Comments on

"The Influence of geometry and topology on Helicity"

presented by

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1. The definition of magnetic Helicity (a 3-form)

First, the definition of Magnetic Helicity with which I am comfortable is that given by the 3D (volume) integral of the vector potential dotted with its curl:

Definition 1:

$$\textit{Magnetic Helicity} := \iiint \mathbf{A} \circ curl\mathbf{A}\, dVol = \iiint \mathbf{A} \circ \mathbf{B}\, dVol = \iiint A\hat{}F = \iiint A\hat{}dA$$

For me this object is a well defined object on any variety $\{x,y,z\}$. On every 3 dimensional *closed* manifold for which $\mathbf{A} = \mathbf{A}(x,y,z)$, it is easy to show that the *closed* integrals of Magnetic Helicity density, $\mathbf{A} \circ \mathbf{B}$, are deformation invariants of **any** continuous evolutionary process that can be described by a singly parametrized vector direction (or flow) field, $\mathbf{V}(x,y,z)$. In the language of Poincare-Cartan, the closed integral is a relative integral invariant. All that is required is that the points which make up the integration chain in the initial state remain on the same flow fiber in the final state.

This topological conservation law has nothing to do with electromagnetism, per se, but for classical definitions of the Magnetic Field intensity, such as $\mathbf{B} = curl\,\mathbf{A}$, it works for electromagnetism. It is also true for fluids or any other system that can be described by a 1-form of Action, A, and can be extrapolated to higher dimensions. The proof is easy:

$$L_{(\beta\mathbf{V})} \iiint_{closed} \mathbf{A} \circ \mathbf{B}\, dVol = \iiint_{closed} \{d(i(\beta\mathbf{V})A\hat{}F) \ + i(\beta\mathbf{V})d(A\hat{}F)\} \ =$$

$$\iiint_{closed} \{d(i(\beta\mathbf{V})A\hat{}F) \ + i(\beta\mathbf{V})F\hat{}F\}$$

$$= 0 + 0 \supset \ \textit{evolutionary invariance for} \text{ any } \beta\mathbf{V}$$

The first integral vanishes because the integral of an exact form over a closed integration chain vanishes. The second integral vanishes as $F\hat{}F = 0$ on a 3 dimensional manifold; all 3-dimensional volume elements are closed. As $\beta(x,y,z)$ is arbitrary, it acts as a possible deformation parameter. The result is true for any gauge, does not depend upon metric, and is independent of any geometrical connection, and certainly does not depend upon a constitutive constraint. A true topological quantity. Note that the closed 3-dimensional integration domain can be a cycle and does not have to be a boundary (of a higher dimensional space). It is important to realize (for application to domains that do not have a euclidean topology) that the 3-form $A\^F$ has components that transform as a covariant tensor of rank 3.

Perhaps it is more important that the theorem of Topological deformation invariance of the closed integrals of the 3-form is also valid in **any** dimension for which $F\^F$ is zero. In higher dimensions the integral must be evaluated over 3-sub_manifolds that need not be space-like. For 4 dimensions the 3-form $A\^F$ has 4 components and can be constructed as

$$A\^F = i(\mathbf{T}_4)dx\^dy\^dz\^dt$$

where

$$\mathbf{T}_4 = [\mathbf{E}\times\mathbf{A} + \phi\mathbf{B}, \mathbf{A}\circ\mathbf{B}]$$

From this formulation it must be remembered that the components of this "vector" transform as a third rank covariant tensor field; $\mathbf{A}\circ\mathbf{B}$ is merely the fourth component. In the literature such objects are often described as pseudo vectors.

The criteria $F\^F = 0$ for topological invariance with respect to any continuous evolutionary process is equivalent to the statement that $\mathbf{E}\circ\mathbf{B} = 0$. The condition is sufficient, but not necessary, for topological invariance of the Helicity integral, even when the fields are explicitly time dependent. The relative integral invariant in 4D is:

$$H = \iiint_{closed} A\^F = \iiint_{closed} \mathbf{T}^x dy\^dz\^dt - \mathbf{T}^y dx\^dz\^dt + \mathbf{T}^z dx\^dy\^dt - \mathbf{A}\circ\mathbf{B}dx\^dy\^dz$$

In all cases the 3 divergence of $\mathbf{B}$ vanishes, and the 2-form F is closed, for it is exact, $dF = 0$. For isochronous domains, $dt = 0$, and the integral reduces to the standard spatial format of plasma physics. However, there are example electromagnetic fields for which $\mathbf{A}\circ\mathbf{B} = 0$, and yet $\mathbf{T}_4$ is not zero.

