CONTINUOUS TOPOLOGICAL EVOLUTION AND THERMODYNAMICS

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Examples and details at

http://**www.uh.edu**/~**rkiehn**

Cartan’s Corner on CD rom

is available at this conference

Primary Results

- **Differential Topology can be used to determine when a flow process is irreversible - without statistics**!
- **Cartan's exterior calculus describes Continuous Topological Evolution including Anholonomic Fluctuations**.
- **Continuous Topological Evolution can be used to describe the decay**, **but not the creation**, **of turbulence**.
- **Cartan's Magic formula of Continuous Topological Evolution is a dynamical equivalent to the First Law of Thermodynamics**.
- **Thermodynamic Irreversibility is an artifact of 4 topological dimensions**.

Cartan’s Magic Formula

In Cartan’s Calculus, a Physical system is represented by a 1-form of Action:

$$\mathit{A} = A_\mu dx^\mu - \phi dt \quad \approxeq \quad p_\mu dq^\mu - hdt$$

A process is represented by a vector field:

$$\mathbf{V} = [\mathbf{v}, 1] \quad \text{relative to } \{x,y,z,t\}$$

Evolution of the Action, A, relative to the process, $\mathbf{V}$, is described by Cartan’s Magic Formula using the Lie derivative $\mathbf{L_V}(\boldsymbol{A}) = \mathbf{i}(\mathbf{V})\mathbf{d}\boldsymbol{A} + \mathbf{d}(\mathbf{i}(\mathbf{V})\boldsymbol{A}) \Rightarrow \boldsymbol{Q}$.
Rewriting:

$$\boldsymbol{L}_V(\boldsymbol{A}) = \boldsymbol{W} + \boldsymbol{d}(\boldsymbol{U}) = \boldsymbol{Q}$$

Cartan’s Magic Formula is a dynamical equivalent to the First Law of Thermodynamics!

Note independence from a metric or a connection!

An Electromagnetic Example

The Action 1-form is composed from the electromagnetic vector and scalar potentials:

$$A = \mathbf{A} \circ d\mathbf{r} - \phi dt$$

with

$$\mathbf{B} = curl\,\mathbf{A} \quad \text{and} \quad \mathbf{E} = -\partial\mathbf{A}/\partial t - \nabla\phi$$

Then the virtual work $W = i(\mathbf{V})dA$ becomes:

$$W = \{\mathbf{E} + \mathbf{v} \times \mathbf{B}\} \circ d\mathbf{r} - (\mathbf{E} \circ \mathbf{v})dt$$

where the spatial part is the Lorentz force times the differential displacement - the classical definition of differential work.

If $W = 0$ then the plasma is ”Force Free”

To a Topologist

The First Law is a statement of Cohomology

$$\mathbf{Q - W = dU}$$

Definition: A process that produces a 1-form of Heat, Q, is reversible when Q admits an integrating factor. Then,

$$\mathbf{Q = TdS}, \text{ and } \mathbf{Q\hat{}dQ = 0}$$

It follows that the 3-form

$$\mathbf{W\hat{}dW = dU\hat{}dT\hat{}dS}$$

and

$$\mathbf{d(W\hat{}dW) = 0}$$

$\therefore$ Reversible work, W, is of Pfaff dimension ≤ 3

It follows that for an

IRREVERSIBLE PROCESS

$$d(W \wedge dW) \neq 0$$

Which implies that the Work 1-form is of

Pfaff Dimension ≥ 4

CONCLUSIONS:

- **2D+1 Turbulence is a myth**
- **Irreversibility is an artifact of $\geq$4D**

Cartan's Magic Formula

of

Topological Evolution

establishes a non-statistical link between

Thermo and Dynamical Systems

and may be used to test if a

Dynamical System, V,

represents a(n)

(ir)Reversible Process

Use Cartan's Magic formula to compute $Q\hat{}dQ = L_{(V)}A\hat{}L_{(V)}dA$ for a given physical system, A, and a given process, V. The process is irreversible if

$$L_{(V)}A\hat{}L_{(V)}dA \neq 0.$$

Anholonomic Fluctuations

are defined as the deviations from kinematic perfection. They carry topological content.

$$\Delta^{\mu} = (dq^{\mu} - V^{\mu}dt) \neq 0.$$

Cartan's Magic formula of topological evolution can handle anholonomic fluctuations:

Consider a Physical system represented by a Lagrange 1-form of Action:

$$A = L(q^{\mu}, V^{\mu}, t)dt + \quad p_{\mu} \circ (dq^{\mu} - V^{\mu}dt)$$

For now, treat the p_{μ} as Lagrange multipliers

QUESTION

What is the Pfaff topological dimension D (the minimum number of independent functions) of this action 1-form, A? At first glance A has **3N**+**1** independent variables.

but **Surprise** ! $\mathbf{D = 2N + 2}$

Anholonomic fluctuations
produce symplectic manifolds

of even dimension which admit a

unique **Torsion direction field**

that describes an **irreversible** process.

