2D Turbulence is a Myth

R. M. Kiehn
Physics Department
University of Houston
http://www.uh.edu/~rkiehn

Fundamental Assumption:
Turbulence is
thermodynamically
irreversible

Primary Objectives

To demonstrate:

1. **Differential Topology can be used to determine when a flow process is irreversible - without statistics!**

2. **Cartan's exterior calculus describes Continuous Topological Evolution.**

3. **Continuous Topological Evolution can be used to describe the decay, but not the creation, of turbulence.**

4. **Cartan's Magic formula of Continuous Topological Evolution is a dynamical equivalent to the First Law of Thermo.**

5. **Thermodynamic Irreversibility is an artifact of 4 topological dimensions.**

As Turbulence is Irreversible,
and Irreversibility requires 4D,
time dependent 2D turbulence is a myth

Secondary Objectives

A. Importance of the Topological Torsion Vector (helicity density is a 4th component)

B. Examples of Irreversible Evolution in the direction of the

Topological Torsion Vector

THERMODYNAMICS

Physical Systems and Processes

To a topologist the First Law is a statement of Cohomology

$$\mathbf{Q} - \mathbf{W} = \mathbf{dU}$$

Definition: A process that produces a 1-form of Heat, **Q**, is reversible when **Q** admits an integrating factor: $\mathbf{Q} = \mathbf{TdS}$.

(Two independent functions $\supset$ Pfaff topological dimension of **Q** $= 2$)

Frobenius theorem: **Q** admits an integration factor iff

$$\mathbf{Q \hat{} dQ} = \mathbf{0}$$

Hence if $\mathbf{Q \hat{} dQ} \neq \mathbf{0}$, then the process is thermodynamically

IRREVERSIBLE

Cartan’s Magic Formula

In Cartan’s Calculus, a Physical system is represented by a 1-form of Action:

$$A = A_\mu dx^\mu - \phi dt \quad or \quad p_\mu dq^\mu - hdt$$

A process is represented by a vector field:

$$V = [\mathbf{v}, 1] \quad relative\,to \ \{x, y, z, t\}$$

Evolution of the Action, $A,$ relative to the process, $V,$ is described by Cartan’s Magic Formula using the Lie differential

$$\mathbf{L}_V(\mathbf{A}) = \mathbf{i}(\mathbf{V})\mathbf{dA} + \mathbf{d}(\mathbf{i}(\mathbf{V})\mathbf{A}) = \mathbf{Q}$$

$$\mathbf{L}_V(\mathbf{A}) = \quad \mathbf{W} \quad + \quad \mathbf{d}(\mathbf{U}) \quad = \mathbf{Q}$$

Cartan’s Magic Formula is a dynamical equivalent to the

First Law of Thermodynamics !!!

No geometric metric or a connection is required!

An Electromagnetic Example

The Action 1-form is composed from the electromagnetic vector and scalar potentials:

$$A = \mathbf{A} \bullet d\mathbf{r} - \phi dt$$

with

$$\mathbf{B} = curl\,\mathbf{A} \quad \text{and} \quad \mathbf{E} = -\partial\mathbf{A}/\partial t - \nabla\phi$$

Then the virtual work $W = i(V)dA$ becomes:

$$W = \{\mathbf{E} + \mathbf{v} \times \mathbf{B}\}\bullet d\mathbf{r} - (\mathbf{E} \bullet \mathbf{v})dt$$

where the spatial part is the Lorentz force times the differential displacement - the classical definition of differential work.

If $W = 0$ then have a ”Force Free Plasma”

The Cartan Topology

The functional coefficients of the 1-form of Action used to represent a physical system induces a (Cartan) Topology on space time.

The Pfaff (topological) dimension is defined as the mininum number of functions required to define the topological features of the Action 1-form A.

The Pfaff dimension is easy to compute.From $A,$ construct $dA,$ and then the Pfaff Sequence:

$$\{A, dA, A\text{^}dA, dA\text{^}dA, \ldots\}$$

The sequence terminates at an integer N, which is the Pfaff dimension (or Class) of the domain.

Theorem : Potential and Streamline Flows are possible iff the Pfaff dimension is less than 3, or $A\text{^}dA = 0.$

THE CARTAN TOPOLOGY

$A = v_k dx^k - \mathcal{H} dt$

$F = dA,\ H = A\hat{}dA,\ K = dA\hat{}dA$

Basis = { A, A^c, H, H^c } = { A, A∪ F, H, H∪K}

Open Sets: = {X, 0, A, H, A^c, H^c, A∪ H, A∪ H^c, A^c∪H}.

