# TOPOLOGY AND TOPOLOGICAL EVOLUTION OF ELECTROMAGNETIC FIELDS AND CURRENTS.

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**Abstract** Classical electromagnetism is shown to be equivalent to a course topology defined on a set of independent variables in terms of two fundamental exterior differential systems. The domains of support for finite non-zero electromagnetic field intensities, and finite non-zero electromagnetic currents, in general cannot be compact without boundary. The only exceptions occur when the Euler characteristic of the compact domain is zero. On a domain of four independent variables, the course topology can induce two other exterior differential systems that lead to the independent concepts of topological torsion and topological spin. The exterior derivative of these two 3-forms define the Poincare deformation invariants of the electromagnetic system. The vanishing of the two 3-forms can be used to define the concepts of transverse magnetic and transverse electric modes on topological grounds. The four dimensional lines in space time associated with the 3-forms of topological torsion and topological spin can exhibit linking and separation into component domains. The possible evolution of these topological properties is studied with respect to classes of processes that can be defined in terms of singly parameterized vector fields. Non-zero values of the Poincare invariants are the source of topological and thermodynamic change.

## 1. Introduction

In the language of exterior differential systems [1] it becomes evident that classical electromagnetism is equivalent to a set of topological constraints on a variety of independent variables. Certain integral properties of an electromagnetic system are deformation invariants with respect to all continuous evolutionary processes that can be described by a singly parameterized vector field. These deformation invariants lead to the fundamental topological conservation laws described in the physical literature as the conservation of charge and the conservation of flux. Recall the definitions:

> A continuous process is defined as a map from an initial state of topology $T_{initial}$ into a final state of perhaps different topology $T_{final}$ such that the limit points of the initial state are permuted among the limit points of the final state. [2]
>
> A deformation invariant is defined as an integral over a closed manifold, $\int\cdots\int_{closed} \omega$ such that the Lie derivative of the closed integral with respect to a singly parameterized vector field, $\beta V^k$, vanishes, for any choice of parametrization, $\beta$.

The idea of a deformation invariant comes from the Cartan concept of a tube of trajectories as applied to Hamiltonian mechanics. Under certain conditions (when the virtual work vanishes) the integrand (an exterior 1-form of Action, $pdq - H(p, q, t)dt$) and the closed integration domain (a closed line integral) can be deformed in any continuous manner along the tube of trajectories, and yet the value of the closed integral remains the same. Cartan used this idea for demonstrating that the tube of trajectories is uniquely defined on a contact manifold as a Hamiltonian flow that conserves energy. [3] He thereby defined conservative Hamiltonian processes in a topological manner by requiring that processes be the subsets of singly parameterized vector fields that leave the closed integral of the 1-form of Action a deformation invariant.

However, for physical systems that can be defined by a 1-form of Action, $A$, the derived 2-form $F = dA$ is a deformation invariant with respect to *all* continuous processes that can be defined by a singly parameterized vector field. This concept is at the basis of the Helmholtz theorems in hydrodynamics, and the conservation of flux in classical electromagnetism. Herein, this topological constraint will be called the postulate of potentials. When written as the equation, $F - dA = 0$, the postulate of potentials is to be recognized as an exterior differential system constraining the topology of the independent variables. From Stokes theorem, the (2 dimensional) domain of finite support for $F$ can not in general be compact without boundary, unless the Euler characteristic vanishes. There are two exceptional cases for two dimensional domains, the torus and the Klein-Bottle, but these situations require the additional topological constraint that $F\hat{}F = 0$. The fields in these exceptional cases must reside on these exceptional compact surfaces, which form topological coherent structures in the electromagnetic field. For an electromagnetic action, the exceptional compact cases can only exist if $\mathbf{E} \circ \mathbf{B} = 0$. The resulting statement is that there do not exist compact domains of support without boundary when $\mathbf{E} \circ \mathbf{B} \neq 0$, a statement that will be of interest to thermodynamics of irreversible systems.

An electromagnetic system will be defined in terms of a second topological constraint imposed upon the domain of independent variables. This postulate will be called the postulate of conserved currents. The electromagnetic domain not only supports the 1-form *A*, but also supports an N-1=3 form, *J*, which is exact. The equivalent differential system, $J - dG = 0$, requires that the (N-1 dimensional) domain of support for *J* cannot be compact without boundary. However, the closed integrals of *J* are deformation invariants for *any* continuous evolutionary process that can be defined in terms of a singly parameterized vector field. For the 3-forms of charge current, a similar arguement indicates that the compact domains of support are limited to those of zero Euler characteristic. The classic example is the three sphere, $S^3$. The three sphere (that will support currents with out zeros) has a famous map to compact two sphere. Hence, there can exist domains of field excitations on compact two spheres, such that the induced current, $J = dG$, resides on the three sphere. The image is the Hopf map, which can have torsion. Such currents are in the direction of the torsion vector, $A\hat{}dA = A\hat{}F$, and have extraordinary properties, as will be shown below.

In section 2, the classical Maxwell system will be displayed in terms of the vector formalism of Sommerfeld and Stratton. The key feature is to note that the fields of intensities (**E** and **B**) are considered as separate and distinct from the fields of excitation (**D** and **H**), a historical distinction that is often masked in modern exposes of electromagnetic theory.

In section 3, it will be demonstrated explicitly that the classic formalism of electromagnetism in section 2 is a consequence of a system of two fundamental topological constraints

$$F - dA = 0, \qquad J - dG = 0. \tag{1.1}$$

defined on a domain of four independent variables. The theory requires the existence of four fundamental exterior differential forms, $\{A, F, G, J\}$, which can be used to construct the complete Pfaff sequence [4] of forms by the processes of exterior differentiation and exterior multiplication. On a domain of four independent variables, the complete Pfaff sequence contains three 3-forms: the classic 3-form of charge current density, $J$, and the (apparently novel to many researchers) 3-forms of Spin Current density, $A$^$G$, [5] and Topological Torsion-Helicity, $A$^$F$ [6].

As the charge current 3-form, $J$, is a deformation invariant by construction, it is of interest to determine topological refinements or constraints for which the 3-forms of Spin Current and Topological Torsion will define physical topological conservation laws in the form of deformation invariants. The additional constraints are equivalent to the topological statement that the closure (exterior derivative) of each of the three forms is empty (zero). It will be demonstrated in section 4 that these closure conditions define the two classic Poincare invariants (4-forms) as deformation invariants, and when each of these invariants vanish the corresponding 3-form generates a topological quantity (Spin or Torsion respectively) which is also a deformation invariant. The possible values of the topological quantities, as deRham period integrals [7], form rational ratios.

The concepts of Spin Current and the Torsion vector have been utilized hardly at all in applications of classical electromagnetic theory. Just as the vanishing of the 3-form of charge current, $J = 0$, defines the topological domain called the vacuum, the vanishing of the two other 3-forms will refine the fundamental topology of the Maxwell system. Such constraints permit a definition of transversality to be made on topological (rather than geometrical) grounds. If both $A$^$G$ and $A$^$F$ vanish, the vacuum state supports topologically transverse modes only (TTEM). Examples lead to the conjecture that TTEM modes do not transmit power, a conjecture that has been verified when the concept of geometric transversality (TEM) and topological transversality (TTEM) coincide. A topologically transverse magnetic (TTM) mode corresponds to the topological constraint that $A$^$F = 0$. A topologically transverse electric mode (TTE) corresponds to the topological constraint that $A$^$G = 0$. Examples, both novel and well-known, of vacuum solutions to the electromagnetic system which satisfy (and which do not satisfy) these topological constraints are given in section 4. The ideas should be of interest to those working in the field of Fiber Optics. Recall that classic solutions which are geometrically and topologically transverse (TEM≡TTEM) do not transmit power [8]. However, in section 4 an example vacuum wave solution is given which is geometrically transverse (the fields are orthogonal to the field momentum and the wave vector), and yet the geometrically transverse wave transmits power at a constant rate: the example wave is not topologically transverse as $A \hat{} F \neq 0$.

In section 5, an additional topological constraint will be used to define the plasma process as a restriction on all processes which can be described in terms of a singly parameterized vector field. The plasma process (which is to be distinguished from a Hamiltonian process) will be restricted to those vector fields which leave the closed integrals of $G$ a deformation invariant. (Compare to the Cartan definition that a Hamiltonian process is a restriction on arbitrary processes such that the closed integrals of $A$ are deformation invariants with respect to Hamiltonian processes). A plasma process need not conserve energy. A *perfect* plasma process is a plasma process which is also a Hamiltonian process. Again, the three forms, $J$, $A \hat{} G$ and $A \hat{} F$ are of particular interested for their tangent manifolds define "lines" in the 4-dimensional variety of space and time. Relative to plasma processes, the topological evolution associated with such lines, and their entanglements, is of utility in understanding solar corona and plasma instability. [9]

## 2. The Domain of Classical Electromagnetism

2.1 The classical Maxwell-Faraday and the Maxwell-Ampere equations.

Using the notation and the language of Sommerfeld and Stratton [10], the classic definition of an electromagnetic system is a domain of space-time $\{x, y, z, t\}$ which supports both the Maxwell-Faraday equations,

$$curl\, \mathbf{E} + \partial\mathbf{B}/\partial t = 0, \quad div\, \mathbf{B} = 0, \tag{2.1}$$

and the Maxwell-Ampere equations,

$$curl\, \mathbf{H} - \partial\mathbf{D}/\partial t = \mathbf{J}, \quad div\, \mathbf{D} = \rho. \tag{2.2}$$

2.2 The conservation of charge current

In every case, the charge current density for the Maxwell system satisfies the conservation law,

$$div\, \mathbf{J} + \partial\rho/\partial t = 0. \tag{2.3}$$

The charge-current densities are subsumed to be zero [$\mathbf{J}$, $\rho$]= 0 for the vacuum state.

