ELECTROMAGNETIC WAVES IN THE VACUUM WHICH HAVE TORSION AND SPIN.

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Abstract. Wave solutions to the homogeneous Maxwell equations have been found that are not transverse, exhibit both torsion and spin, and for which the second Poincare invariant $\mathbf{E} \circ \mathbf{B} \neq 0$. The concept of topological transversality is introduced in terms of two topological 4-vectors of spin current and torsion. The divergence of each 4 vector generates the Poincare invariants of the electromagnetic system. When the Poincare invariants vanish the closed integrals of each 4-vector over a closed 3 dimensional domain creates a topological conserved quantity with values that have ratios which are rational. When both topological 4-vectors are zero, the Maxwell system does not radiate

1. Introduction, the Domain of Classical Electromagnetism

Using the notation and the language of Sommerfeld and Stratton [1], the classic definition of an electromagnetic system is a domain of space-time independent vraiables, $\{x, y, z, t\}$, which supports both the Maxwell-Faraday equations,

$$curl\ \mathbf{E} + \partial\mathbf{B}/\partial t = 0, \quad div\ \mathbf{B} = 0, \tag{1.1}$$

and the Maxwell-Ampere equations,

$$curl\ \mathbf{H} - \partial\mathbf{D}/\partial t = \mathbf{J}, \quad div\ \mathbf{D} = \rho. \tag{1.2}$$

In every case, the charge current density for the Maxwell system satisfies the conservation law,

$$div\ \mathbf{J} + \partial\rho/\partial t = 0. \tag{1.3}$$

For the vacuum state, the charge-current densities are subsumed to be zero $[\mathbf{J}, \rho] = 0$. The Lorentz vacuum state is distinguished by the assumption that the field excitations, **D** and **H**, are linearly connected to the field intensities, **E** and **B**, by means of the homogeneous and isotropic constitutive relations:

$$\mathbf{D} = \varepsilon\mathbf{E}, \quad \mathbf{B} = \mu\mathbf{H}. \tag{1.4}$$

The two vacuum constraints imply that the solutions to the homogeneous Maxwell equations also satisfy the vector wave equation, typically of the form

$$grad\ div\ \mathbf{B} - curl\, curl\, \mathbf{B} - \varepsilon\mu\partial^2\mathbf{B}/\partial t^2 = 0. \tag{1.5}$$

Wave solutions typically can have both a phase velocity, v_p, and a group velocity, v_g, with a product equal to square of the "speed" of light:

$$v_p v_g = 1/\varepsilon\mu \equiv c^2 \tag{1.6}$$

Similar results can be obtained for the solid state where the constitutive constraints can be more complex [2], and for the plasma state where the charge-current densities are not zero.

It is further subsumed that the classic Maxwell electromagnetic system is constrained by the statement that the field intensities are deducible from a system of twice differentiable potentials, $[\mathbf{A}, \phi]$:

$$\mathbf{B} = curl\ \mathbf{A}, \quad \mathbf{E} = -grad\ \phi - \partial\mathbf{A}/\partial t. \tag{1.7}$$

This constraint topologically implies that domains that support non-zero values for the covariant field intensities, $\mathbf{E}$ and $\mathbf{B}$, can *not* be compact domains without a boundary. It is this constraint that distinguishes classical electromagnetism from Yang Mills theories. Two other classical 3-vector fields of interest are the Poynting vector, $\mathbf{E} \times \mathbf{H}$, representing the flux of electromagnetic radiative energy, and the field momentum flux, $\mathbf{D} \times \mathbf{B}$.

Besides the charge current 4-vector density, $[\mathbf{J}, \rho]$, whose integral over any closed dimensional manifold is a deformation invariant of the Maxwell system, there exist two other algebraic combinations of the fields and potentials that can lead to similar topological quantities. These objects are the Spin 4 vector, or current, defined as

$$\mathbf{S}_4 = [\mathbf{A} \times \mathbf{H} + \mathbf{D}\phi, \mathbf{A} \circ \mathbf{D}] \equiv [\mathbf{S}, \sigma]. \tag{1.8}$$

and the Torsion 4 vector defined as

$$\mathbf{T}_4 = [\mathbf{E} \times \mathbf{A} + \mathbf{B}\phi, \mathbf{A} \circ \mathbf{B}] \equiv [\mathbf{T}, h]. \tag{1.9}$$