Example: Define $\mathbf{A} = [0, 0, (x^2 + y^2)/2]$, $\phi = z$. Then

$$\mathbf{B} = [-y, x, 0]$$
$$\mathbf{E} = [0, 0, -1]$$
$$\mathbf{A}\circ\mathbf{B} = 0, \quad \mathbf{E}\circ\mathbf{B} = 0,$$
$$\mathbf{T}_4 = z\mathbf{B}$$

I call $A\^F$ the topological torsion of the field, and $d(A\^F) = F\^F$, the topological parity.

Special situations become evident when the integration domain is compact with a boundary, for then Stokes law may be applied, and deformation invariance requires that $i(\beta\mathbf{V})A\^F=0$ *on the boundary*. These are interesting but special cases which are invariants of only a special choice of boundary conditions. For example, if $\beta(x,y,z) = 0$ defines the boundary, then for deformation invariance of the integral it must be true that the function β also must be an evolutionary invariant, such that $L_{(\mathbf{V})}d\beta = i(\mathbf{V})d\beta = 0$. Classically the function β which is used to define the boundary, is not arbitrary, but must be a first integral of the evolutionary vector field. For such special cases, the field on the boundary need not be tangential! There are other special situations as well.

For integration domains which are open, the criteria for absolute integral invariance is much more severe, and requires that $d(i(\beta\mathbf{V})A\hat{}F) = 0$. This constraint is to be recognized as the criteria that the evolutionary vector field $\beta\mathbf{V}$ be an element of the symplectic group. I have demonstrated that all such evolutionary processes are thermodynamically reversible.

Some remarks

To relate the above definition of a 3-form of Helicity with the six dimensional formulation involving the Biot-Savart substitution (to me) is an extraordinary constraint on the topology of the domain. The substitution effectively mixes a tensor and a tensor density, where definition 1 mixes a tensor with a tensor. There is however, another well defined electromagnetic 3-form, $A\hat{}G$, which mixes a tensor (A) and a tensor density (G). I call $A\hat{}G$ with physical dimensions of angular-momentum, the Spin 3-form. The 3-form $A\hat{}F$ has physical dimensions of Angular momentum divided by Ohms. I call $A\hat{}F$ the 3-form of topological Torsion_helicity.

In electrical engineering notation,

$$Spin\ :\ \mathbf{S}_4 = [\mathbf{A}\times\mathbf{H} + \phi\mathbf{D}, \mathbf{A}\circ\mathbf{D}]$$

$$Torsion\_Helicity\ :\ \mathbf{T}_4 = [\mathbf{E}\times\mathbf{A} + \phi\mathbf{B}, \mathbf{A}\circ\mathbf{B}]$$

The closure of $A\hat{}G$ defines a measure known as the first Poincare invariant,

$$P1 := (\mathbf{B}\circ\mathbf{H} - \mathbf{D}\circ\mathbf{E}) - (\mathbf{A}\circ\mathbf{J} - \rho\phi) \equiv d(A\hat{}G)$$

while the closure of $A\hat{}F$ yields the second Poincare invariant.

$$P2 := 2(\mathbf{E}\circ\mathbf{B}) \equiv d(A\hat{}F).$$

When these measures vanish (the divergences of the 4-vectors vanish), then there exist separate topological conservation laws (of Spin and Helicity)

For the above example with the assumption that $\mathbf{D} = \varepsilon\mathbf{E}$ and $\mathbf{B} = \mu\mathbf{H}$

$\mathbf{A} = [0, 0, (x^2 + y^2)/2], \quad \phi = z$. Then

$\mathbf{B} = [y, -x, 0]$

$\mathbf{E} = [0, 0, -1]$

$\mathbf{A}\circ\mathbf{B} = 0,$

$\mathbf{T}_4 = [zy, -zx, 0]$

$P2 = \mathbf{E}\circ\mathbf{B} = 0,$

$$\mathbf{A} \circ \mathbf{D} = -\varepsilon(x^2 + y^2)/2$$
$$\mathbf{S}_4 = [x(x^2 + y^2)/2, y(x^2 + y^2)/2, -\varepsilon\mu z, -\varepsilon\mu(x^2 + y^2)/2]/\mu$$
$$P1 = (2(x^2 + y^2) - \varepsilon\mu)/\mu$$
$$\mathbf{J} = [0, 0, -2/\mu]$$
$$\rho = 0$$

which implies that Helicity is conserved but Spin is not conserved in the example field.

2. Link Integrals (a 2-form) and Obstructions

The basic issue is that not all divergence free fields are exact. It is true that all divergence free fields in a 3 dimensional *euclidean topology* are exact, but that is precisely where the topological features enter into the picture. A euclidean topology is simply connected and without obstructions.