HOWEVER, if the Lagrange multipliers are constrained to be

CANONICAL MOMENTA, **such that**

$$p_\mu - \partial L(q^\mu, V^\mu, t)/\partial V^\mu = 0$$

Then

$$\mathbf{D = 2N + 1}$$

and the physical manifold is an odd dimensional **Contact Manifold** (state space) and admits a

Unique Extremal direction field

that is thermodynamically **Reversible**.

IN THIS SENSE

ANHOLONOMIC FLUCTUATIONS are the TOPOLOGICAL SOURCE of THERMODYNAMIC IRREVERSIBILITY

EQUATIONS OF MOTION
and topological
EQUIVALENCE CLASSES
of PROCESSES

Topological properties of the 1-form of Virtual Work $W = i(V)dA$, form 2 categories of processes.

Either dW $=$ 0 or dW $\neq$ 0.

THEOREM: All processes such that **dW $=$ 0** are thermodynamically reversible.

The category $dW = 0$ includes extremal, Hamiltonian, Bernoulli-Casimir and all Symplectic processes.

Proof:

$$\mathbf{L}_{(V)}\mathbf{dA} = \mathbf{dW} + \mathbf{0} = \mathbf{dQ}$$

If $dW = 0$, it follows that

$$\mathbf{L}_{(V)}\mathbf{A}\hat{}\mathbf{L}_{(V)}\mathbf{dA} = \mathbf{Q}\hat{}\mathbf{dQ} = \mathbf{0}.$$

Hence an integrating factor exists for the Heat 1-form Q and the process is thermodynamically reversible

Classes of

Reversible Processes

- **Extremal -** Unique Hamiltonian:

  Virtual Work is zero.$\mathbf{W} = \mathbf{i}(\mathbf{V})\mathbf{dA} = \mathbf{0}$

  1. Potential flows (no lift)

  $\supset\ dA = 0,\ \oint A = 0$ no lift.

  2. Joukowski flow

  $\supset\ dA = 0,\ \oint A \neq 0$

  Circulation with translation $\Rightarrow$ lift

  3. Lamb Flows $\supset A\hat{}dA = 0$

  No Helicity

- **Bernoulli-Casimir**-Hamiltonian:

  Virtual Work is exact, Cyclic work is zero (Eulerian fluid, barotropic fluids).

  $$\mathbf{W} = \mathbf{i}(\mathbf{V})\mathbf{dA} = \mathbf{d}\Theta, \qquad \oint W = 0$$

- **Helmholtz-Symplectic**: Virtual Work is closed, but cyclic work is not zero!

  $$\mathbf{dW} = \mathbf{0}, \quad \oint W \neq 0$$

IRREVERSIBLE PROCESSES
and the
TOPOLOGICAL TORSION VECTOR

Irreversible processes exist only on domains that support a disconnected Cartan topology,

$$\mathbf{A\hat{\ }dA} \neq \mathbf{0},$$

For thermodynamic irreversibility:

$$\mathbf{dW\hat{\ }dW} \neq \mathbf{0} \supset \mathbf{dA\hat{\ }dA} \neq \mathbf{0}.$$

$\therefore$ The irreversible Action Manifold of support is symplectic of rank 4.

On the symplectic domain there exists a unique vector direction field, $\mathbf{T}_4$**,** with components such that

$$\mathbf{i}(\mathbf{T}_4)\mathbf{dx\hat{\ }dy\hat{\ }dz\hat{\ }dt} = \mathbf{A\hat{\ }dA}$$

As $\mathbf{dA\hat{\ }dA} = \{\mathbf{div}_4\mathbf{T}_4\}\mathbf{dx\hat{\ }dy\hat{\ }dz\hat{\ }dt} \neq \mathbf{0}$

this vector field has a non-zero divergence almost everywhere except on (defect) regions where the manifold is no longer symplectic. ($D \neq 4$)

Define (on the symplectic 4D domain)

$$\Gamma = \mathbf{div}_4 \mathbf{T}_4 \neq \mathbf{0}$$

Then it is possible to show that

$$\mathbf{W} = \mathbf{i}(\mathbf{T}_4)\mathbf{dA} = \Gamma \bullet \mathbf{A}$$

$$\mathbf{L}_{(\mathbf{T}_4)}\mathbf{A} = \Gamma \bullet \mathbf{A}$$

and

$$\mathbf{dQ \hat{} dQ} = \Gamma^2 \bullet \mathbf{dA \hat{} dA}$$

Hence, Evolution in the direction of the Topological Torsion vector is

Thermodynamically

IRREVERSIBLE

The irreversible evolution decays on a symplectic manifold (2n+2) to an attractor which is a contact manifold (2n+1)

Examples of $\mathbf{T}_4$

ELECTROMAGNETISM

The Action 1-form of potentials:

$$A = \mathbf{A} \circ d\mathbf{r} - \phi dt$$

with

$$\mathbf{B} = curl\,\mathbf{A} \quad \text{and} \quad \mathbf{E} = -\partial\mathbf{A}/\partial t - \nabla\phi$$

forming the components of dA.