Note that the classical helicity, $h = \mathbf{A} \circ \mathbf{B}$, forms only the fourth component of this third rank tensor. The derivation of these 4-component fields and their topological implications is developed in section 3, below. The 4-divergence of these 4-component vectors leads to the Poincare projective invariants of the Maxwell system:

$$\begin{aligned} Poincare\ Invariant\ 1 &= div_3(\mathbf{A} \times \mathbf{H} + \mathbf{D}\phi) + \partial(\mathbf{A} \circ \mathbf{D})/\partial t \\ &= (\mathbf{B} \circ \mathbf{H} - \mathbf{D} \circ \mathbf{E}) - (\mathbf{A} \circ \mathbf{J} - \rho\phi) \end{aligned} \tag{1.10}$$

$$\begin{aligned} Poincare\ Invariant\ 2 &= div_3(\mathbf{E} \times \mathbf{A} + \mathbf{B}\phi) + \partial(\mathbf{A} \circ \mathbf{B})/\partial t \\ &= -2\mathbf{E} \circ \mathbf{B} \end{aligned} \tag{1.11}$$

Most of the time-dependent vacuum solutions to the Maxwell system of equations which are found in the historic literature are composed (typically) of simple geometrically transverse waves, for which the wave vector defining the direction of the phase propagation is transverse to the magnetic field $\mathbf{B}$, or transverse to the electric excitation $\mathbf{D}$, or transverse to both. In the latter case the wave vector is in the direction of the field momentum flux,

$\mathbf{D}\times\mathbf{B}$, where not only the first and the second Poincare invariants are identically zero, but also the Torsion and the Spin vectors, as defined above, are identically zero. This clue permits the concept of transversality to be described in a topological manner. A topologically transverse wave (TTEM) is defined as a wave for which both the Torsion vector and the Spin vector vanish: $\mathbf{A}\circ\mathbf{B} = 0, \mathbf{A}\circ\mathbf{D} = 0$. A topologically transverse magnetic wave (TTM) is defined as a wave for which the Torsion vector vanishes and the Spin vector does not: $\mathbf{A}\circ\mathbf{B} = 0, \mathbf{A}\circ\mathbf{D} \neq 0$. A topologically transverse electric wave (TTE) is defined as a wave for which the Spin vector vanishes and the Torsion vector does not: $\mathbf{A}\circ\mathbf{B} \neq 0, \mathbf{A}\circ\mathbf{D} = 0$. There exisit vacuum waves for which neither the Torsion nor the Spin 4-vector vanishes.

When the Spin vector is non-zero, but the first Poincare invariant vanishes, it is possible to construct a deformation invariant topological quantity in terms of the deRham period integral over a closed 3 dimensional submanifold:

$$Spin = \iiint_{closed} \{S^x dy\hat{\ }dz\hat{\ }dt - S^y dx\hat{\ }dz\hat{\ }dt + S^z dx\hat{\ }dy\hat{\ }dt - \sigma dx\hat{\ }dy\hat{\ }dz\}. \qquad (1.13)$$

This closed integral is an invariant of any evolutionary process that can be described by a singly parameterized vector field, independent of the choice of parameterization. The values of the Spin integral form rational ratios. The proof of this statement is given in section 3

When the Torsion vector is non-zero, but the second Poincare invariant vanishes, it is possible to construct a deformation invariant topological quantity in terms of the deRham period integral over a closed 3 dimensional submanifold:

$$Torsion - Helicity = \iiint_{closed} \{T^x dy\hat{\ }dz\hat{\ }dt - T^y dx\hat{\ }dz\hat{\ }dt + T^z dx\hat{\ }dy\hat{\ }dt - h dx\hat{\ }dy\hat{\ }dz\}. \qquad (1.14)$$

This closed integral is an invariant of any evolutionary process that can be described by a singly parameterized vector field, independent of the choice of parameterization. The values of the Torsion integral form rational ratios.

It is the purpose of this article to display exact time dependent radiative vacuum solutions to the above system of equations which are not TTEM, TTE, or TTM waves. In particular topologically non-transverse wave solutions to the vacuum Maxwell equations will be presented for which $\mathbf{E}\circ\mathbf{B} \neq 0$. Such field configurations have been associated with time reversal symmetry breaking in certain Lagrangian field theories[3] and, more recently, with thermodynamic irreversibility [4].

In addition it will be demonstrated that there exist superpositions of non-transverse vacuum waves for which the Torsion 4-vector vanishes identically, but the Spin 4 vector does not, and yet both Poincare invariants are zero. For such superposed solutions, the Spin topological quantity is non-zero and is conserved as an evolutionary invariant. Moreover, in the example to be displayed, the Poynting vector of radiative energy flow is proportional to the spatial components of the Spin 4-vector:

$$\mathbf{E}\times\mathbf{H} \approx factor \bullet (\mathbf{A}\times\mathbf{H} + \mathbf{D}\phi) = factor \bullet \mathbf{S}. \qquad (1.14)$$

These special solutions are transverse magnetic, $h = \mathbf{A}\circ\mathbf{B} = 0$, but are not transverse electric, $\sigma = \mathbf{A}\circ\mathbf{D} \neq 0$. The topological quantity of Spin is quantized to the integers, and the energy flux is proportional to the Spin current. Therefore, these special vacuum solutions appear to give classical credence to the Planck concept of a photon as an object of unit spin, with radiated energy proportional to the spin.