Subsets (σ)	Limit Pts (dσ)	Interior	Boundary (∂σ)	Closure (σ∪dσ)
0	0	0	0	0
A	F	A	F	A∪F
F	0	0	F	F
H	K	H	K	H∪K
K	0	0	K	K
A∪F	F	A∪F	0	A∪F
A∪H	F,K	A∪H	F∪K	X
A∪K	F	A	F∪K	A∪F∪K
F∪H	K	H	F∪K	F∪H∪K
F∪K	0	0	F∪K	F∪K
H∪K	K	H∪K	0	H∪K
A∪F∪H	F,K	A∪F∪H	K	X
F∪H∪K	K	H∪K	F	F∪H∪K
A∪H∪K	F,K	A∪H∪K	F	X
A∪F∪K	F	A∪F	K	A∪F∪K
X	F,K	X	0	X

TOPOLOGICAL CONCLUSIONS:

Pfaff dimension D implies the existence of a (submersive) map to a space of D dimensions.

Turbulent flows must have a Pfaff dimension greater than 2! The suggestion is:

$$Chaos \supset Pfaff\ Dimension\ D = 3$$

$$Turbulence \supset Pfaff\ Dimension\ D = 4$$

More importantly,can show:

$$A\hat{\ }dA = 0 \supset Pfaff\ Dimension\ D < 3$$

$\supset$ a Connected Topology

$$A\hat{\ }dA \neq 0 \supset Pfaff\ Dimension\ D > 2$$

$\supset$ **a Disconnected Topology**

Anholonomic Fluctuations

Consider a Physical system represented by a Lagrange 1-form of Action:

$$A = L(q^{\mu}, V^{\mu}, t)dt + \quad p_{\mu} \bullet (dq^{\mu} - V^{\mu}dt)$$

Anholonomic fluctuations are defined as the deviations from kinematic perfection:

$$\Delta q^{\mu} = (dq^{\mu} - V^{\mu}dt) \neq 0$$

The p_{μ} are Lagrange multipliers

QUESTION: What is the **P**faff

Topological **D**imension of A?

Note that A has **3N+1** independent geometric variables, but the Pfaff Topological dimension is only:

$$\mathbf{PTD}(\mathbf{A}) = \mathbf{2N} + \mathbf{2}$$

Anholonomic fluctuations produce symplectic manifolds

HOWEVER, if the Lagrange multipliers are constrained to be

CANONICAL MOMENTA, such that

$$p_{\mu} - \partial L(q^{\mu}, V^{\mu}, t)/\partial V^{\mu} = 0$$

Then **D** $=$ **2N** $+$ **1** and the physical

(Contact) manifold is of odd dimension (state space), and admits a

Unique Extremal Reversible Hamiltonian Direction Field.

IN THIS SENSE

ANHOLONOMIC FLUCTUATIONS are

the SOURCE OF IRREVERSIBILITY

Note:

$$W = i(V)dA = Q - dU$$

such that always

$$i(V)W = i(V)i(V)dA \Rightarrow 0$$

which implies

$$\supset i(V)dQ = dU$$

Work is always transversal, Heat is NOT.

Transition to/from Turbulence

Theorem: Continuous Evolution from a streamline flow (connected topology), **A^dA** $=$ **0**, to a turbulent flow, (disconnected topology), **A^dA** $\neq$ **0**, is impossible. However, the converse is possible, so,

Continuous decay of Turbulence

$$\mathbf{A \wedge dA = 0 \Leftarrow A \wedge dA \neq 0}$$

is possible.

Continuous creation of Turbulence

$$\mathbf{A \wedge dA = 0 \Rightarrow A \wedge dA \neq 0}$$

is impossible.

When is a Dynamical System Irreversible?

For physical systems encoded by a 1-form of Action, **A**, and those processes tencoded by a vector field, **V**, Cartan's Magic formula yields:

$$\mathbf{L}_{(V)}\int_a^b \mathbf{A} = \int_a^b \{\mathbf{i}(\mathbf{V})\mathbf{dA} + \mathbf{d}(\mathbf{i}(\mathbf{V})\mathbf{A})\} \quad = \int_a^b \mathbf{Q}$$

Those adiabatic solutions, **V**, such that $\mathbf{L}_{(V)}\int_a^b \mathbf{A} = \mathbf{0}$ are equivalent to those paths in the calculus of variations that leave the integral stationary, and may be interpreted as an equation describing **Continuous Topological Evolution.**

By defining the 1-form of virtual work, $\mathbf{W} = \mathbf{i}(\mathbf{V})\mathbf{dA}$, and the internal energy as $\mathbf{U} = \mathbf{i}(\mathbf{V})\mathbf{A}$, Cartan's Magic formula becomes the **First Law of Thermodynamics.**

Cartan's Magic Formula

may be used to test if a

Dynamical System, V,

represents a

Reversible Process.