In 3-dimensions (and with euclidean dogma) there are two species of 3 component fields. Even in euclidean domains every one learns that the 3 vector of angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ is never to be added to the 3 vector of momentum, $\mathbf{p}$. They are "two different species" of vectors. However, with regard to non-euclidean topological domains there is also another concept defined as a vector density. There is a topological difference even in 3D between and a covariant tensor and a contravariant tensor density, but not if volume deforming processes are excluded (the typical non-dissipative case). As an example consider a C1 map onto a euclidean domain:

$$\phi : \{\xi^1, \xi^2, \xi^3\} \Rightarrow \{x, y, z\} = \phi^j(\xi^k)$$

with a Jacobian matrix defined by the equations

$$d\phi : \{d\xi^1, d\xi^2, d\xi^3\} \Rightarrow \{dx, dy, dz\} = [\mathbf{J}] \circ |d\xi\rangle = [\partial\phi^j/\partial\xi^k]|d\xi^k\rangle$$

Construct a 3 component vector field $\mathbf{Z}$ on $\{x, y, z\}$

$$\mathbf{Z} = [U(x, y, z), V(x, y, z), W(x, y, z)]$$

Now ask what is the pre-image of $\mathbf{Z}$ on $\{\xi^1, \xi^2, \xi^3\}$. The functional form of the pre-image is not unique! There are two tensor fields whose functional forms are well defined by functional substitution; one is a contravariant tensor density, and the other is a covariant tensor, on $\{\xi^1, \xi^2, \xi^3\}$. The two pre-images can be represented by the formulas,

$$Contra-variant\ tensor\ density\ \ \left|\mathbf{D}^k\right\rangle_{pb} = [\partial\phi^j/\partial\xi^k]^{adjoint} \circ |\mathbf{Z}\rangle,$$

and

$$Co-variant\ tensor\ \ |\mathbf{A}_k\rangle_{pb} = [\partial\phi^j/\partial\xi^k]^{transpose} \circ |\mathbf{Z}\rangle,$$

where *pb* means pullback from $\{x, y, z\}$ to $\{\xi^1, \xi^2, \xi^3\}$ by means of functional substitution (without the use of an inverse map, which may not exist globally). Note that the arguments made above are still valid even if the initial domain is of higher dimension than the final domain (say $\{\xi^1,\xi^2,\xi^3,\xi^4\} \Rightarrow \{x,y,z\}$ is a projection onto). The pre-images are well defined in a functional sense even if the topology of the initial state is different from the topology of the final state. The map ϕ need not have an inverse!

The notation above is deliberate, for it distinguishes the electromagnetic Intensities, **E** (as components of a covariant tensor deduced from $\mathbf{A}_k$) from the electromagnetic Quantities (or excitations) **D**, **H** (as components of a contravariant tensor density).

The assumption of a euclidean domain masks these topological features. The topological closure of $|\mathbf{D}\rangle$ is the concept of zero divergence; the topological closure of $|\mathbf{A}\rangle$ is a zero curl concept. In 3D, for c2 differentiable fields where $|\mathbf{B}\rangle = curl\,|\mathbf{A}\rangle$, it follows that the closure of the 2-form generated from the components of $|\mathbf{B}\rangle$ is always empty, in a global manner! The closure of $|\mathbf{D}\rangle$ is not globally empty!.

However it is possible to define integrating factors, λ, for *any* $|\mathbf{Z}\rangle$ such that

$$|\mathbf{D}_c^k\rangle = [\partial\phi^j/\partial\xi^k]^{adjoint} \circ |\mathbf{Z}/\lambda\rangle$$

is closed, *but not necessarily in a global manner*. That is, the two form with components $\mathbf{D}_c$ is closed but not exact :

$$d(D_c) = d(D_c^z dy\wedge dz - D_c^y dx\wedge dz + D_c^z dx\wedge dy)/\lambda) = (div\mathbf{D}_c^k)d\xi^1\wedge d\xi^2\wedge d\xi^3 = 0$$

Any homogeneous function of the Holder type and of degree n (with any anisotropic signature defined by constants a,b,c, and any exponent p) will act as integrating divisor for any 3D vector **Z** when n=3. (The theorem is also valid in dimension n)

$$\textbf{Theorem}:\quad If \quad \lambda(U,V,W) = \{aU^p + bV^p + cW^p\}^{3/p} \quad then \quad div(\mathbf{Z}/\lambda) = 0$$