2. Example Radiative Vacuum Solutions which are not topologically transverse.

2.1 A First Example.

As an example of a radiative solution which is not a simple transverse wave, consider system of potentials given by the equations

$$\mathbf{A} = [+y, -x, -ct]/\lambda^4\ , \quad \phi = cz/\lambda^4, \quad where\ \lambda^2 = -c^2t^2 + x^2 + y^2 + z^2.$$

Compute the field intensities and find that

$$\mathbf{E} = [-2(cty - xz), +2(ctx + yz), -(c^2t^2 + x^2 + y^2 - z^2)]2c/\lambda^6$$

and

$$\mathbf{B} = [-2(cty + xz), +2(ctx - yz), +(c^2t^2 + x^2 + y^2 - z^2)]2/\lambda^6.$$

These equations for the electromagnetic field intensities satisfy the Maxwell-Faraday equations (1.1).

Next compute the field excitations using the Lorentz vacuum constitutive relations. Substitute these fields into the Maxwell-Ampere equations (1.2) and determine that $\mathbf{J} = 0$ and $\rho = 0$, subject to the dispersion relation, $\varepsilon\mu c^2 = 1$. The solutions presented therefore satisfy the homogeneous Maxwell equations without charge currents, and are therefore acceptable vacuum solutions. The electromagnetic fields are also solutions to the vector wave equation.

The Spin current density for this first non-transverse wave example is evaluated as:

$$\mathbf{S}_4 = [x(3\lambda^2 - 4y^2 - 4x^2), y(3\lambda^2 - 4y^2 - 4x^2), z(\lambda^2 - 4y^2 - 4x^2), t(\lambda^2 - 4y^2 - 4x^2)](2/\mu)/\lambda^{10}$$

The Torsion current may be evaluated as

$$\mathbf{T}_4 = -[x, y, z, t]2c/\lambda^8.$$

The 4-divergence of the Spin Current is zero. Hence the first Poincare invariant vanishes. As the interaction energy density, $(\mathbf{A}\circ\mathbf{J} - \rho\phi)$, for the vacuum is zero, the vanishing of the first Poincare invariant implies that the magnetic energy density is exactly equal to the electric energy density, typical of oscillator systems. However, the divergence of the Torsion vector is not zero, and the second Poincare invariant has the non-zero value:

$$Poincare\ 2 = -2\mathbf{E} \circ \mathbf{B} = +8c/\lambda^8.$$

These solutions are not simple transverse waves for both $\mathbf{A} \circ \mathbf{B} \neq 0$, and $\mathbf{A} \circ \mathbf{D} \neq 0$. Note that the physical units of the second Poincare invariant are that of an energy density multiplied by an impedance (ohms).

However, as the first Poincare invariant is zero it is possible to construct a deformation invariant in terms of the deRham period integral over a closed 3 dimensional submanifold. That is, the closed integral of Spin is an evolutionary deformation invariant for this solution.

2.2 A Second Example.

It is to be noted that the example solution given above is but one of a class of vacuum wave solutions that have similar non transverse properties. For a second example, consider the fields that can be constructed from the potentials,

$$\mathbf{A} = [+ct, -z, +y]/\lambda^4\ , \quad \phi = cx/\lambda^4, \quad where\ \lambda^2 = -c^2t^2 + x^2 + y^2 + z^2.$$

These potentials will generate the field intensities

$$\mathbf{E} = [+(-c^2t^2 + x^2 - y^2 - z^2), +2(ctz + yx), -2(cty - zx)]2c/\lambda^6$$

and

$$\mathbf{B} = [+(-c^2t^2 + x^2 - y^2 - z^2), +2(-ctz + yx), +2(cty + zx)]2/\lambda^6.$$