For the Lorentz vacuum state, the field excitations, **D** and **H**, are linearly connected to the field intensities, **E** and **B**, by means of the Lorentz (homogeneous and isotropic) constitutive relations:

$$\mathbf{D} = \varepsilon\mathbf{E}, \quad \mathbf{B} = \mu\mathbf{H}. \tag{2.4}$$

The two vacuum constraints imply that the solutions to the homogeneous Maxwell equations also satisfy the vector wave equation, typically of the form

$$grad\ div\ \mathbf{B} - curl\,curl\,\mathbf{B} - \varepsilon\mu\partial^2\mathbf{B}/\partial t^2 = 0. \tag{2.5}$$

The constant wave phase velocity, $v_p$, is taken to be

$$v_p^2 = 1/\varepsilon\mu \equiv c^2 \tag{2.6}$$

Similar results can be obtained for the solid state where the constitutive constraints can be more complex [11], and for the plasma state where the charge-current densities are not zero.

2.2 The existence of potentials

It is further subsumed that the classic Maxwell electromagnetic system is constrained by the statement that the field intensities are deducible from a system of twice differentiable potentials, $[\mathbf{A}, \phi]$:

$$\mathbf{B} = curl\ \mathbf{A}, \quad \mathbf{E} = -grad\ \phi - \partial\mathbf{A}/\partial t. \tag{2.7}$$

This constraint topologically implies that domains that support non-zero values for the covariant field intensities, $\mathbf{E}$ and $\mathbf{B}$, can *not* be compact domains without a boundary. It is this constraint that distinguishes classical electromagnetism from Yang Mills theories. Two other classical 3-vector fields are of interest, the Poynting vector $\mathbf{E}\times\mathbf{H}$ representing the flux of electromagnetic radiative energy, and the field momentum flux, $\mathbf{D}\times\mathbf{B}$.

## 3. The Fundamental Exterior Differential Systems.

The formulation of Maxwell theory in section 2 is relative to a choice independent variables $\{x, y, z, t\}$ using classical vector analysis developed in euclidean 3-space. The topological features of the formalism are not immediately evident. However, electromagnetism has a formulation in terms of Cartan's exterior differential forms [12]. Exterior differential forms do not depend upon a choice of coordinates, do not depend upon the a choice of metric, and are independent of the constraints imposed by gauge groups and connections. In such a formulation the equations of an electromagnetic system become recognized as consequences of topological constraints on a domain of independent variables.

The use of differential forms should not be viewed as just another formalism of fancy. The technique goes beyond the methods of tensor calculus, and admits the study of topological evolution. Recall that if an exterior differential system is valid on a final variety of independent variables {x,y,z,t}, then it is also true on any initial variety of independent variables that can be mapped onto {x,y,z,t}. The map need only be differentiable, such that the Jacobian matrix elements are well defined *functions*. The Jacobian matrix does not have to have an inverse, so that the exterior differential system is not restricted to the equivalence class of diffeomorphisms. The field intensities on the initial variety are functionally well defined by the pullback mechanism, which involves algebraic composition with components of the Jacobian matrix transpose, and the process of functional substitution. This independence from a choice of independent variables (or coordinates) for Maxwell's equations was first reported by Van Dantzig [13]. It follows that the Maxwell differential system is well defined in a covariant manner for both Galilean transformations as well as Lorentz transformations, or any other diffeomorphism. (The singular solution sets to the equations do not enjoy this universal property). In addition, it should be noted that the ideas of the exterior differential system imply that the closure equations of the Maxwell-Faraday type form a nested set, with exactly the same format, independent of the choice of the *number* of independent variables. Every physical system (such as fluid) that supports a 1-form of Action, also has its version of the Maxwell-Faraday equations.

3.1 The Maxwell-Faraday exterior differential system.

The Maxwell-Faraday equations are a consequence of the exterior differential system

$$F - dA = 0, \qquad (3.1)$$

where $A$ is a 1-form of Action, with twice differentiable coefficients (potentials proportional to momenta) which induce a 2-form, $F$, of electromagnetic intensities ($\mathbf{E}$ and $\mathbf{B}$, related to forces). The exterior differential system is a topological constraint that in effect defines field intensities in terms of the potentials. On a four dimensional space-time of independent variables, $(x, y, z, t)$ the 1-form of Action (representing the postulate of potentials) can be written in the form

$$A = \Sigma_{k=1}^{3} A_k(x,y,z,t)dx^k - \phi(x,y,z,t)dt = \mathbf{A} \circ d\mathbf{r} - \phi dt. \qquad (3.2)$$

Subject to the constraint of the exterior differential system, the 2-form of field intensities, $F$, becomes:

$$F = dA = \{\partial A_k/\partial x^j - \partial A_j/\partial x^k\} dx^j \wedge dx^k = F_{jk} dx^j \wedge dx^k = \mathbf{B}_z dx \wedge dy ... \mathbf{E}_x dx \wedge dt ... \qquad (3.3)$$

where in usual engineering notation,

$$\mathbf{E} = -\partial \mathbf{A}/\partial t - grad\phi, \qquad \mathbf{B} = curl\ \mathbf{A} \equiv \partial A_k/\partial x^j - \partial A_j/\partial x^k. \qquad (3.4)$$

The closure of the exterior differential system, $dF = 0$,

$$dF = ddA = \{curl\ \mathbf{E} + \partial \mathbf{B}/\partial t\}_x dy \wedge dz \wedge dt - .. + .. - div\,\mathbf{B} dx \wedge dy \wedge dz\} \Rightarrow 0, \qquad (3.5)$$

generates the Maxwell-Faraday partial differential equations.:

$$\{curl\ \mathbf{E} + \partial \mathbf{B}/\partial t = 0, \quad div\,\mathbf{B} = 0\}. \qquad (3.6)$$

The component functions ($\mathbf{E}$ and $\mathbf{B}$) of the 2-form, $F$, transform as covariant tensor of rank 2. The topological constraint that $F$ is exact, implies that the domain of support for the field intensities cannot be compact without boundary, unless the Euler characteristic vanishes. These facts distinguish classical electromagnetism from Yang-Mills field theories. Moreover, the fact that $F$ is subsumed to be exact and C1 differentiable excludes the concept of magnetic monopoles from classical electromagnetic theory on topological grounds. The integral of the 2-form $F$ over any closed 2-manifold is a deformation (topological) invariant of any evolutionary process that can be described by a singly parameterized vector field, for

$$L_{\mathbf{V}}(\iint_{closed} F) = \iint_{closed} \{i(V)dF + d(i(V)F)\} = \iint_{closed} \{0 + d(i(V)F)\} = \iint_{closed} d(i(V)F) = 0. \qquad (3.7)$$

3.2 The Maxwell Ampere exterior differential system

The Maxwell Ampere equations are a consequence of second exterior differential system,

$$J - dG = 0, \qquad (3.8)$$

where $G$ is an N-2 form *density* of field excitations ($\mathbf{D}$ and $\mathbf{H}$, related to sources), and $J$ is the N-1 form of charge-current densities. The partial differential equations equivalent to the exterior differential system are precisely the Maxwell-Ampere equations. This second postulate, on a four dimensional domain of independent variables, assumes the existence of a N-2 form density given by the expression,

$$G = G^{34}(x,y,z,t)dx\hat{}dy... + G^{12}(x,y,z,t)dz\hat{}dt... = \mathbf{D}^z dx\hat{}dy...\mathbf{H}^z dz\hat{}dt... \tag{3.9a}$$

Exterior differentiation produces an N-1 form,

$$J = \mathbf{J}^z(x,y,z,t)dx\hat{}dy\hat{}dt... - \rho(x,y,z,t)dx\hat{}dy\hat{}dz. \tag{3.9b}$$

Matching the coefficients of the exterior expression $dG = J$ leads to the Maxwell-Ampere equations,

$$curl\,\mathbf{H} - \partial\mathbf{D}/\partial t = \mathbf{J} \quad and \quad div\,\mathbf{D} = \rho. \tag{3.10}$$

The fact that $J$ is exact leads to the charge conservation law, $dJ = ddG = 0$, or

$$\partial\mathbf{J}^x/\partial x + \partial\mathbf{J}^y/\partial y + \partial\mathbf{J}^z/\partial z + \partial\rho/\partial t = 0. \tag{3.11}$$

The exterior differential system is a topological constraint for by Stokes theorem the support for $G$ can be compact without boundary only if the domain is without charge-currents. The closure of the exterior differential system, $dJ = 0$, generates the charge-current conservation law. The integral of $J$ over a closed 3 dimensional domain is a relative integral invariant (a deformation invariant) of any process that can be described in terms of a singly parametrized vector field. The formal statement is given by Cartan's magic formula [14], which describes continuous topological evolution in terms of the action of the Lie derivative, with respect to a vector field, acting on the exterior differential 3-form, $J$ :

$$L_V(\iiint_{closed} J) = \iiint_{closed}\{i(V)dJ + d(i(V)J)\} = \iiint_{closed}\{0 + d(i(V)J)\} = 0. \tag{3.12}$$

The Lie derivative is equal to zero for any 4-vector field $V$, when $dJ = 0$. The integral is then a deformation invariant, for the result is valid even if the 4-vector field is distorted by an arbitrary function, $f\{x,y,z,t\}$, such that $\mathbf{V} \Rightarrow f(x,y,z,t)\mathbf{V}$.