Although the 2-form with components $|\mathbf{B}\rangle$ is closed and exact, the 2-form with components $|\mathbf{D}_c\rangle$ is closed, BUT NOT EXACT. The fundamental idea is that for a non-bounding closed cycle (*nbcc*) (such as a closed twisted ribbon),

$$\iint_{nbcc} \mathbf{B} \circ d(Area) = 0, \qquad but \quad \iint_{nbcc} \mathbf{D} \circ d(Area) \neq 0$$

where for a boundary (such as toroidal surface)

$$\iint_{boundary} \mathbf{B} \circ d(Area) = 0, \qquad and \quad \iint_{boundary} \mathbf{D} \circ d(Area) = 0.$$

If the integration chain is a closed in the sense of cycle, and is not a boundary, then there must exist points of the integration domain which must be excluded. These points form the topological defects (the point charges in EM theory or "topological holes") or the topological obstructions that are of interest to the theory of Links and Braids. In particular, the theory of links depends upon such obstructions and is represented by integrals of the form:

$$Lk = \iint_{nbcc} (D^z dy\wedge dz - D^y dx\wedge dz + D^z dx\wedge dy)/\lambda = \iint_{nbcc} G \neq 0, \quad dG = 0.$$

and should have nothing to do with magnetic flux,

$$\Phi_m = \iint_{nbcc} (B_x dy\wedge dz - B_y dx\wedge dz + B_z dx\wedge dy) = \iint_{nbcc} F = 0, \quad dF = 0,$$

which has no obstructions, as the integrand is globally exact. If F was to have obstructions, the pre-images global postulate of potentials $F - dA = 0$ must fail, and the conservation of flux would not be true. Such a failure implies the existence of magnetic monopoles (the obstructions to F being globally exact). My personal view (along with E.J.Post and many others) subsumes that the failure to detect magnetic monopoles is proof that classical electromagnetism is defined by the postulate of potentials; i.e., $F - dA = 0$, globally. On the other hand, the 2-form of field excitations, G, is not exact.

Example 1. **The Gauss Linking integral**

As an example of the Gauss integral, Lk, consider the case where the displacement vector is the difference of two position vectors to two separate space curves. Define

$$\mathbf{Z} = (\mathbf{R}_2 - \mathbf{R}_1) \quad \mathbf{R}_2 = [x_2, y_2, z_2] \quad \mathbf{R}_1 = [x_1, y_1, z_1]$$
$$\lambda = (a(x_2 - x_1)^p + b(y_2 - y_1)^p - +c(z_2 - z_1)^p)^{3/p}$$

where $\mathbf{R}_1$ defines the position vector to one field of space curves, and $\mathbf{R}_2$ defines the position vector to a second field of space curves. Space curves from different families can have different parametrizations. Hence, the vector $\mathbf{Z}$ represents the vector difference of points on two different space curves which cannot be synchronized parametrically. Next assume that the displacements of interest are constrained by two parametric curves given by the exterior differential system

$$d\mathbf{R}_1 - \mathbf{V}_1 dt = 0 \quad and \quad d\mathbf{R}_2 - \mathbf{V}_2 dt' = 0,$$

where the parameters dt and dt' are not functionally related, such that

$$dt\wedge dt' \neq 0, \; but \; dt\wedge dt = 0 \; and \; dt'\wedge dt' = 0.$$

The vector $\mathbf{D}$ can be interpreted as the displacement vector between points on the space curve C1 parametrized by t, and the points on another space curve C2 parametrized by t'. The integral to be evaluated is

$$Lk = \iint_{closed} \Gamma = \iint i(\mathbf{Z}/\lambda) d(x_2 - x_1)\wedge d(y_2 - y_1)\wedge d(z_2 - z_1)$$

From Stokes theorem, if the 2 dimensional integration domain is a boundary, then the Link integral vanishes. However, if a particular integration chain is a closed cycle (not a boundary) then the linking integral has values with rational ratios. The closed integrals are deRham period integrals in two dimensions. Points where $\mathbf{D}$ vanishes are excluded.

It is important to realize that kinematic constraints are topological constraints on the domain that refine the Cartan topology. Direct substitution of these differential expressions for $\{dx_{1.....}dy_3\}$ into the expression for the 2-form G, using $dt\^dt' \neq 0$, *but* $dt\^dt = 0$ and $dt'\^dt' = 0$, leads to the classic (1833) triple vector product representation for the Gauss Linking integral for two closed curves. (The formula as stated below is for the isotropic euclidean case, a=b=c=1, p=2, but the general formula works for any p and any anisotropy)

$$Lk = \iint_{closed} G = \oint_t \oint_{t'} \{(\mathbf{R}_2 - \mathbf{R}_1)\circ\mathbf{V}_1 \times \mathbf{V}_2\} dt\^dt'/\lambda$$

$$\lambda = (\mathbf{R}_1 \circ \mathbf{R}_1 - 2\mathbf{R}_1 \circ \mathbf{R}_2 + \mathbf{R}_2 \circ \mathbf{R}_2)^{3/2}.$$