As before, these fields satisfy the Maxwell-Faraday equations, and the associated excitations satisfy the Maxwell-Ampere equations without producing a charge current 4-vector. However, it follows by direct computation that the second Poincare invariant, and the Torsion 4-vector are of opposite signs to the values computed for the first example:

$$\mathbf{T}_4 = +[x, y, z, t]2c/\lambda^8.$$

$$-2\mathbf{E} \circ \mathbf{B} = -8c/\lambda^8 \quad .$$

2.3 Superposition of the two examples.

When the two examples are combined by addition (or subtraction), the resulting wave is transverse magnetic, but not transverse electric. Not only does the second Poincare invariant vanish under superposition, but so also does the Torsion 4 vector. Conversely, the examples above show that there can exist transverse magnetic waves which can be decomposed into two non-transverse waves. A notable feature of the superposed solutions is that the Spin 4 vector does not vanish, hence the example superposition is a wave that is not transverse electric. For the examples presented above and their superposition, the first

Poincare invariant vanishes, which implies that the Spin remains a conserved topological quantity for the superposition. The spin current density for the combined examples is given by the formula:

$$\mathbf{S}_4 = [-2x(y+ct)^2, (y+ct)(x^2-y^2+z^2-2cty-c^2t^2), -2z(y+ct)^2, \\ -(y+ct)(x^2+y^2+z^2+2cty+c^2t^2)](4/\mu)/\lambda^{10}$$

while the Torsion current is a zero vector

$$\mathbf{T}_4 = [0,0,0,0].$$

In addition, for the superposed example, the spatial components of the Poynting vector are equal to the Spin current density vector multiplied by γ, such that

$$\mathbf{E}\times\mathbf{H} = \gamma\,\mathbf{S}, \quad with\ \gamma = -(x^2+y^2+z^2+2cty+c^2t^2)/2c(y+ct)\lambda^2.$$

These results seem to give classical credence to the Planck assumption that vacuum state of Maxwell's electrodynamics supports quantized angular momentum, and that the energy flux must come in multiples of the spin quanta. In otherwords, these combined solutions have the appearance of the photon.

3. **The 3-form of Topological Torsion and the 3-form of Topological Spin**.

3.1 Maxwell's theory as a consequence of two topological constraints

The formulation of Maxwell theory in section 1 is relative to a choice of coordinates {x,y,z,t} using classical vector analysis developed in euclidean 3-space. The topological features of the formalism are not immediately evident. However, electromagnetism has a formulation in terms of exterior differential systems composed of exterior differential forms [5]. If an exterior differential system is valid on a variety of independent variables, {x,y,z,t}, then it is also valid on any other variety of independent variables that can be mapped onto {x,y,z,t}. The map need only be differentiable, such that the Jacobian coefficients are well defined as functions. The Jacobian matrix does not have to have an inverse, which implies that the exterior differential system is not restricted to a set of diffeomorphisms. The component differential forms are functionally well defined by the pullback mechanism, an algebraic process that involves the components of the Jacobian matrix transpose or adjoint, and the process of functional substitution. In this sense the Maxwell theory does not depend upon a choice of coordinates, does not depend upon the choice of metric, and is independent of the constraints imposed by gauge groups and connections that can impose other perhaps interesting but not necessary topological refinements. These metric free, coordinate free, properties of Maxwell theory apparently were first recognized in the Schouten school by van Dantzig. [6] The formulation in terms of exterior differential systems makes it apparent that Maxwell's electrodynamics is a consequence of topological constraints on a domain of independent variables.

The Maxwell-Faraday equation is a consequence of the exterior differential system

$$F - dA = 0,$$

where A is a 1-form of Action, with twice differentiable coefficients which induce a 2-form, F, of electromagnetic intensities ($\mathbf{E}$ and $\mathbf{B}$, related to forces). The exterior differential system is a topological constraint on the doamin of independent variables that defines field intensities in terms of the potentials. From Stokes theorem, the domain of support for the 2-form F of field intensities can not be compact without boundary. The closure of the exterior differential system, $dF = 0$, generates the Maxwell-Faraday partial differential equations. The component functions ($\mathbf{E}$ and $\mathbf{B}$) of the 2-form transform as covariant tensor of rank 2. The topological fact that F is exact, which implies that the domain of support for the field intensities cannot be compact without boundary, is the fact that distinguishes classical electromagnetism from Yang-Mills field theories. Moreover, the fact that F is subsumed to be C1 differentiable excludes the concept of magnetic monopoles from classical electromagnetic theory on topological grounds. The integral of F over any closed 2-manifold is a deformation (topological) invariant of any evolutionary process that can be described by a singly parameterized vector field.

The Maxwell Ampere equations are a consequence of second exterior differential system,

$$J - dG = 0,$$

where G is an N-2 form *density* of field excitations ($\mathbf{D}$ and $\mathbf{H}$, related to sources), and J is the N-1 form of charge-current densities. The partial differential equations equivalent to the exterior differential system are precisely the Maxwell-Ampere equations. The exterior differential system is a topological constraint on the domain that implies that the domain of support for G can be compact without boundary only if the domain is without charge-currents. The closure of the exterior differential system, $dJ = 0$, generates the charge-current conservation law. The integral of J over a closed 3 dimensional domain is a relative integral invariant (a deformation invariant) of any process that can be described in terms of a singly parametrized vector field.