### 3.3 The Torsion and Spin 3-forms

As mentioned above, the method of exterior differential forms goes beyond the domain of classical tensor analysis, for it admits of maps from initial to final state that are without inverse. (Tensor analysis and coordinate transformations require that the Jacobian map from initial to final state has an inverse - the method of exterior differential forms does not.) Hence the theory of electromagnetism expressed in the language of exterior differential forms admits of topological evolution, at least with respect to continuous processes without Jacobian inverse. With respect to such non-invertible maps, both tensor fields and differential forms are not functionally well defined in a predictive sense [15]. Given the functional forms of a tensor field on an initial state, it is impossible to predict uniquely the functional form of the tensor field on the final state unless the map between initial and final state is invertible. However differential forms are functionally well defined in a retrodictive sense, by means of the pullback. Covariant anti-symmetric tensor fields pull back retrodictively with respect to the transpose of the Jacobian matrix (of functions) and functional substitution, and contravariant tensor densities pullback retrodictively with respect to the adjoint of the Jacobian matrix, and functional substitution. The transpose and the adjoint of the Jacobian exist, even if the Jacobian inverse does not.

The exterior differential forms that make up the electromagnetic system consist of the primitive 1-form, *A*, and the primitive N-2 form density, *G*, their exterior derivatives, and their algebraic intersections defined by all possible exterior products. The complete Maxwell system of exterior differential forms (the Pfaff sequence for the Maxwell system) is given by the set:

$$\{A, F = dA, G, J = dG, A\wedge F, A\wedge G, A\wedge J, F\wedge F, G\wedge G\}. \qquad (3.13)$$

These forms and their unions may be used to form a topological base on the domain of independent variables. The Cartan topology constructed on this system of forms has the useful feature that the exterior derivative may be interpreted as a limit point, or closure, operator in the sense of Kuratowski [16]. The exterior differential systems that define the Maxwell-Ampere and the Maxwell-Faraday equations above are essentially topological constraints of closure. Note that the complete Maxwell system of differential forms (which assumes the existence of *A*) also generates two other exterior differential systems.

$$d(A\wedge G) - (F\wedge G - A\wedge J) = 0, \qquad (3.14)$$

and

$$d(A\wedge F) - F\wedge F = 0. \qquad (3.15)$$

The two objects, $A\wedge G$ and $A\wedge F$ are three forms, not usually found in discussions of classical electromagnetism. The closed components of the first 3-form (density) were called topological spin [17] and the closed components of the second 3-form were called topological torsion (or helicity) [18]. By direct evaluation of the exterior product, and on a domain of 4 independent variables, each 3-form will have 4 components that can be symbolized by the 4-vector arrays

$$Spin - Current \, : \, \mathbf{S}_4 = [\mathbf{A}\times\mathbf{H} + \mathbf{D}\phi, \mathbf{A}\circ\mathbf{D}] \equiv [\mathbf{S}, \sigma], \qquad (3.16)$$

and

$$Torsion - vector \, : \, \mathbf{T}_4 = [\mathbf{E}\times\mathbf{A} + \mathbf{B}\phi, \mathbf{A}\circ\mathbf{B}] \equiv [\mathbf{T}, h], \qquad (3.17)$$

which are to be compared with the charge current 4-vector density:

$$Ch\arg e - Current \, : \, \mathbf{J}_4 = [\mathbf{J}, \rho],$$

The 3-forms then can be defined by the equivalent contraction processes

$$Topological \ Spin \ 3 - form \doteq \ A\hat{}G \qquad (3.18)$$

$$= i(\mathbf{S}_4)dx\hat{}dy\hat{}dz\hat{}dt = \mathbf{S}^x dy\hat{}dz\hat{}dt..... - \sigma dx\hat{}dy\hat{}dz$$

and

$$Topological \ Torsion - helicity \ \ 3 - form \doteq A\hat{}F \qquad (3.19)$$

$$= i(\mathbf{T}_4)dx\hat{}dy\hat{}dz\hat{}dt = \ \mathbf{T}^x dy\hat{}dz\hat{}dt..... - hdx\hat{}dy\hat{}dz.$$

The vanishing of the first 3-form is a topological constraint on the domain that defines topologically transverse electric (TTE) waves: the vector potential, **A**, is orthogonal to **D**, in the sense that $\mathbf{A}\circ\mathbf{D} = 0$. The vanishing of the second 3-form is a topological constraint on the domain that defines topologically transverse magnetic (TTM) waves: the vector potential, **A**, is orthogonal to **B**, in the sense that $\mathbf{A}\circ\mathbf{B} = 0$. When both 3-forms vanish, the topological constraint on the domain defines topologically transverse (TTEM) waves. For classic real fields this double constraint would require that vector potential, **A**, is collinear with the field momentum, $\mathbf{D}\times\mathbf{B}$, and in the direction of the wave vector, **k**.

The geometric notion of distinct transversality modes of electromagnetic waves is a well known concept experimentally, but the association of transversality to topological issues is novel herein. For certain examples that appear in section 4, it is apparent that the concept of geometric and topological transversality are the same. In the classic case, often considered in fiber optic theory, it is known that the TEM modes do transmit power. However, in section 4 a vacuum wave solution is given which satisfies the geometric concept of transversality ( it is both a TM and a TE solution) but the mode radiates for it is not both a TTM and a TTE solution. The conjecture obtained from examples is that a TTEM solution does not radiate.

Note that if the 2-form $F$ was not exact, such topological concepts of transversality would be without meaning, for the 3-forms of Topological Spin and Topological Torsion depend upon the existence of the 1-form of Action. The torsion vector $\mathbf{T}_4$ and the Spin vector $\mathbf{S}_4$ are associated vectors to the 1-form of Action in the sense that

$$i(\mathbf{T}_4)A = 0 \;\; and \;\; i(\mathbf{S}_4)A = 0 \tag{3.20}$$

3.4 The Poincare Invariants

The exterior derivatives of the 3-forms of Spin and Torsion produce two 4-forms, $F\hat{}G - A\hat{}J$ and $F\hat{}F$, whose integrals over closed 4 dimensional domains are deformation invariants for the Maxwell system. These topological objects are related to the conformal invariants of a Lorentz system as discovered by Poincare and Bateman. In the format of independent variables $\{x,y,z,t\}$, the exterior derivative corresponds to the 4-divergence of the 4-component Spin and Torsion vectors, $\mathbf{S}_4$ and $\mathbf{T}_4$. The functions so created define the Poincare conformal invariants of the Maxwell system:

$$Poincare\ 1 = d(A\hat{}G) = F\hat{}G - A\hat{}J \tag{3.21a}$$

$$= \{div_3(\mathbf{A}\times\mathbf{H} + \mathbf{D}\phi) + \partial(\mathbf{A}\circ\mathbf{D})/\partial t\}dx\hat{}dy\hat{}dz\hat{}dt$$

$$= \{(\mathbf{B}\circ\mathbf{H} - \mathbf{D}\circ\mathbf{E}) - (\mathbf{A}\circ\mathbf{J} - \rho\phi)\}dx\hat{}dy\hat{}dz\hat{}dt$$

$$Poincare\ 2 = d(A\hat{}F) = F\hat{}F \tag{3.21b}$$

$$= \{div_3(\mathbf{E}\times\mathbf{A} + \mathbf{B}\phi) + \partial(\mathbf{A}\circ\mathbf{B})/\partial t\}dx\hat{}dy\hat{}dz\hat{}dt$$