When the two curves are distinct, the integration is over the two bounding cycles of a closed ribbon. The ribbon surface is closed but it is not a boundary of any volume. The the two non-intersecting cycles (that form the boundary of the ribbon area) are defined by the two distinct parameters, dt, and dt'. When integrations are computed along these closed curves whose tangent vectors are $\mathbf{V}_1$ and $\mathbf{V}_2$, then the integer values of the closed integral may be interpreted as how many times the two curves are linked. The interpretation of the closed surface integral as a orientable ribbon works if the triple product divided by lambda does not change sign as t and t' are varied. If the integrand changes sign, then the ribbon in non-orientable.

The constraint that $dt\^dt' \neq 0$ implies that the "motion" along the curve generated by $\mathbf{R}_1$ is independent of the "motion" along the curve generated by $\mathbf{R}_2$. If the curve generated by $\mathbf{R}_1$ is a conic in the *xy* plane and the curve generated by $\mathbf{R}_2$ is a conic in the *xz* plane, then the surface swept out by the vector **D** is a Dupin cyclide. Such surfaces have application to the propagation of waves in electromagnetic systems.

Example 2: **Flat tangential developables**

From another point of view, consider the ruled surface defined by the vector field of two parameters, $\{t, \mu\}$ (isotropic, a=b=c=1 , p=2). The ruled surface will be defined by the position vector $\mathbf{R}(t)$ to a space curve and a ruling parameter μ times the tangent Velocity vector to the space curve, $\mathbf{V}(t)$.

Use the general methods above to create the doubly parametrized divergence free vector field:

$$\mathbf{Z}(\mu, t) = \{\mathbf{R}(t) \pm \mu\mathbf{V}(t)\ \}$$

$$\lambda(\mu, t) = (\mathbf{R} \circ \mathbf{R} \pm 2\mu\mathbf{R} \circ \mathbf{V} + \mu\mathbf{V} \circ \mu\mathbf{V})^{3/2}.$$

Vector fields of this type are primitive examples of "strings" for fixed values of the parameter, t, and string parameter, μ. Direct substitution of the physical constraints, $d\mathbf{R} - \mathbf{V}dt = 0$, and $d\mathbf{V} - \mathbf{A}dt = 0$, such that $d\mathbf{Z} = d\{\mathbf{R}(t) \pm \mu\mathbf{V}(t)\ \}$ into the definition of the linking integral

$$\iint_{closed\, on\, N} i(\mathbf{Z}/\lambda) dZ^1\^dZ^2\^dZ^3$$

leads to yet another realization and interpretation of the Gauss formula:

$$Q = \iint_{closed} G = \iint_{closed\,on\,\mu t} \{\mathbf{R} \circ \mu\mathbf{V} \times \mathbf{A}\} dt\hat{}d\mu/\lambda$$

$$= \iint_{closed} \{\mathbf{A} \circ \mathbf{R} \times \mu\mathbf{V}\} dt\hat{}d\mu/(\mathbf{R} \circ \mathbf{R} \pm 2\mu\mathbf{R} \circ \mathbf{V} + \mu\mathbf{V} \circ \mu\mathbf{V})^{3/2}.$$

It is apparent that the interaction of the ”angular” momentum, $\mathbf{L} = \mathbf{R} \times \mu\mathbf{V}$, and the acceleration, **A**, produces a topological invariant whose values are ”quantized” (in the sense that the ratios of the closed integrals are rational). Note that the triple vector product of the integrand numerator is proportional to the Frenet torsion of the orbit. For an orbit that is planar the Frenet torsion is zero everywhere, and the Gauss integral vanishes. In this sense the Gauss integral is more related to spin values, $A\hat{}G$, rather that to helicity values, $A\hat{}F$.

Recall that if a the space curve is an edge of regression, then the ruled surfaces according to the forward and backward motions are not same to second order. Such a result is an obvious distinction between forward and backward motion that breaks time reversal symmetry. Such effects have been observed in dual polarized ring lasers.

Example 3. **Scrolls**

The ruled surface described above is closely related to the ruled surface known as the tangential developable. Such ruled surfaces (parametrized by arc length *s*, with the directrix in the direction of the unit tangent vector, and multiplied by μ) have zero Gauss curvature. Though bent, they can be rolled out flat. By constructing the ruled surfaces in terms of the normal and/or binormal to a space curve, other forms of ruled surfaces yield negative values for the Gauss integral, and are not ”flat”. They are defined as Scrolls.