A deformation invariant is defined as an integral over a closed manifold, $\int\cdots\int_{closed}\omega$ of a differential form, ω, such that the Lie derivative of the closed integral with respect to a singly parameterized vector field, βV, vanishes for any parameterization function β. By using Cartan's magic formula [7] it follows that for an electromagnetic system both the 3-form of charge current, and the 2-form of field intensities define topological quantities (deformation invariants) on the domain of independent variables.

$$L_{\beta V}(\iiint_{closed} J) = \iiint_{closed}\{i(\beta V)dJ + d(i(\beta V)J)\} = \iiint_{closed}\{0 + d(i(\beta V)J)\} = 0.$$

The integral is a deformation invariant, for the result is valid even if the 4-vector field, V, is distorted by an arbitrary function, $\beta\{x, y, z, t\}$ such that $\mathbf{V} \Rightarrow \beta(x, y, z, t)\mathbf{V}$.

As mentioned above, the method of exterior differential forms goes beyond the domain of classical tensor analysis for it admits of maps from initial to final state that are without inverse. (Tensor analysis and coordiante transformations require that the Jacobian map from intial to final state has an inverse - the method of exterior differential forms does not.) Hence the theory of electromagnetism expressed in the language of exterior differential forms admits of topological evolution, at least with respect to continuous processes without Jacobian inverse. With respect to such non-invertible maps, both tensor fields and differential forms are not functionally well defined in a predictive sense. Given the functional forms of a tensor field on an initial state, it is impossible to predict uniquely the functional form of the tensor field on the final state unless the map between initial and final state is invertible. However differential forms are functionally well defined in a retrodictive sense, by means of the pullback [8]. Covariant anti-symmetric tensor fields pull back retrodictively with respect to the transpose of the Jacobian matrix, and functional substitution, and contravariant tensor densities pullback retrodictively with respect to the adjoint of the Jacobian matrix, and functional substitution. The transpose and the adjoint of the Jacobian exist, even if the Jacobian inverse does not.

The exterior differential forms that make up the electromagnetic system consist of the primitive 1-form, *A*, and the primitive N-2 form density, *G*, their exterior derivatives, and their algebraic intersections defined by all possible exterior products. The complete Maxwell system of forms is given by the set:

$$\{A, F = dA, G, J = dG, A\hat{}F, A\hat{}G, A\hat{}J, F\hat{}F, G\hat{}G\}.$$

These forms and their unions may be used to form a topological base on the domain of independent variables. The Cartan topology constructed on this system of forms has the useful feature that the exterior derivative may be interpreted as a limit point, or closure, operator in the sense of Kuratowski [9]. The exterior differential systems that define the Maxwell-Ampere and the Maxwell-Faraday equations above are essentially topological constraints of closure. Note that the complete system of forms also generates two other exterior differential systems.

$$d(A\hat{}G) - (F\hat{}G - A\hat{}J) = 0,$$

and

$$d(A\hat{}F) - F\hat{}F = 0.$$

The two objects, $A\hat{}G$ and $A\hat{}F$ are three forms, not usually found in discussions of classical eletromagnetism. The closed components of the first 3-form (density) were used to define topological spin [10] and the closed components of the second 3-form were used to define topological torsion (or helicity) [11]. When the engineering notation of section 1 is employed, the explicit notation of the first 3-form is given by the 4 component vector (see equation 1.8)

$$Topological\ Spin\ 3-form = A\hat{}G = i(\mathbf{S}_4)dx\hat{}dy\hat{}dz\hat{}dt$$
$$Spin = \iiint_{closed} A\hat{}G$$

and the explicit notation of the second 3-form is (see equation 1.9)

$$Topological\ Torsion-helicity\ 3-form = A\hat{}F = i(\mathbf{T}_4)dx\hat{}dy\hat{}dz\hat{}dt$$
$$Torsion-Helicity = \iiint_{closed} A\hat{}F$$

The vanishing of the first 3-form is a topological constraint of refinement on the domain that defines topologically transverse electric (TTE) waves: the vector potential $\mathbf{A}$ is orthogonal to $\mathbf{D}$. The vanishing of the second 3-form is a topological constraint on the domain that defines topologically transverse magnetic (TTM) waves: the vector potential $\mathbf{A}$ is orthogonal to $\mathbf{B}$. When both 3-forms vanish, the topological constraint on the domain defines topologically transverse (TTEM) waves. In certain cases, the concept of topological transversality and geometric transversality are equivalent, but there are cases where the two concepts are distinct. The different geometric transversality modes of electromagnetic waves (defined in terms of the transversality of the wave vector, $\mathbf{k}$, not the vector potential, $\mathbf{A}$) are well known concepts experimentally, but the association of transversality to topological issues is novel herein. If the 2-form F was not exact, such concepts of topological transversality would be without meaning.