By direct computation,

$$\mathbf{T}_4 = [\mathbf{E} \times \mathbf{A} + \mathbf{B}\phi, \ \ \mathbf{A} \circ \mathbf{B}]$$

$$\Gamma = \mathbf{div}_4\mathbf{T}_4 = 2\ \mathbf{E} \circ \mathbf{B} \neq \mathbf{0}$$

Note 1: The Helicity density $\mathbf{A} \circ \mathbf{B}$ is the 4th component of the Topological Torsion Vector, $\mathbf{T}_4$.

Note 2: Γ is the second Poincare Invariant for an electromagnetic system.

Examples of $\mathbf{T}_4$

HYDRODYNAMICS

Consider an Action 1-form of potentials:

$$A = \mathbf{v} \circ d\mathbf{r} - Hdt$$

with H defined as

$$H = (\mathbf{v} \circ \mathbf{v}/\mathbf{2} - \lambda div\mathbf{v} + \int dP/\rho) = \mathbf{v} \circ \mathbf{v} - L$$

and Vorticity defined as $\omega = curl\mathbf{v}$.

The equivalence class of solutions that satisfy the topological constraint on the virtual work W,

$$W = i(V)dA = \nu\{curlcurl\ \mathbf{v} \circ (d\mathbf{r} - \mathbf{v}dt)\},$$

are solutions to the Navier Stokes equations of motion.

By direct computation,

$$\mathbf{T}_4 = [h\mathbf{v} - L\omega - \nu\ curl\ \omega,\ h]$$

where $h = helicity = \mathbf{v} \circ \omega$

and

$$\Gamma = \mathbf{div}_4\mathbf{T}_4 = 2\nu\ \{\omega \circ curl\,\omega\} \neq \mathbf{0}$$

Hence, for a Navier Stokes fluid,domains where the vorticity field does not satisfy the Frobenius integability condition

$$\omega \circ\ curl\ \omega \neq 0$$

are domains of thermodynamic irreversibility, and are therefore candidates for turbulent flow.

More detail and references can be found at

CARTAN's CORNER

http://www.uh.edu/~rkiehn

Other topics on Cartan's Corner

http://www22.pair.com/csdc/car/carhomep.htm

- **The Skidding Bowling Ball**.

An example of anholonomic irreversible decay from a symplectic manifold to a contact manifold.

- **A Solution to Maxwell's electrodynamics that is not time reversal invariant**.

A wave solution for t
which is not a wave solution
if $t \Rightarrow -t$

- **2-D Turbulence is a myth**.
- **Topological Torsion Waves**.

The Ubiquitous van der Waals Gas.

Three dimensional dynamical systems can be deformed into a universal van der Waals gas, with a spinodal and binodal line and a critical point. The singularities are determined from differential topology properties of the similarity invariants of the Jacobian matrix. For example, the spinodal line is the locus of points where the Gauss curvature of the Gibbs function vanishes.

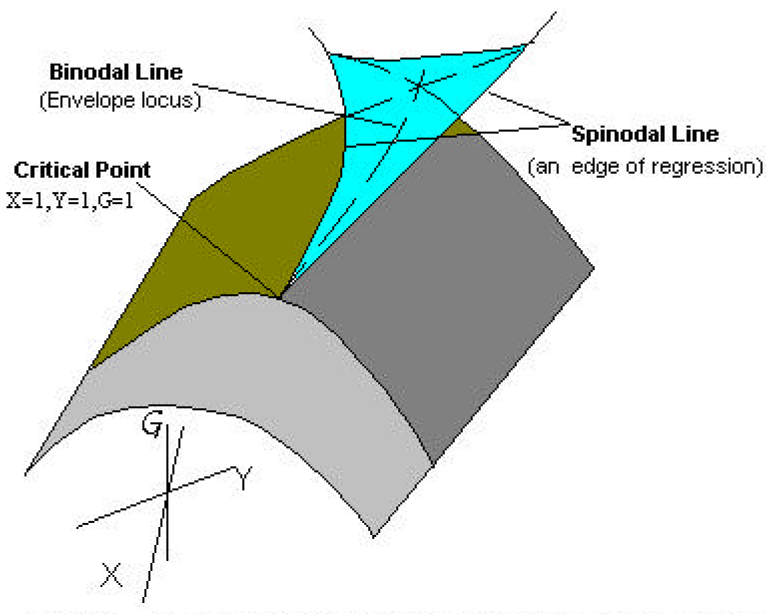

Fig. 1   A Universal Thermodynamic Swallowtail
The Gibbs surface of a Van der Waals gas