$$= \{-2\mathbf{E}\circ\mathbf{B}\}dx\hat{}dy\hat{}dz\hat{}dt$$

For the vacuum state, with $J = 0$, zero values of the Poincare invariants require that the magnetic energy density is equal to the electric energy density ($1/2\mathbf{B}\circ\mathbf{H} = 1/2\mathbf{D}\circ\mathbf{E}$), and, respectively, that the electric field is orthogonal to the magnetic field ($\mathbf{E}\circ\mathbf{B} = 0$). Note that these constraints often are used as elementary textbook definitions of what is meant by electromagnetic waves. When either Poincare invariant vanishes, the corresponding closed 3-dimensional integral becomes a topological quantity in the sense of a deRham period integral. For example, when the first Poincare invariant vanishes, the closed integral of the 3-form of spin becomes a deformation invariant with quantized values:

$$Define\ :\ Spin = \iiint_{closed} A\hat{}G \qquad Let\ d(A\hat{}G) = 0,\ then \qquad (3.22)$$

$$L_{\mathbf{V}}(Spin) = \iiint_{closed} \{i(V)d(A\hat{}G) + d(i(V)(A\hat{}G)\}$$

$$= \iiint_{closed} \{0 + d(i(V)(A\hat{}G)\} \ = 0.$$

Similarly, when the second Poincare invariant vanishes, the closed integral of the 3-form of Torsion-helicity becomes a deformation invariant with quantized values:

$$Define\ :\ Torsion\text{-}Helicity = \iiint_{closed} A\hat{}F \qquad Let\ d(A\hat{}F) = 0, \quad then$$

$$(3.23)$$

$$L_{\mathbf{V}}(Torsion\text{-}Helicity) = \iiint_{closed} \{i(V)d(A\hat{}F) + d(i(V)(A\hat{}F)\}$$

$$= \iiint_{closed} \{0 + d(i(V)(A\hat{}F)\} \ = 0.$$

It is important to realize that these topological conservation laws are valid in a plasma as well as in the vacuum, subject to the conditions of zero values for the Poincare invariants. On the other hand, topological transitions require that the Poincare invariants are not zero.

## 4. Electromagnetic Waves in the Vacuum with Spin and Torsion

### 4.1 Solutions Old and New

As the Spin 4-vector and the Torsion 4-vector formalism may be unfamiliar to many readers, it is useful to compare four classes of unusual vacuum wave solutions with the usual waveguide solutions. The "unusual waves" have their vector potential, **A**, orthogonal to the wave vector, **k**, describing the direction of the wave front. In each unusual example, the current density is in the direction of the vector potential and therefore also orthogonal to the wave vector. The usual wave solutions have their vector potential parallel to the wave vector. The four unusual cases belong to equivalence classes defined by the constraints

$$(A\hat{}F = 0, A\hat{}G \neq 0)$$
$$(A\hat{}F \neq 0, A\hat{}G = 0)$$
$$(A\hat{}F = 0, A\hat{}G = 0)$$
$$(A\hat{}F \neq 0, A\hat{}G \neq 0).$$

Each component of the potentials satisfies the wave equation subject to the dispersion relation, $\omega/k \pm \sqrt{1/\xi\mu} = 0$. The examples do not generate any charge current distributions when the vacuum dispersion equation is satisfied (the phase velocity equals to the group velocity equals the speed of light as determined by the constitutive equations). The choice of dispersion equation solution determines the direction of wave propagation.

In each example given below, the 1-form of Action is specified and the field intensities are computed. Then the Spin Current and the Torsion vector are evaluated. The functions have been chosen to satisfy the Lorentz vacuum conditions of zero charge current densities, subject to a dispersion relation. The Poynting vector is computed, and the Poincare invariants are evaluated.

The four classes of these simple (but unusual) wave types correspond to:

Example 1. Real Linear Polarization:

Consider the Potentials

$$A = [\cos(kz - \omega t), \cos(kz - \omega t), 0, 0] \tag{4.1}$$

and their induced fields:

$$\mathbf{E} = [-\sin(kz - \omega t), -\sin(kz - \omega t), 0]\omega$$

$$\mathbf{B} = [+\sin(kz - \omega t), -\sin(kz - \omega t), 0]k$$

$$\mathbf{J}_4 = [\cos(kz - \omega t), \cos(kz - \omega t), 0, 0](k^2 - \varepsilon\mu\omega^2)/\mu$$

$$\mathbf{S}_4 = [0, 0, -k/\mu, -\varepsilon\omega]\, 2\cos(kz - \omega t)\sin(kz - \omega t).$$

$$\mathbf{T}_4 = [0, 0, 0, 0].$$

$$\mathbf{E} \times \mathbf{H} = [0, 0, 1](\omega k/\mu)(2\cos(kz - \omega t)^2 - 1)$$

$$(\mathbf{B} \circ \mathbf{H} - \mathbf{D} \circ \mathbf{E}) = -2\{\cos(kz - \omega t)^2 - \sin(kz - \omega t)^2\}(k^2 - \varepsilon\mu\omega^2)/\mu \qquad (\mathbf{E} \circ \mathbf{B}) = 0$$

This class of potentials generates a set of complex field intensities and excitations, and a current density proportional to the vector potential. If the dispersion relation $(k^2 - \varepsilon\mu\omega^2) = 0$ is satisfied, then the solutions are acceptable vacuum solutions, with a vanishing charge current density. The

Torsion vector vanishes indentically, independent from the dispersion condition, but the Spin vector does not. The first Poincare invariant vanishes subject to the constraint of the dispersion relation. The second Poincare invariant vanishes identically. The solution corresponds to a linear state of polarization at 45° with respect to the x-axis, with the electric and the magnetic fields in phase. There is a non-zero Poynting vector along the z axis., which is orthogonal to the vector potential. Note that the radiated power has a time average which is zero. If the charge current density is not zero (due to a fluctuation in the dispersion relation) the charge current vector is orthogonal to the Spin current vector.

Example 2. Real Circular Polarization:

Consider the Potentials

$$A = [\cos(kz-\omega t), \sin(kz-\omega t), 0, 0] \tag{4.2}$$

and their induced fields:

$$\mathbf{E} = [-\sin(kz-\omega t), +\cos(kz-\omega t), 0]\omega$$

$$\mathbf{B} = [-\cos(kz-\omega t), -\sin(kz-\omega t), 0]k$$

$$\mathbf{J}_4 = [\cos(kz-\omega t), \sin(kz-\omega t), 0, 0](k^2-\varepsilon\mu\omega^2)/\mu$$

$$\mathbf{S}_4 = [0, 0, 0, 0].$$

$$\mathbf{T}_4 = [0, 0, -\omega, -k].$$

$$\mathbf{E}\times\mathbf{H} = [0, 0, 1]\,\omega k/\mu$$

$$(\mathbf{B}\circ\mathbf{H}-\mathbf{D}\circ\mathbf{E}) = 0 \qquad (\mathbf{E}\circ\mathbf{B}) = 0$$

This class of potentials generates a set of complex field intensities and excitations, and a current density proportional to the vector potential. If the dispersion relation $(k^2-\varepsilon\mu\omega^2) = 0$ is satisfied, then the solutions are acceptable vacuum solutions, with a vanishing charge current density. The Spin vector vanishes identically, but the Torsion vector does not. In fact, the torsion vector is constant. The solution corresponds to a circular state of polarization with the constant magnetic and electric amplitudes rotating about the z axis. The Poynting vector is not zero and is a constant, time independent, vector. This wave solution is geometrically transverse (TEM), yet it produces power as it is not topologically transverse (TTEM). If the dispersion relation is not precisely satisfied, the current vector is orthogonal to the Torsion vector and parallel to the vector potential. Both Poincare invariants vanish identically. The soliton like solution should be compared to the wave guide solution of example 5 below, which is also TEM, but does not radiate.

Example 3. Complex Linear Polarization:
Consider the Potentials

$$A = [\cos(kz - \omega t), i\cos(kz - \omega t), 0, 0] \tag{4.3}$$

and their induced fields:

$$\mathbf{E} = [-\sin(kz - \omega t), -i\sin(kz - \omega t), 0]\omega$$

$$\mathbf{B} = [+i\sin(kz - \omega t), -\sin(kz - \omega t), 0]k$$

$$\mathbf{J}_4 = [\cos(kz - \omega t), i\cos(kz - \omega t), 0, 0](k^2 - \varepsilon\mu\omega^2)/\mu$$

$$\mathbf{S}_4 = [0, 0, 0, 0].$$

$$\mathbf{T}_4 = [0, 0, 0, 0].$$

$$\mathbf{E} \times \mathbf{H} = [0, 0, 0]$$

$$(\mathbf{B} \circ \mathbf{H} - \mathbf{D} \circ \mathbf{E}) = 0 \qquad (\mathbf{E} \circ \mathbf{B}) = 0$$

This class of potentials generates a set of complex field intensities and excitations, and a current density proportional to the vector potential. The fields are said to be complex linearly polarized because the complex **B** field is a complex scalar multiple of the complex **E** field. If the dispersion relation $(k^2 - \varepsilon\mu\omega^2) = 0$ is satisfied, then the solutions are acceptable vacuum solutions, with a vanishing charge current density. Note that both the Torsion vector and the Spin vector vanish identically. The complex square of both the electric and the magnetic field vectors vanish. Both Poincare invariants vanish independent from the dispersion constraint. Although the fields are propagating, there is no momentum flux and the Poynting vector is zero. The **E** and **B** fields are (complex) collinear. This example is perhaps the simplest member of the class of Bateman-Whittaker complex solutions described in Example 10, below.