3.1 Torsion, Spin and the Poincare Invariants.

Consider the 3-form of Spin Current, $A\hat{}G$, which has the physical dimensions of angular momentum, h. The closure of this 3-form is evaluted in terms of its exterior derivative. By direct computation,

$$Poincare\ 4-form\#1 = d(A\hat{}G) = F\hat{}G - A\hat{}J = \{(\mathbf{B}\circ\mathbf{H} - \mathbf{D}\circ\mathbf{E}) - (\mathbf{A}\circ\mathbf{J} - \rho\phi)\}dx\hat{}dy\hat{}dz\hat{}dt.$$

which defines the first Poincare invariant 4-form of classical electromagnetism. This 4-form is a deformation invariant on the domain of 4 independent variables. The functional coefficient of the 4-volume element is equivalent to the 4-divergence of the Spin 4-vector:

$$div_3(\mathbf{A}\times\mathbf{H} + \mathbf{D}\phi) + \partial(\mathbf{A}\circ\mathbf{D})/\partial t = (\mathbf{B}\circ\mathbf{H} - \mathbf{D}\circ\mathbf{E}) - (\mathbf{A}\circ\mathbf{J} - \rho\phi)$$

For both the example non-transverse wave solutions given in sections 2.1 and 2.2, the Spin density 3-form is not zero, but the first Poincare invariant vanishes. Therefore it is possible to demonstrate that for these wave solutions, the quantity of Spin, defined as the integral

$$Spin \doteq \iiint_{closed} A\hat{}G,$$

is a deRham period integral. As such, the quantity of Spin is a topological quantity whose possible values form rational ratios. The Spin is also a deformation invariant of any evolutionary process that can be described by a singly parameterized vector field, for when the first Poincare invariant vanishes,

$$L_{\mathbf{V}}(\iiint_{closed} A\hat{}G) = 0.$$

For the example solutions of section 2.1 and 2.2, Spin is an evolutionary invariant while Torsion-Helicity is not. Such waves are neither TTE or TTM.

Next consider the 3-form of Topological Torsion, $A\hat{}F$, which has the physical dimensions of angular momentum times an impedance $(h/e)^2$. The closure of this 3-form is evaluted in terms of its exterior derivative. By direct computation,

$$Poincare\ 4-form\,\#2 = d(A\hat{}F) = F\hat{}F = -2\mathbf{E}\circ\mathbf{B}dx\hat{}dy\hat{}dz\hat{}dt.$$

an equation which defines the second Poincare invariant 4-form of classical electromagnetism. This 4-form is a deformation invariant on the domain of 4 independent variables. The functional coefficient of the 4-volume element is equivalent to the 4-divergence of the Torsion 4-vector:

$$div_3(\mathbf{E}\times\mathbf{A}+\mathbf{B}\phi)+\partial(\mathbf{A}\circ\mathbf{B})/\partial t = -2\mathbf{E}\circ\mathbf{B}.$$

When the second Poincare invariant vanishes, the Torsion-Helicity integral (1.18) becomes a conserved topological quantity (a deformation invariant). It is formally a deRham period integral with values whose ratios are rational. For the example non-transverse wave solutions presented in section 2.1 and 2.2, $\mathbf{E}\circ\mathbf{B}\neq 0$, and the Torsion (or helicity) is not conserved. That is

$$L_{\mathbf{V}}(\iiint_{closed} A\hat{}F) = \iiint_{closed} -2\mathbf{E}\circ\mathbf{B}i(V)dx\hat{}dy\hat{}dz\hat{}dt \neq 0.$$

It is remarkable, hoever, that if the example solutions of section 2.1 and 2.2 are combined, the two torsion currents cancel identically, and not only does $F\hat{}F \Rightarrow 0$, but also $A\hat{}dA \Rightarrow 0$. Hence the superposed solution generates a TTM solution that conserves the quantity of Torsion (with value zero). As mentioned above the solutions of section 2 cause the first Poincare invaraint to vanish. Hence the superposed solution is a TTM wave for which a non-zero quantity of Spin is preserved as an evolutionary deformation invaraint. The example given in section 2.3 above is the first known (TM) solution to the vacuum Maxwell equations where the Torsion vanishes, but the conserved Spin does not, and in addition the Poynting vector is in the direction of the Spin vector,