From thermodynamics and Frobenius, if Q^dQ $=$ 0, then the process is reversible

Hence, use Cartan's Magic formula to compute

$$\mathbf{Q \wedge dQ} = \mathbf{L}_{(V)}\mathbf{A} \wedge \mathbf{L}_{(V)}\mathbf{dA}$$

for a given physical system, A, and a given process, V.

The process is reversible if

$$\mathbf{L}_{(V)}\mathbf{A} \wedge \mathbf{L}_{(V)}\mathbf{dA} = \mathbf{0}.$$

Equations of Motion
and topological classes
of Processes

Topological properties of the 1-form of Virtual Work $W = i(V)dA$, form 2 categories of processes. **Either**

$dW = 0$ **or** $dW \neq 0$.

THEOREM: All processes such that $dW = 0$ are thermodynamically reversible.

The category $dW = 0$ includes extremal, Hamiltonian, Bernoulli - Casimir and all Symplectic processes. Proof:

$$\mathbf{L}_{(V)}\mathbf{dA} = \mathbf{dW} + \mathbf{0} = \mathbf{dQ}$$

If $dW = 0$, it follows that

$$\mathbf{L}_{(V)}\mathbf{A}\hat{}\mathbf{L}_{(V)}\mathbf{dA} = \mathbf{Q}\hat{}\mathbf{dQ} = \mathbf{0}.$$

Hence all processes where $dW = dQ = 0$ are thermodynamically reversible.

Classes of Reversible Processes

Extremal - Unique Hamiltonian: Virtual Work is zero. $\mathbf{W} = \mathbf{i}(\mathbf{V})\mathbf{dA} = \mathbf{0}$

Potential flows
$\supset \mathbf{dA} = \mathbf{0},\ \int_{cyclic} \mathbf{A} = \mathbf{0}$ no lift.

Joukowski flow
$\supset \mathbf{dA} = \mathbf{0},\ \int_{cyclic} \mathbf{A} \neq \mathbf{0}$ Circulation without vorticity produces lift.

Lamb Flows $\supset \mathbf{A\wedge dA} = \mathbf{0}$ No Helicity

Bernoulli-Casimir-Hamiltonian:
Cyclic work is zero (Eulerian fluid).

$$\mathbf{W} = \mathbf{i}(\mathbf{V})\mathbf{dA} = \mathbf{d\Theta},\ \int_{cyclic} \mathbf{W} = \mathbf{0}$$

Helmholtz-Symplectic: Cyclic work is not zero!

$$\mathbf{dW} = \mathbf{0},\ \int_{cyclic} \mathbf{W} \neq \mathbf{0}$$

Topological Quirks

Extremal (Hamiltonian) processes are unique on domains of 2n+1 dimensions

Hamiltonian processes are not unique on domains of dimension 2n+2.

REMARK:

If D = 3, then there exists a unique extremal field which nature selects by the principal of Least Action.

An extremal field is reversible, hence cannot be used to represent turbulence.

Conclusion:

Turbulence must be an artifact of Pfaff dimension 4 (or more)

IRREVERSIBLE PROCESSES

and the

TOPOLOGICAL TORSION VECTOR

Irreversible processes exist only on domains that support a disconnected Cartan topology,

$A \wedge dA \neq 0$, (with non-uniqueness,

envelopes, regressions, and projectivized tangent bundles) Least Action hypotheses $\supset$ Pfaff D $=$ 4, such that for thermodynamic irreversibility:

$$\mathbf{dA \wedge dA \neq 0}.$$

$\therefore$ **The irreversible Action Manifold of support is symplectic of rank 4**

On the symplectic domain there exists a unique vector direction field, $\mathbf{T}_4$, with components determined by the functions used to define the physical system (the 1-form of Action, A).

$$\mathbf{i}(\mathbf{T}_4)\mathbf{dx\hat{}dy\hat{}dz\hat{}dt} = \mathbf{A\hat{}dA}$$