Example 4. Complex Circular Polarization:
Consider the Potentials

$$A = [\cos(kz - \omega t), i\sin(kz - \omega t), 0, 0] \tag{4.4}$$

and their induced fields:

$$\mathbf{E} = [-\sin(kz-\omega t), +i\cos(kz-\omega t), 0]\omega$$

$$\mathbf{B} = [-i\cos(kz-\omega t), -\sin(kz-\omega t), 0]k$$

$$\mathbf{J}_4 = [\cos(kz-\omega t), i\sin(kz-\omega t), 0, 0](k^2-\varepsilon\mu\omega^2)/\mu$$

$$\mathbf{S}_4 = [0, 0, -k/\mu, -\varepsilon\omega]\, 2\cos(kz-\omega t)\sin(kz-\omega t).$$

$$\mathbf{T}_4 = i[0, 0, -\omega, -k].$$

$$\mathbf{E}\times\mathbf{H} = [0, 0, -1]\,(\omega k/\mu)(2\cos(kz-\omega t)^2 - 1)$$

$$(\mathbf{B}\circ\mathbf{H} - \mathbf{D}\circ\mathbf{E}) = -2\{\cos(kz-\omega t)^2 - \sin(kz-\omega t)^2\}(k^2-\varepsilon\mu\omega^2)/\mu \qquad (\mathbf{E}\circ\mathbf{B}) = 0$$

This class of potentials generates a set of complex field intensities and excitations, and a current density proportional to the vector potential. If the dispersion relation $(k^2-\varepsilon\mu\omega^2) = 0$ is satisfied, then the solutions are acceptable vacuum solutions, with a vanishing charge current density. Both the Torsion vector (imaginary) and the Spin vector (real) do not vanish. The second Poincare invariant vanishes identically, and the first Poincare invariant vanishes subject to the dispersion constraint. The current vector, if non-zero due to fluctuations in the dispersion relation, is orthogonal to both the Torsion vector and the Spin vector.

Examples 1 through 4 above are geometrically transverse waves in the engineering sense that the propagation direction (along the z axis) is in the direction of the momentum flux, $\mathbf{D}\times\mathbf{B}$. However, the waves are not "topologically transverse" in that the sense that the **D** and **B** fields are not necessarily transverse to the components of the vector potential **A**.

Example 5. Waveguide TEM modes

Consider the Potentials

$$A = [0, 0, \phi(x,y), (\omega/k)\phi(x,y)]\cos(kz-\omega t) \qquad (4.5)$$

and their induced fields:

$$\mathbf{E} = [-(\omega/k)\partial\phi/\partial x, -(\omega/k)\partial\phi/\partial y, 0]\cos(kz-\omega t)$$

$$\mathbf{B} = [\partial\phi/\partial y, -\partial\phi/\partial x, 0]\cos(kz-\omega t)$$

$$\begin{aligned}\mathbf{J}_4 = [&\partial\phi/\partial x(\varepsilon\mu(\omega/k)^2 - 1)\sin(kz - \omega t),\\ &\partial\phi/\partial y(\varepsilon\mu(\omega/k)^2 - 1)\sin(kz - \omega t),\\ &\nabla^2\phi\cos(kz - \omega t),\\ &(\varepsilon\mu\omega/k)\nabla^2\phi\cos(kz - \omega t)]/\mu\end{aligned}$$

$$\begin{aligned}\mathbf{S}_4 = [&\phi\partial\phi/\partial x\cos(kz - \omega t)^2(1 - \varepsilon\mu(\omega/k)^2),\\ &\phi\partial\phi/\partial y\cos(kz - \omega t)^2(1 - \varepsilon\mu(\omega/k)^2),\\ &0,\\ &0]/\mu\end{aligned}$$

$$\mathbf{T}_4 = [0, 0, 0, 0].$$

$$\begin{aligned}\mathbf{E}\times\mathbf{H} = [&\phi(\partial\phi/\partial x)k\cos(kz - \omega t)\sin(kz - \omega t)(v_g - v_p),\\ &\phi(\partial\phi/\partial y)k\cos(kz - \omega t)\sin(kz - \omega t)(v_g - v_p),\\ &(v_g)\cos(kz - \omega t)^2(\nabla^2\phi)]/\mu\end{aligned}$$

$$(\mathbf{B}\circ\mathbf{H} - \mathbf{D}\circ\mathbf{E}) \neq 0 \qquad (\mathbf{E}\circ\mathbf{B}) = 0$$

Note that the vector potential, $\mathbf{A}$, is parallel to both the wave vector, $\mathbf{k}$, and the field momentum, $\mathbf{D}\times\mathbf{B}$. The Torsion vector and the second Poincare invariant are indentically zero. The solution produces current and spin densities unless a dispersion relation, $\varepsilon\mu(\omega/k)c^2 = 1$, is satisfied. Subject to the dispersion constraints, this classic solution has both a zero Torsion vector and a zero Spin vector. Both $\mathbf{A}\circ\mathbf{D} = 0$ and $\mathbf{A}\circ\mathbf{B} = 0$. The wave front is in the spatial direction of the potential, by construction. The candidate solution subject to the dispersion relation is both topologically transverse TTEM and geometrically transverse, TEM .

However, even if the dispersion relations are satisfied, the geometric TEM solution produces finite charge current densities, unless the function $\phi(x, y)$ is a solution of the two dimensional Laplace equation, $\nabla^2\phi = 0$. This further constrain implies that the TEM solution produces no radiated power in the charge free state, for $\mathbf{E}\times\mathbf{H} \Rightarrow 0$ as $\nabla^2\phi \Rightarrow 0$. In the next example the constraint that the system be TTEM is relaxed, and radiated power is achieved. in a TTM mode.

Example 6. Waveguide TM modes

Consider the Potentials

$$A = [0, 0, \phi(x,y)\cos(kz - \omega t), v_g\phi(x,y)\cos(kz - \omega t) \qquad (4.6)$$

and their induced fields (note that example 6 differes from example 5 in that a "group" velocity $v_g$ is used in the definition of the potentials, instead of the phase velocity, $v_p = \omega/k$):

$$\mathbf{E} = [-v_g\partial\phi/\partial x, -v_g\partial\phi/\partial y, \phi(x,y)\tan(kz - \omega t)(v_g k - \omega)]\cos(kz - \omega t)$$

$$\mathbf{B} = [\partial\phi(x,y)/\partial y\cos(kz - \omega t), -\partial\phi(x,y)/\partial x\sin(kz - \omega t), 0]$$

$$\begin{aligned}\mathbf{J}_4 = [&k\partial\phi/\partial x(\varepsilon\mu v_g v_p - 1)\sin((kz - \omega t),\\ &k\partial\phi/\partial y\sin((kz - \omega t)(\varepsilon\mu v_g v_p - 1),\\ &-(\nabla^2\phi + \alpha\phi)\cos(kz - \omega t),\\ &-v_g\varepsilon\mu(\nabla^2\phi + \beta\phi)\cos(kz - \omega t)]/\mu\end{aligned}$$

$$\alpha = k^2\varepsilon\mu v_p(v_p - v_g), \qquad \beta = k^2 v_g(v_p/v_g - 1)$$

$$\begin{aligned}\mathbf{S}_4 = [&-(v_g/v_p - 1)\phi\partial\phi/\partial x\cos(kz - \omega t)^2,\\ &-(v_g/v_p - 1)\phi\partial\phi/\partial x\cos(kz - \omega t)^2,\\ &-k(v_g/v_p - 1)\phi^2\sin(kz - \omega t),\\ &-\mu k(v_g - v_p)\phi^2\sin(kz - \omega t)]/\mu\end{aligned}$$

$$\mathbf{T}_4 = [0, 0, 0, 0].$$

$$\begin{aligned}\mathbf{E}\times\mathbf{H} = [&(v_p/v_g - 1)\phi\partial\phi/\partial x\sin(kz - \omega t),\\ &(v_p/v_g - 1)\phi\partial\phi/\partial y\sin(kz - \omega t),\\ &((\partial\phi/\partial x)^2 + (\partial\phi/\partial y)^2)\cos(kz - \omega t)](v_g/\mu)\cos(kz - \omega t)\end{aligned}$$

$$(\mathbf{B}\circ\mathbf{H} - \mathbf{D}\circ\mathbf{E}) = -(\{\varepsilon\mu(\omega/k)^2 - 1\}/\mu)\cos(kz - \omega t)^2\{(\nabla\phi)^2 + \phi(\nabla^2\phi)\} \qquad (\mathbf{E}\circ\mathbf{B}) = 0$$