4. The Hopf Map

The example solutions given in section 2 above were inspired by the work of Ranada [12] who investigated the applications of the Hopf map to the problem of finding solutions to the Maxwell equations. Recall that the Hopf map can be written as the common constraint on the map Φ from $R4(x,y,z,s)$ to $R3(X,Y,Z)$ given by the expressions:

$$X = 2(ys - xz)$$
$$Y = -2(yz + xs)$$
$$Z = -(z^2 + s^2) + (x^2 + y^2)$$

such that

$$R^2_{(3)} = X^2 + Y^2 + Z^2 = (x^2 + y^2 + z^2 + s^2)^2 = (R^2_{(4)})^2$$

Fixing the value of $R^2_{(4)} = \pm A$ determines a sphere in R4 and also a sphere, $R^2_{(3)} = A^2$, in R3. Permuting symbols and changing signs of the components in R4 give other similar expressions relating the quadratic form in R3 to the Quartic form in R4.

Ranada suggested the 4-potential

$$\mathbf{A} = [y, -x, -s](2/\pi)/\lambda^4 \ , \quad \phi = 0/\lambda^4, \quad where\ \lambda^2 = s^2 + x^2 + y^2 + z^2.$$

This 4 potential will generate the field intensities

$$\mathbf{E} = [0, 0, 0]$$

and

$$\mathbf{B} = [-2(sy + zx), +2(sx - yz), +(-s^2 + x^2 + y^2 - z^2)](4/\pi)/\lambda^6.$$

When the right hand side of the Hopf map is mapped projectively (by choosing $s = 1$), then the format for the potentials used by Ranada becomes apparent. The components of the induced B field are precisely the coefficients of the Hopf Map. Ranada discusses the knottedness of the magnetic field lines of such a solution to the Maxwell-Faraday equations, by computing the Torsion current. Only the fourth component of the Torsion vector survives, yielding the helicity value

$$h = \mathbf{A} \circ \mathbf{B} = -8s/\pi^2\lambda^8$$

Unfortunately, as it stands, the Ranada 4-potential does not satisfy the Maxwell-Ampere equation for the vacuum. Substitution of the field intensities into the Maxwell-Ampere equation, using the Lorentz vacuum constraints, generates a finite current (although the charge density is zero). Although the Ranada suggestion does generate a conserved finite Torsion-helicity integral, it is not an acceptable vacuum solution to the Maxwell equations, as the charge current 4-vector is not zero. Moreover, the solution is static. (It is to be noted that the Ranada suggestion also generates a finite spin current, but the Spin current is not a conserved quantity in the Ranada example). Be that as it may be, the Ranada suggestion inspired the search for a vacuum solution, and led to the results of section 2 above.

Consider the modification of the Hopf map obtained by the substitution, $\{x, y, z, ct\} \to \{x, y, z, ict\}$, such that the complex map relates the position vector to a Hyperbolic surface of one sheet in a Minkowski space to the position vector of a sphere in R3; i.e., consider

$$X = 2(y \bullet ict - xz)$$
$$Y = -2(yz + x \bullet ict)$$
$$Z = -(z^2 - c^2t^2) + (x^2 + y^2)$$

such that

$$R_{(3)}^2 = X^2 + Y^2 + Z^2 = (x^2 + y^2 + z^2 - c^2t^2)^2 = (R_{(m)}^2)^2$$

A similar result occurs when the original Hopf map undergoes the complex linear transformation, $\{x, y, z, ct\} \to \{ix, iy, iz, ct\}$. However, in this case the map is from a hyperbolic surface of three sheets to the sphere in R3. The light cone in R4 maps to the origin in R3.

5. **Globally Transverse Waves and Wave Solutions that are related to Minimal surfaces**.

It is pertinent to discuss other vacuum-solution equivalence classes for which both the divergence of the Torsion vector and the divergence of the Spin vector vanish identically. The transverse equivalence class of solutions was developed by Bateman (and Whittaker) based upon the concept of self dual fields. In particular Bateman demonstrated that if two complex functions $\alpha(x, y, z, t)$ and $\beta(x, y, z, t)$ could be found that satisfied the Bateman conditions, then the complex potential 1-form

$$A = \alpha d\beta - \beta d\alpha$$

would generate a solution to the vacuum Maxwell equations. The Bateman conditions are given by the constraints

$$\nabla\alpha \times \nabla\beta = \pm\sqrt{-1}/c\{(\partial\beta/\partial t)\nabla\alpha - (\partial\alpha/\partial t)\nabla\beta\}$$