As

$\mathbf{dA\hat{}dA} = \{\mathbf{div}_4\mathbf{T}_4\}\mathbf{dx\hat{}dy\hat{}dz\hat{}dt} \neq \mathbf{0}$
this vector field $\mathbf{T}_4$ has a non-zero divergence almost everywhere except on regions (defect ”holes”) where the manifold is no longer symplectic. $(D \neq 4)$

Evolution in the direction of $\mathbf{T}_4$

is

Thermodynamically IRREVERSIBLE

Proof:

Define (on the symplectic 4D domain)

$$\Gamma = \mathbf{div}_4 \mathbf{T}_4 \neq \mathbf{0}$$

Then it is possible to show that

$$\mathbf{W} = \mathbf{i}(\mathbf{T}_4)\mathbf{dA} = \Gamma \bullet \mathbf{A}$$

$$\mathbf{L}_{(\mathbf{T}_4)}\mathbf{A} = \Gamma \bullet \mathbf{A}$$

and

$$\mathbf{dQ \hat{} dQ} = \Gamma^2 \bullet \mathbf{dA \hat{} dA}$$

Hence, Evolution in the direction of the Topological Torsion vector is

Thermodynamically IRREVERSIBLE

Examples of $\mathbf{T}_4$

ELECTROMAGNETISM

The Action 1-form of potentials:

$$A = \mathbf{A} \bullet d\mathbf{r} - \phi dt$$

with

$$\mathbf{B} = curl\,\mathbf{A} \text{ and } \mathbf{E} = -\partial\mathbf{A}/\partial t - \nabla\phi$$

forming the components of dA.

By direct computation,

$$\mathbf{T}_4 = -[\mathbf{E} \times \mathbf{A} + \mathbf{B}\phi,\ \mathbf{A} \bullet \mathbf{B}]$$

$$\Gamma = \mathbf{div}_4\mathbf{T}_4 = 2\ \mathbf{E} \bullet \mathbf{B} \neq \mathbf{0}$$

Note 1: The Helicity density $\mathbf{A} \bullet \mathbf{B}$ is the 4th component of the Topological Torsion Vector, $\mathbf{T}_4$.

Note 2: Γ is the second Poincare Invariant for an electromagnetic system.

Examples of $\mathbf{T}_4$

HYDRODYNAMICS

Consider an Action 1-form of potentials:
$\mathbf{A} = \mathbf{v} \bullet \mathbf{dr} - H\mathbf{dt}$

with H defined as

$$H = (\mathbf{v} \bullet \mathbf{v/2} - \lambda \mathbf{div v} + \int \mathbf{dP}/\rho) = \mathbf{v} \bullet \mathbf{v} - L$$

and Vorticity defined as $\boldsymbol{\omega} = curl\, \mathbf{v}$.

The equivalence class of solutions that satisfy the topological constraint on the virtual work W,

$$\begin{aligned} W &= i(V)dA \\ &= \nu\{curlcurl\, \mathbf{v} \bullet (d\mathbf{r} - \mathbf{v}dt)\}, \end{aligned}$$

are solutions to the Navier Stokes equations of motion. By direct computation,

$$\mathbf{T}_4 = [h\mathbf{v} - L\boldsymbol{\omega} - \nu\ curl\ \boldsymbol{\omega},\ h]$$

where $h = helicity = \mathbf{v} \bullet \boldsymbol{\omega}$

$$\Gamma = \mathbf{div}_4\mathbf{T}_4 = 2\nu \; \{\boldsymbol{\omega} \bullet curl\boldsymbol{\omega}\}$$

Hence, for a Navier Stokes fluid, domains where the vorticity field does not satisfy the Frobenius integability condition

$$\boldsymbol{\omega} \bullet curl \; \boldsymbol{\omega} \neq 0$$

are domains of thermodynamic irreversibility, and are therefore candidates for turbulent flow.

More detail and references can be found at

CARTAN's CORNER

http://www.cartan.pair.com

CONCLUSIONS:

1. **2 D Turbulence is a myth**

2. **Irreversibility is an artifact of 4 D.**

3. **Cartan's methods may be used to describe continuous topological evolution.**

4. **Anholonomic fluctuations in kinematic perfection is a topological source of irreversibility.**

5. **Evolution along the direction field of the Topological Torsion is Thermodynamically irreversible.**

6. **There exist solutions to the Navier-Stokes equations which are thermodynamically irreversible.**

7. **Thermodynamics and dynamics are unified by topological methods (not statistical methods).**

An invited talk presented at the

EGS XXIV IUTAM conference

at the HAGUE

April 19-23, 1999

R. M. Kiehn
http://www.cartan.pair.com