Note that in this solution, the fourth component of the Action is scaled by the "group velocity", $v_g$, not the "speed of light", as determined by the constitutive properties: $c = \sqrt{1/\xi\mu}$. This class of potentials requires that the function $\phi(x,y)$ be a solution of the two dimensional Helmholtz equation, $\nabla^2\phi + \lambda^2\phi = 0$ . The phase velocity, $v_p = \omega/k$, differs from the group velocity, $v_g$. Again, two constraint conditions (dispersion relations) are required for the solution to be a vacuum solution without charge currents. One of the constraint conditions demands that the product of the group and the phase velocity, $v_p = \omega/k$, to be equal to the square of the speed of light as determined from the constitutive properties:

$$v_p \bullet v_g = 1/\varepsilon\mu = c^2.$$

The second constraint required for the vacuum state $(\mathbf{J} = 0, \rho = 0)$ is determined by the Helmholtz parameter, $\lambda$, and is satisfied when

$$\lambda^2 = k^2(v_p/v_g - 1).$$

Such TM modes are also TTM modes; the Torsion vector is identically zero, but the Spin vector is not. Note that the solution becomes a TEM mode solution when the phase velocity equals the group velocity, and the function $\phi$ satisfies the Laplace equation, $\nabla^2\phi = 0$. Further note that the $\mathbf{E}$ field has a longitudinal component when the group velocity and the phase velocity are not the same. For the transverse magnetic mode, $\mathbf{A} \circ \mathbf{B} = 0$, but $\mathbf{A} \circ \mathbf{D} \neq 0$. The second Poincare invariant vanishes, $\mathbf{E} \circ \mathbf{B} = 0$, but for this solution, the first Poincare invariant does not vanish. Not only is the Spin vector not zero, but also its divergence is not zero. The energy flow is in the direction of the wave vector, $\mathbf{k}$, but not in the direction of the field momentum, $\mathbf{D} \times \mathbf{B}$, and the energy propagates with the group velocity $v_g$.

Example 7. A vacuum solution for which $\mathbf{E} \circ \mathbf{B} \neq 0$

Consider the potentials

$$\mathbf{A} = [+y, -x, ct]/\lambda^4, \quad \phi = cz/\lambda^4, \quad where\ \lambda^2 = -c^2t^2 + x^2 + y^2 + z^2. \tag{4.7}$$

and their induced fields:

$$\mathbf{E} = [-2(cty - xz), +2(ctx + yz), -(c^2t^2 + x^2 + y^2 - z^2)]2c/\lambda^6$$

$$\mathbf{B} = [-2(cty + xz), +2(ctx - yz), +(c^2t^2 + x^2 + y^2 - z^2)]2/\lambda^6.$$

$$\mathbf{S}_4 = [x(3\lambda^2 - 4y^2 - 4x^2), y(3\lambda^2 - 4y^2 - 4x^2), z(\lambda^2 - 4y^2 - 4x^2), t(\lambda^2 - 4y^2 - 4x^2)](2/\mu)/\lambda^{10}$$

$$\mathbf{T}_4 = -[x, y, z, t]2c/\lambda^8.$$

$$(\mathbf{B} \circ \mathbf{H} - \mathbf{D} \circ \mathbf{E}) = 0 \qquad and \qquad (\mathbf{E} \circ \mathbf{B}) = -4c/\lambda^8$$

Both the Spin current and the Torsion vector are non-zero, which implies that this solution represents waves which are neither TTM nor TTE. They are not transverse waves in any sense. However, the first Poincare invariant vanishes, implying that the Spin integral is a deformation invariant, and is conserved. The second Poincare invariant is not zero, which implies that the Torsion-Helicity integral is not a topological invariant. These solutions are not simple transverse waves for both $\mathbf{A}\circ\mathbf{B} \neq 0$, and $\mathbf{A}\circ\mathbf{D} \neq 0$. Note that the physical units of the second Poincare invariant are that of an energy density multiplied by an impedance (ohms). As the second Poincare invariant is not zero, it is impossible to find a compact without boundary two surface that contains non-zero lines of magnetic field. That is, a closed 2-torus of magnetic field lines does not exist.

However, as the first Poincare invariant is zero it is possible to construct a deformation invariant in terms of the deRham period integral over a closed 3 dimensional submanifold:

$$Spin = \iiint_{closed} \{S_x dy\wedge dz\wedge dt - S_y dx\wedge dz\wedge dt + S_z dx\wedge dy\wedge dt - \sigma dx\wedge dy\wedge dz\}.$$

Example 8. Another vacuum solution for which $\mathbf{E}\circ\mathbf{B} \neq 0$, complimentary to example 7.
Consider the potentials

$$\mathbf{A} = [+ct, -z, +y]/\lambda^4\ , \quad \phi = cx/\lambda^4, \quad where\ \lambda^2 = -c^2t^2 + x^2 + y^2 + z^2 \tag{4.8}$$

and their induced fields:

$$\mathbf{E} = [+(-c^2t^2 + x^2 - y^2 - z^2), +2(ctz + yx), -2(cty - zx)]2c/\lambda^6$$

$$\mathbf{B} = [+(-c^2t^2 + x^2 - y^2 - z^2), +2(-ctz + yx), +2(cty + zx)]2/\lambda^6.$$

As in example 7, these fields satisfy the Maxwell-Faraday equations, and the associated excitations satisfy the Maxwell-Ampere equations without producing a charge current 4-vector. However, it follows by direct computation that the second Poincare invariant, and the Torsion 4-vector are of opposite signs to the values computed for example 7:

$$\mathbf{E}\circ\mathbf{B} = +4c/\lambda^8 \quad and \quad \mathbf{A}\circ\mathbf{B} = +2ct/\lambda^8\ .$$

Example 9 Superposition of the two complimentary examples 7 and 8.

When the potentials of examples 7 and 8 are combined by addition or subtraction, the resulting wave is topologically transverse magnetic, but not topological transverse electric. Not only does the second Poincare invariant vanish under superposition, but so also does the Torsion 4 vector. Conversely, the examples above show that there can exist topologically transverse magnetic waves which can be decomposed into two non-transverse waves. A notable feature of the superposed solutions is that the Spin 4 vector does not vanish, hence the example superposition is a wave that is not topologically transverse electric. However, for the examples above and their superposition, the first Poincare invariant vanishes, which implies that the Spin remains a conserved topological quantity for the superposition. The spin current density for the combined examples is given by the formula:

$$\mathbf{S}_4 = [-2cx(y+ct)^2, cy(y+ct)(x^2-y^2+z^2-2cty-c^2t^2), -2cz(y+ct)^2, \tag{4.9}$$
$$-(y+ct)(x^2+y^2+z^2+2cty+c^2t^2)]4c/\lambda^{10}$$

while the Torsion current is a zero vector

$$\mathbf{T}_4 = [0,0,0,0].$$

In addition, for the superposed example, the spatial components of the Poynting vector are equal to the Spin current density vector multiplied by $\gamma$, such that

$$\mathbf{E}\times\mathbf{H} = \gamma\,\mathbf{S}, \quad with\ \gamma = -(x^2+y^2+z^2+2cty+c^2t^2)/2c(y+ct)\lambda^2.$$

These results seem to give classical credence to the Planck assumption that vacuum state of Maxwell's electrodynamics supports quantized angular momentum, (the conserved spin integral) and that the energy flux must come in multiples of the spin quanta. In other words, these combined solutions of examples 7 and 8 have the appearance of the photon.

Example 10. Bateman-Whittaker solutions.

In the modern language of differential forms, Bateman [19] (and Whittaker) determined that if two *complex* functions $\alpha(x,y,z,t)$ and $\beta(x,y,z,t)$ are used to define the 1-form of Action,

$$A = \alpha d\beta - \beta d\alpha \Rightarrow \mathbf{A} = \alpha\nabla\beta - \beta\nabla\alpha, \quad \phi = -(\alpha\partial\beta/\partial t - \beta\partial\alpha/\partial t) \tag{4.10}$$

then the derived 2-form

$$F = 2d\alpha\wedge d\beta$$

generates the complex field intensities,

$$\mathbf{E} = (\partial\alpha/\partial t)\nabla\beta - (\partial\beta/\partial t)\nabla\alpha \ \ and\ \ \mathbf{B} = \nabla\alpha\times\nabla\beta,$$

which of course satisfy the Maxwell-Faraday equations. If in addition, the functions $\alpha$ and $\beta$ satisfy the complex Bateman constraints:

$$\nabla\alpha \times \nabla\beta = \pm(i/c)[(\partial\alpha/\partial t)\nabla\beta - (\partial\beta/\partial t)\nabla\alpha],$$

then the complex field excitations computed from the Lorentz vacuum constitutive constraints will satisfy the Maxwell-Ampere equations for the vacuum, without charge currents. It is apparent immediately that the second Poincare invariant is identically zero for such solutions. It is also apparent immediately that the Torsion vector is identically zero. What is not immediately apparent is that first Poincare invariant and the Spin vector vanish identically as well. In fact, the constrained complex solutions of the Bateman type are examples of topologically transverse (TTEM) waves. The Bateman solutions do not radiate!