Of particular interest to physics are those ruled surfaces of negative Gauss curvature, which are also minimal surfaces. These objects are double stranded ribbon configurations for constant fixed values of μ. The equations for the ruled surface of a scroll, with a directrix in the direction of the binormal, $\mathbf{b}(s)$, is :

$$\mathbf{D}(\mu, s) = \{\mathbf{R}(s) \pm \mu\mathbf{b}(s)\}/\lambda$$

$$\lambda = (\mathbf{R} \circ \mathbf{R} \pm 2\mu\mathbf{R} \circ \mathbf{b} + \mu\mathbf{b} \circ \mu\mathbf{b})^{3/2}.$$

When the parameter μ takes on the constant values $\mu = \kappa/\tau^3$, the ruled surface is a minimal surface, and the binormal field twists about the space curve generated by $\mathbf{R}(s)$.

Another interesting scroll is that generated by the Darboux vector.

$$\mathbf{D}(\theta, s) = \{\mathbf{R}(s) \pm (\mathbf{n}(s)\cos(\theta) + \mathbf{b}(s)\sin(\theta)\}/\lambda$$

$$\lambda = (\mathbf{D} \circ \mathbf{D})^{3/2}.$$

which seems to be of interest to Longcope.

The Torsion 3-form and the Braid integral

For n = 4 the same procedures described above may be used to produce a period integral over a closed 3-dimensional domain. The technique is to define a 4 dimensional vector field, $\mathbf{Z} = [Z_1, Z_2, Z_3, Z_4]$. Use the general renormalization function,

$$\lambda = \{\alpha Z_1^p + \beta Z_2^p + \gamma Z_3^p + \epsilon Z_4^p\}^{n/p}$$

and set n=4, for zero four divergence. Construct the closed 3 -form,

$$\Gamma = i(\mathbf{Z}/\lambda) dZ_1 \hat{} dZ_2 \hat{} dZ_3 \hat{} dZ_4$$

Assume the 4 component vector has a realization as $\mathbf{Z} = \mathbf{P_1} + \mathbf{P_2} + \mathbf{P_3}$, where the three independent fields $\mathbf{P}$ represent three space-time curves that obey the kinematic constraints:

$$d\mathbf{P_1} - \mathbf{f_1} ds = 0, \quad d\mathbf{P_2} - \mathbf{f_2} ds' = 0, \quad d\mathbf{P_3} - \mathbf{f_3} ds'' = 0.$$

Substitute for each of the differentials in Γ (and further assume that the domain {x,y,z,t} of interest is further constrained such that $dt = 0$) to yield the three form

$$G = \{\mathbf{f}_1 \circ (\mathbf{f}_2 \times \mathbf{f}_3)\} ds \hat{} ds' \hat{} ds''/\lambda$$

$$\lambda = \{\alpha Z_1^p + \beta Z_2^p + \gamma Z_3^p + \epsilon Z_4^p\}^{4/p}$$

The spatial braid integral becomes equal to

$$Br := \oint_t \oint_{t'} \oint_{t''} \mathbf{f}_1 \circ (\mathbf{f}_2 \times \mathbf{f}_3)\} ds \hat{} ds' \hat{} ds''/\lambda$$

The integrations are now over three closed curves whose tangents are the ”Newtonian forces”, **f**, on three ”particles”. Where in the two dimensional Gauss integral, of the previous section, the evaluation was along the closed curves of a ribbon, in this case the integrations are along the closed trajectories of three ”particles” which form the vertices of a triangle.

The idea that three ”lines” are used to form the integral (whose values form rational ratios) is the reason that this topological integral in the format given above is defined as the braid integral. Of course the three form of topological torsion is a variant of the braid integral, but applies to those topologies where the system is not reducible to three factors dt, dt' and dt”. An example of a period 3 braid with Braid integral zero (chaotic) and Braid integral 2 (non-chaotic) is given in Figure 5. It is illuminating to construct the two braids by wrapping a long flat ribbon of paper smoothly around the palm of your hand. Close the ribbon surface by pasting the ends together. Then make another example, where this time thread the loose end underneath the middle wrap, rather than over the middle wrap, before gluing the ends together. Take the two examples from your hand and note that one is continuously deformable into a closed cylinder (Tw = zero) while the other has a 4 pi twist (Tw =2). What is surprising is that it is the Tw =0 configuration that has a chaotic neighborhood, while the Tw = 2 structure is not chaotic. To test for chaos construct the equivalent of the closed braid form copper tubes. Then link any pair of tubes with a large loop of elastic or thread. Push the looping thread around the period three copper tube, and note that for a Tw=2 configuration, the looping thread becomes untangled after 6 pi revolutions about the central axis. For the Tw =0 configuration, the looping thread never unwinds, but becomes more and more twisted and complex.

