)dx + N_y(x,y,z)dy + N_z(x,y,z)dz$$

The tangent vectors to the surface are defined algebraically as

$$i(\mathbf{e}_k)\omega(x,y,z) = 0.$$

Note that the 1-form ω can be multiplied by an arbitrary function $\lambda(x,y,z)$ without modification of the surface ”tangent vectors”. Of particular interest is the case when the function $\lambda(x,y,z)$ is homogenous in the coefficients of ω. That is, consider the projective divisors such that

$$\omega \Rightarrow \varpi = \omega/\lambda(N_x,N_y,N_z)$$

The normal field becomes ”projectivized” :

$$\mathbf{n} = \{n_x, n_y, n_z\} = \mathbf{N}/\lambda$$

Consider homogeneous divisors of degree m such that $\lambda = (\sum a^k (N_k)^p\}^{m/p}$.

A fundamental observation

It is remarkable that if such divisors are homogeneous of degree 1 ($m = 1$ any p), then the 3x3 Jacobian matrix of the projectivized normal field at any point has similarity invariants that are special to the ”surface”. The three Cayley-Hamilton invariants are:

1. The trace of the Jacobian, which is equal to the Mean curvature, H, of the surface at any point.
2. The trace of the adjoint matrix to the Jacobian, which is equal to the Gauss curvature, G, of the surface at any point.
3. The determinant of the Jacobian, K, which is always zero, and thereby implies that the topological domain of the projectivized normal field is of rank = dimension 2; i.e., a surface in a 3-D space.

MORE LATER