As an explicit example, consider

$$\alpha = (x \pm iy)/(z - r), \quad \beta = (r - ct), \quad r = \sqrt{x^2 + y^2 + z^2}.$$

These functions satisfy the Bateman conditions (and, it should be mentioned, the Eikonal equation subject to the dispersion relation $\varepsilon\mu c^2 = 1$). The **E** and the **B** fields are complex (and complicated algebraically)

$$\mathbf{B} = [yx + \sqrt{-1}(z^2 + y^2 - rz), -(z^2 + x^2 - rz) - \sqrt{-1}\,xy, (r^2 + z^2 - 2rz)/(r - z))(y - \sqrt{-1}\,x)]2/(r(z -$$

$$\mathbf{E} = [-\sqrt{-1}\,yx + (y^2 + z^2 - rz), \sqrt{-1}(x^2 + z^2 - rz) - xy, (z - r)(x + \sqrt{-1}\,y)]2c/(r(z - r)^2)$$

$$\mathbf{S}_4 = [0, 0, 0, 0].$$

$$\mathbf{T}_4 = [0, 0, 0, 0].$$

$$\mathbf{E} \times \mathbf{H} = [0, 0, 0], \quad \mathbf{D} \times \mathbf{B} = [0, 0, 0], \quad \mathbf{E} \circ \mathbf{E} = 0, \quad \mathbf{B} \circ \mathbf{B} = 0$$

$$(\mathbf{B} \circ \mathbf{H} - \mathbf{D} \circ \mathbf{E}) = 0 \qquad (\mathbf{E} \circ \mathbf{B}) = 0$$

The functions $\alpha$ and $\beta$ that satisfy the Bateman condition may be used to construct an arbitrary function, $F(\alpha, \beta)$, and remarkably enough, the arbitrary function $F(\alpha, \beta)$ satisfies the Eikonal equation,

$$(\nabla F)^2 - \varepsilon\mu(\partial F/\partial t)^2 = 0.$$

From experience with Eikonal solutions and wave equations, it might be thought that Eikonal solutions are sufficient. However, the Bateman conditions are necessary, for both the candidate solutions

$$\alpha = (x \pm iy)/(z - ct), \quad \beta = (r - ct), \quad r = \sqrt{x^2 + y^2 + z^2}.$$

satisfy the Eikonal equation, but not the Bateman conditions. They do not generate TTEM modes in the vacuum. For arbitrary functions the algebra can become quite complex. A Maple symbolic mathematics program for computing the various terms is available (see references below)

4.2 Self dual solutions

It is possible to construct a two-form $G$ (without using the Lorentz vacuum constitutive definitions) in terms of two arbitrary functions, $\alpha$ and $\beta$, from the dual relations:

$$G = i(*d\alpha)\hat{}i(*d\beta)\Omega = i(*d\alpha)\hat{}i(*d\beta)dx\hat{}dy\hat{}dz\hat{}dt.$$

The functions $\alpha$ and $\beta$ used in the dual construction are not required to be solutions of the Bateman condition. However, the resulting "self-dual" field excitations are **not** the same as those generated by the Bateman method, unless the functions also satisfy the Bateman conditions of complex collinearity. In the self dual formulas the * operator is the Hodge * operator with respect to the Lorentz metric modified by the impedance of free space. The resulting self-dual excitations constructed from the two arbitrary functions indeed satisfy the Maxwell-Ampere equations, in virtue of the Maxwell-Faraday equations and the dispersion relation. The construction yields:

$$\mathbf{H} = \sqrt{-1}/\mu c(\partial\alpha/\partial t)\nabla\beta - (\partial\beta/\partial t)\nabla\alpha \quad and \quad \mathbf{D} = -\sqrt{-1}\,\varepsilon/c\nabla\alpha \times \nabla\beta.$$

The self-dual construction, however, implies a chiral (non-Lorentz) constitutive relation of the type $\mathbf{D} = -[\gamma]\circ\mathbf{B}$ and $\mathbf{H} = [\gamma^{\dagger}]\circ\mathbf{E}$, and will not be considered further in this article.

## 5. **Deformation Invariants and the Plasma State**.

5.1 Special evolutionary processes. The plasma process

As described in section 3, the fundamental equation of topological evolution is given by Cartan's magic formula, acting as a propagator on the forms that make up the exterior differential system. As stated in the first paragraph, an evolutionary process is defined herein as a map that can be described by a singly parameterized vector field. If the Action of the Lie derivative on the complete system of Maxwell exterior differential forms vanishes for a particular choice of process, then that process leaves the entire Maxwell system absolutely invariant. As a topology can be constructed in terms of an exterior differential system, and if a special process leaves that system of forms invariant, then the topology induced by the system of forms is invariant, and the process must be a homeomorphism.

However, for a given Maxwell system, it is more likely that only some of the exterior differential forms that make up the Maxwell system are invariant relative to an arbitrary process; others are not. Of particular interest are those forms which are relative integral invariants of continuous deformations. The closed integral of the form is not only invariant with respect to a process represented by particular vector field, but also with respect to longitudinal deformations of that process obtained by multiplying the particular vector field by an arbitrary function. For vector fields which are singly parameterized, this concept of longitudinal deformation is equivalent to a reparameterization of the vector field.

The development that follows is guided by Cartan's pioneering work, in which he examined those specialized processes for mechanical systems that leave the 1-form of Action, $A$, a deformation invariant. Cartan proved that such processes always have a Hamiltonian representation. An electromagnetic system has not only the primitive 1-form, $A$, but also the N-2 form, $G$, which can undergo evolutionary processes. electromagnetic systems, a particular interesting choice of specialized processes are those that leave the N-2 form, $G$, of field excitations a deformation (relative) integral invariant. The equations that must be satisfied are of the form

$$L_{\beta\mathbf{V}}(\iint_{closed} G) = \iint_{closed} i(\beta V)dG = \iint_{closed} i(\beta V)J \tag{5.1}$$

$$= \iint_{closed} \beta\{(\mathbf{J}-\rho\mathbf{V})^x dy\wedge dz - ... + (\mathbf{J}\times\mathbf{V})^x dx\wedge dt ... \Rightarrow 0$$

It follows that deformation invariance of the N-2 form $G$ requires that the admissable evolutionary processes be restricted to those that satisfy the definitions of the classical plasma:

$$\mathbf{J} = \rho\mathbf{V}. \tag{5.2}$$

(This constraint is used to define the "Plasma state" in this article). As the closed integrals of $G$ are by Gauss law, the counters of net charge within the closed domain, the classical plasma equation is to be recognized as the statement that in the closed domain the net number of charges is a deformation invariant. That is, charges can be produced only in equal and opposite pairs by a ”plasma process”. A plasma process does not involve net charge production.

This invariance principle is to be compared to the Helmholtz theorem which checks on the validity of the deformation integral invariance of the 2-form $F$.

$$L_{\beta\mathbf{V}}(\iint_{closed} F) = \iint_{closed} i(\beta V)dF = 0 \tag{5.3}$$

The closed integral of Helmholtz is an intrinsic topological (deformation) invariant of an electromagnetic system, for the 2-form $F$ is exact by construction (the postulate of potentials). The Helmholtz integral is a deformation invariant for all evolutionary processes that can be described by a singly parameterized vector field. (This statement is not true for Yang Mills fields). Hence in a plasma, for which the evolutionary processes are constrained such that $\mathbf{J} = \rho\mathbf{V}$, both the closed integrals of $F$ and $G$ are deformation invariants. In the sense, the plasma is a topological refinement of the complete Maxwell system.

In the subsections that follow, various topological categories of plasma processes will be examined. The ideal and semi-ideal plasma processes will obey the plasma master equation, and the non-ideal plasma processes will not. The electromagnetic flux is a local (absolute) invariant of all semi-ideal plasma processes. This statement is similar to the classification of hydrodynamic flows. Ideal and semi-ideal hydrodynamic flows satisfy the Helmholtz theorem, and the ”local” conservation of vorticity.