As an example, Bateman suggested the two primitive functions,

$$\alpha = (x + iy)/(z + r), \quad \beta = r - ct$$

Indeed, such functions produce a vacuum solution to the Maxwell equations, and also satisfy the eikonal expression for propagating discontinuites. The Bateman solutions create TTEM modes and are "globally transverse", for not only are both the first and second Poincare invariants zero, but also the Torsion and Spin currents vanish identically. In addition, the **E** and the **B** fields so generated from the Bateman equivalence class are complex and have zero squares: $\mathbf{E} \circ \mathbf{E} = 0$, $\mathbf{B} \circ \mathbf{B} = 0$, as well as $\mathbf{E} \circ \mathbf{B} = 0$.

These globally transverse TTEM fields, generated above, satisfy Bateman's general formula for self conjugate fields,

$$\mathbf{M} \circ \mathbf{M} = (\mathbf{E} \pm ic\mathbf{B}) \circ (\mathbf{E} \pm ic\mathbf{B}) = 0,$$

a formula which Osserman has shown is related to the existence of two 2-dimensional conjugate minimal surfaces in the four dimensional domain.

References

Although the ideas are straightforward, the algebraic complexity of evaluating examples can be overwhelming. To this end, a symbolic mathematics program based on Maple had been used to generate the objects defined above. Given a set of 4 potentials, the program will compute the field intensities, the field excitations, the charge current densities if they exist, the Torsion 4-vector and the Spin 4-vector, if they exist, and the first and second Poincare invariants. The Maple program may be found at http://www.uh.edu/~rkiehn/pdf/maxwell.zip. A printout of an example worksheet can be found as a pdf file at http://www.uh.edu/~rkiehn/pdf/maxhopf.pdf

1. A. Sommerfeld, Electrodynamics (Academic, New York, 1952). J.A.Stratton, Electromagnetic Theory McGraw Hill N.Y. 1941 Sommerfeld carefully distingushes between intensities and excitations on thermodynamic grounds.

2. Kiehn, R. M., Kiehn, G. P., and Roberds, R. B. (1991) Parity and Time-reversal Symmetry Breaking, Singular Solutions, Phys Rev A, **43**, p. 5665

3. Callan, C. G., Jr., Gilkey, P.B., Hanson, A. J. (1976) Phys. Lett 63B p. 334-. Note the phrase "This ($E \circ B \neq 0$ in the Lagrangian) clearly breaks P and T invariance.. "

4. Kiehn. R.M. (1998) "Coherent Structures are Topological Torsion Defects" to appear

Kiehn, R. M. (1997) When does a dynamical system represent an Irreversible Process, SIAM Snowbird May 1997 poster.

5. Kiehn, R. M. (1991) "Are there three kinds of superconductivity" Int. J. Mod. Phys B Vol. 5 p.1779-17790

Kiehn, R. M. and Pierce, J. F. (1969) "An Intrinsic Transport Theorem" Phys. Fluids 12, p. 1971

6. Van Dantzig. D., Proc. Cambridge Philos. Soc. 30, 421 (1934). Also see:

von Dantzig, D., 1934, "Electromagnetism Independent of metrical geometry", Proc. Kon. Ned. Akad. v. Wet. 37,

pp. 521-531, 644-652, 825-836 .

7. Marsden, J.E. and Riatu, T. S. (1994) *Introduction to Mechanics and Symmetry*, Springer-Verlag, p.122

8. Kiehn, R. M. (1976) "Retrodictive Determinism", Int. J. of Eng. Sci. **14**, p. 749

9. K. Kuratowski, *Topology* (Warsaw, 1948), Vol. I.

Lipschutz, S. (1965) *General Topology* (Schaum, New York), 88.

10. Kiehn, R. M. (1977) Periods on manfolds, quantization and gauge, J. of Math Phys **18**, no. 4, p. 614

11. Kiehn, R. M. (1992) Topological Defects, Coherent Structures and Turbulence in Terms of Cartan's Theory of Differential Topology, in B. N. Antar, R. Engels, A.A. Prinaris and T. H. Moulden, Editors, *Developments in Theoretical and Applied Mathematics*, Proceedings of the SECTAM XVI conference, The University of Tennessee Space Institute, Tullahoma, TN 37388 USA, p. III.IV.2

12. Ranada, A.F. (1992) ”Topological Electromagnetism” J. Phys A. Math Gen. 25 p,.1621-1641

Problem

Find solutions such that there exists a charge current (rotating dipole) which does not radiate.

Is this satisfied by A^F=0 and A^G=0???

Or does A^F = 0, and A^G=0 just define stationary states.