The equivalent to this Figure, and the fact that there are two distinct period 3 configurations one chaotic and one non-chaotic, was brought to my attention during a stimulating lecture given by J. Los at the August, 1991, Pedagogical Workshop on Topological Fluid Mechanics held at the Institute for Theoretical Physics, Santa Barbara UCSB.

It is to be noted that the 3-form of topological torsion is related to the braid integral, a three dimensional thing in four dimensions, and not the Gauss linkage integral, which is a two dimensional thing in three dimensions.

Summary

It is remarkable to me that the solar plasma physics community forces a connection between Helicity and Links, when without further topological constraints there is no clear explicit relationship between the linkage of magnetic field lines **B** and the concepts of magnetic helicity. They two concepts are related to objects of different topological dimension. That is, the exact 2-form, F, that generates a **B** field is not topologically equivalent to the non-exact 2 form, G, that generates a **D** field, unless further topological constraints are put on the domain. The 2-form G is not of the same topological dimension as the 3-form of helicity, $A\hat{}F$. The 3-divergence of $\mathbf{B} = curl\,\mathbf{A}$ is always zero, while the 3-divergence of **D** is not. On the other hand, **D** always admits an integrating factor to yield a zero divergence, but **A** does not always admit an integrating factor to yield zero curl.

The two pre-images ($\mathbf{D}_{pb}$ *and* $\mathbf{A}_{pb}$ of $\mathbf{Z}/\lambda$) are such that $div(\mathbf{D}_{pb}) = 0$ for any Holder type divisor, λ, which is homogeneous of degree n. Also note that $curl\,\mathbf{A}_{pb} = 0$, for the special isotropic case of p=2, a=b=c=1, but $\mathbf{A}_{pb} \circ curl\,\mathbf{A}_{pb} = 0$, for all other values of p, and a,b,c.

The second thing that causes some uneasiness for me, is that for rotating plasma systems like the sun, it would appear that the solar community presumes the constitutive relations between the Intensities and the Excitations are those of a Lorentz vacuum. Such MHD approximations are only valid for parallel translations, while it is known that rotations add a chiral like splitting to the constitutive relations. Could it be that the chiral features of the sun so sought after are related to this oversight? For rotating media, the Constitutive Tensor density does not have the symmetries of the Lorentz frame. (See Jan Post’s article on the Sagnac effect.)

In addition, I see no utilization by the plasma community of the two distinct 3-forms of Spin A^G, and Torsion, A^F. Both 3-forms carry different topological information.

RMK

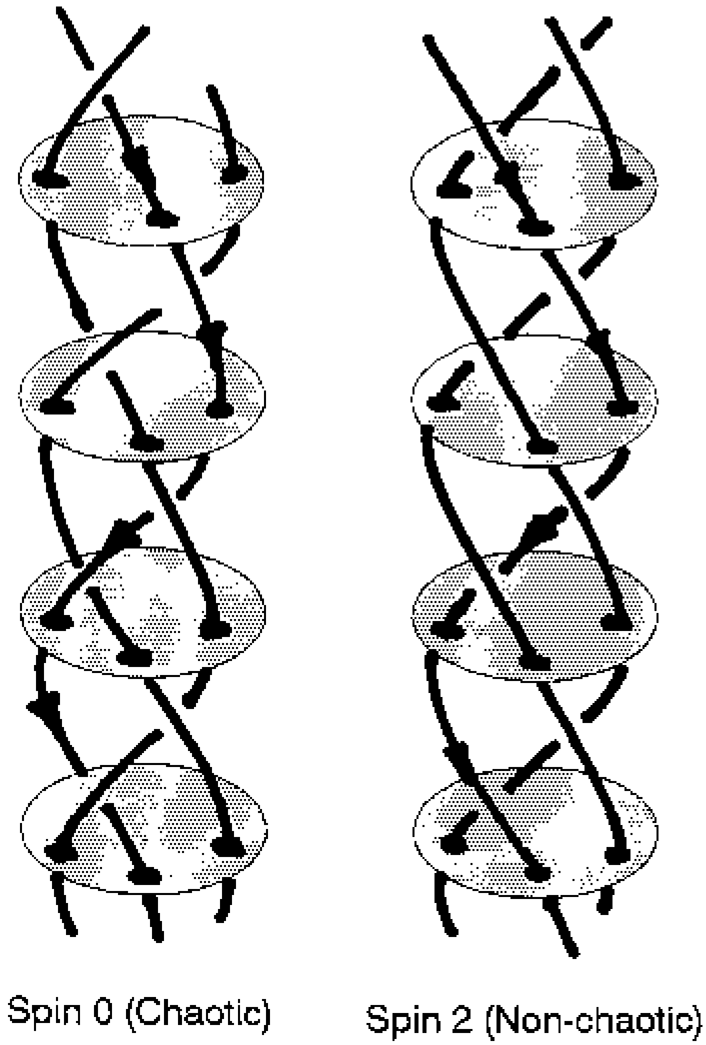

Figure 5. Period 3 Braids

You may be interested in:

Cartan's Corner:

Adventures is Applied Topology – Topological Torsion -Helicity, Topological Parity concepts applied to Electromagnetism (including plasmas), Hydrodynamics, and Thermodynamics. Torsion vs. Spin. Irreversible Processes. Anholonomic fluctuations.

Topology and Topological Evolution of Classical Electromagnetic Fields and Currents: (**Complete Article for Chapman Poster**)

Abstract The theory of classical electromagnetism can be deduced from two topological constraints placed on the variety of independent variables $\{x, y, z, t\}$. The topological constraints are formulated in terms of the exterior differential systems, $F - dA = 0$, and $J - dG = 0$. These topological constraints imply that the domains of support for finite non-zero electromagnetic field intensities, and finite non-zero electromagnetic currents, in general, cannot be compact without boundary. These two fundamental constraints lead to the independent concepts of topological torsion, $A \hat{} F$, and topological spin, $A \hat{} G$, with exterior derivatives that demonstrate that the two Poincare invariants of an electromagnetic system are topological properties. The zero sets of each Poincare 4-form can be used to define the concepts of transverse magnetic and transverse electric waves on topological grounds. The direction fields of the 3-forms $A \hat{} F$ and $A \hat{} G$ can exhibit linking and separation into component domains. The possible evolution of these topological properties is studied with respect to classes of processes that can be defined in terms of singly parameterized vector fields. Non-zero values of the Poincare invariants are the source of topological change and non-equilibrium thermodynamics.

http://www.uh.edu/~rkiehn/pdf/classice.pdf

The Chiral Vacuum:

Abstract. The consequences of modifying the constitutive equations that describe the classical Lorentz Vacuum to include a chiral term in the format, $\mathbf{D} = \varepsilon_0 \mathbf{E} + [\gamma] \circ \mathbf{B}$ and $\mathbf{H} = \mathbf{B}/\mu_0 - [\gamma^{\ddagger}] \circ \mathbf{E}$ are studied. Wave solutions to the Maxwell Faraday and the Maxwell Ampere equations can be found which are free from real charge densities and current densities, and therefor appear to define a Chiral Vacuum state. The assumption of a simple complex scalar form for chiral constitutive matrix, $[\gamma] = (g + i\gamma)$, leads to cases where the only detectable difference between the Chiral vacuum and the Lorentz vacuum is to be found in the value for radiation impedance, Z, a value which depends on the chiral coefficients g and γ, as well as the ratio $\sqrt{\mu_0/\varepsilon_0}$, through the determinant of the constitutive matrix.

http://www.uh.edu/~rkiehn/pdf/chiral.pdf

ElectroMagnetic Waves with Torsion (Helicity) and Spin

Abstract. New time dependent wave solutions to the classical homogeneous Maxwell equations in the vacuum have been found. These waves are not transverse; they exhibit both torsion and spin; they have finite magnetic helicity, $\mathbf{A} \circ \mathbf{B} \neq 0$, a non-zero Poynting vector, $\mathbf{E} \times \mathbf{H} \neq 0$, and a non zero second Poincare invariant, $\mathbf{E} \circ \mathbf{B} \neq 0$. Two four component rank 3 tensors, constructed on topological grounds in terms of the Fields and Potentials, are used to define the concepts of torsion and Spin, even in domains with plasma

currents. The divergence of the spin pseudo vector generates the Poincare invariant equivalent to the Lagrangian of the field, $(\mathbf{B} \circ \mathbf{H} - \mathbf{D} \circ \mathbf{E}) - (\mathbf{A} \circ \mathbf{J} - \rho\phi)$. The divergence of the Torsion pseudo vector generates the second Poincare invariant, $2\mathbf{E} \circ \mathbf{B}$. The Poincare invariants have closed integrals which are deformation invariants, and therefore can be used to define deformable coherent structures in a plasma. When the second Poincare invariant is non-zero, there can exist solutions that are not time-reversal invariant.

http://www.uh.edu/~rkiehn/pdf/helical6.pdf