5.2 The ideal plasma process is a plasma process which is also a Hamiltonian process.

Next consider the evolutionary properties of the 1-form of Action in the plasma state by evaluating the possible deformation invariance of the 1-form of Action, $A$, with respect to motions that preserve the plasma state:

$$L_{\rho\mathbf{V}}(\oint A) = \oint i(\rho V)dA = \oint W = \oint\{(\rho\mathbf{E} + \mathbf{J}\times\mathbf{B})_k dx^k + (\mathbf{J}\circ\mathbf{E})dt\} \Rightarrow 0. \tag{5.4}$$

The 1-form $W$ is the 1-form of virtual work defined in terms of the Lorentz force. The resulting equation demonstrates that the concept of a Lorentz force, $\rho\mathbf{E} + \mathbf{J}\times\mathbf{B}$, has a topological foundation. It is apparent that if the Lorentz force vanishes, $\{\rho\mathbf{E} + \mathbf{J}\times\mathbf{B}) \Rightarrow 0$, and the Plasma current density is NOT ohmic, $(\mathbf{J}\circ\mathbf{E}) = \rho(\mathbf{V}\circ\mathbf{E}) \Rightarrow 0$, then the closed integral of the

Action 1-form is also a deformation topological invariant of the Plasma process. Such a set of constraints, $W = i(\rho V)dA = 0$, topologically defines the "ideal" plasma state as a plasma process for which the 1-form of virtual Work vanishes. By Cartan's theorem, the 1-form of Action then has a unique Hamiltonian representation and the ideal plasma process is uniquely defined as a Hamiltonian process ( the Pfaff dimension of the 1-form, $A$ must processes be 3 or less for uniqueness). The ideal plasma is thereby a restriction of arbitrary processes to that unique process that leaves invariant both the closed integrals of flux and the closed integrals of charge. Ideal plasmas are electromagnetic systems for which the admissable processes are the intersection of a plasma process and a unique Hamiltonian process. The ideal plasma can not exist on a domain of a 4 dimensional variety where the second Poincare invariant is not zero.

5.3 The Bernoulli-Casimir plasma process is a semi-ideal plasma process.

The topological constraint that the 1-form of virtual work vanishes is sufficient but not necessary for a plasma process to preserve the closed integrals of the Action 1-form. Evolutionary invariance of the closed integral of Action does not require that the plasma process be unique. The 1-form of virtual Work, $W$, need not be zero, but only closed: $dW \Rightarrow 0$. By analogy to hydrodynamics, if the virtual Work 1-form is exact,

$$W = d\Theta \tag{5.5}$$

then the Lorentz force is represented by a spatial gradient, $\rho\mathbf{E} + \mathbf{J} \times \mathbf{B} = \nabla\Theta$, and the Power $-\mathbf{J} \circ \mathbf{E} = \partial\Theta/\partial t$. The function $\Theta(x, y, z, t)$ is a Bernouilli-Casimir function, and acts as the generator of a symplectic Hamiltonian flow. The (non-unique) Bernouilli-Casimir function is an evolutionary invariant for each process path, but is not necessarily a constant over the domain:

$$L_{\rho\mathbf{V}}(\Theta) = i(\rho V)d\Theta = i(\rho V)i(\rho V)A = 0. \tag{5.6}$$

The Bernouilli-Casimir function is not the same as the Hamiltonian energy function, but is more closely related to the thermodynamic concept of enthaply. The Bernouilli-Casimir function can be used to generate a "Hamiltonian process", but the process is not uniquely defined.

For such symplectic plasma processes, the gradient of the Bernoulli-Casimir function is transverse to the **B** field only when the second Poincare invariant vanishes.

$$\rho \mathbf{E} \circ \mathbf{B} = \nabla\Theta \circ \mathbf{B}. \qquad (5.7)$$

Similar expression were studied in conjunction with topological conservation in MHD by Hornig and Schidler [20].

$$\rho \mathbf{E} \circ \mathbf{V} = \nabla\Theta \circ \mathbf{V}. \qquad (5.8)$$

If the Ohmic assumption is made for the plasma process, $\mathbf{J} = \rho\mathbf{V} = \sigma(\mathbf{E} + \mathbf{V} \times \mathbf{B})$, then the symplectic condition leads to a thermopower format of the type

$$\mathbf{J} = (1/\rho\sigma) grad(kT) \qquad (5.9)$$

when it is subsumed that the Bernouilli-Casimir function is related to temperature. It would appear that for plasma motion along the **B** field lines, there can exist a dynamo action to produce an **E** field collinear with the magnetic field.

5.4 The Stokes plasma process is a semi-ideal process that obeys the Master equation.

The constraint that the virtual work 1-form, *W*, generated by a plasma process, $W = i(\rho V)dA$, be closed, does not require that it be exact. The constraint of closure yields two vector conditions:

$$dW = 0 \Rightarrow curl(\rho\mathbf{E} + \mathbf{J} \times \mathbf{B}) = 0 \quad and \quad \nabla(\mathbf{J} \circ \mathbf{E}) = \partial(\rho\mathbf{E} + \mathbf{J} \times \mathbf{B})/\partial t. \qquad (5.10)$$

The first vector condition implies that

$$\nabla\rho \times (\mathbf{E} + \mathbf{V} \times \mathbf{B}) + \rho\, curl(\mathbf{E}) + curl(\mathbf{V} \times \mathbf{B}) = 0. \qquad (5.11)$$

By using the Maxwell-Faraday equation, this topological constraint becomes the plasma master equation:

$$-\partial\mathbf{B}/\partial t + curl(\mathbf{V} \times \mathbf{B}) = -\nabla \ln\rho \times (\mathbf{E} + \mathbf{V} \times \mathbf{B}). \qquad (5.12)$$

All of these ideal and semi ideal plasma processes enjoy the property that the electromagnetic flux is conserved locally. That is

$$L_{\rho\mathbf{V}}(dA) = L_{\rho\mathbf{V}}F = d(i(\rho V)F) = 0. \qquad (5.13)$$

5.5 Frozen-in lines.

It is of some interest to examine the evolution of the differential forms that make up an electromagnetic system relative to Plasma processes. The method is to construct the Lie derivative with respect a plasma process, $\mathbf{J} = \rho\mathbf{V}$, of all forms that make up the electromagnetic Pfaff sequence.

For an arbitrary vector field Z whose tangents define a line in space time, the N-1 form

$$W = i(\gamma Z)dx\hat{}dy\hat{}dz\hat{}dt \qquad (5.14)$$

can be tested for evolutionary invariance relative to any other vector $V$. Suppose the effect of the evolutionary process is conformal:

$$L_{(V)}W = i(V)dW + d(i(V)W) = \Gamma(x,y,z,t)W \qquad (5.16)$$

This statement implies that the points that make up the tangent line of the vector field W remain on the tangent line. The points may be permuted but they do not leave the line. Such is the concept of a frozen in field. The lines evolve into lines. If for a given V the evolution of the lines of W is conformal, then there exists a parametrization of V such that the evolution is uniform and invariant. A parametrization function $\beta(x,y,z,t)$ can be found such that

$$L_{(\beta V)}W = \beta L_{(V)}W + L_{(V)}\beta\hat{}W = (\beta \bullet \Gamma + i(V)d\beta)W \Rightarrow 0. \qquad (5.17)$$

For the electromagnetic system there are three N-1 forms, which may or may not be frozen into the evolutionary process. Consider the 3-form of current.

$$L_{(V)}J = i(V)dJ + d(i(V)J) \qquad (5.18)$$

As $dJ = 0$,

$$L_{(V)}J = d\{i(V)i(J)dx\hat{}dy\hat{}dz\hat{}dt\} \qquad (5.19)$$

It follows that if $i(V)i(J)dx\hat{}dy\hat{}dz\hat{}dt\} = 0$, the field lines of $J$ are frozen-in (with $\Gamma = 0$). So the plasma evolutionary process evolutionary, with $\mathbf{J} = \rho\mathbf{V}$, is an example of a process that ”freezes-in” the lines of current. However, there are many other evolutionary processes for which the $J$ lines are frozen in.

The formulas created by 5.16 are valid on any set of independent variables, but expressions on 4 dimensions of space time for ”frozen-in” lines are not quite the same as those that appear in the engineering literature based on euclidean 3-space [21]. Either the time-like component of the 4-vector W must vanish, or the process V must be explicitly time-independent for the general formulas to be in precise agreement with the engineering expressions. [22]

5.6 Evolution of the lines of topological torsion with respect to plasma currents.

Consider the evolution of the lines of topological torsion

$$L_{(\rho V)}A\hat{}F = i(\rho V)d(A\hat{}F) + d(i(\rho V)A\hat{}F) = i(\rho V)d(A\hat{}F) + d\{(i(\rho V)A)\hat{}F - A\hat{}i(\rho V)F\} \qquad (5.20)$$

First consider those systems where the second Poincare invariant vanishes, $F\hat{}F = 0$. The lines in space time which are tangent to the 3-form $A\hat{}F$ then have zero divergence. The lines can only start and stop on boundary points, or they are closed on themselves. The Torsion lines can be either parallel to the plasma current or they can be orthogonal to the plasma current. As the electromagnetic current is exact, any three dimensional domain of support for a finite plasma current cannot be compact without a boundary. If the lines of plasma current start and stop on boundary points, then the lines of torsion can form closed loops that link the current lines. It is the concept of linkages that is of interest to the theory of magnetic knots.

Consider that plasma process such that the evolution is in the direction of the Torsion lines. As in this situation,

$$(i(J)A\hat{}F) = (i(\rho V)A\hat{}F) \Rightarrow (i(\gamma \mathbf{T}_4)A\hat{}F) = \gamma(i(\mathbf{T}_4)(i(\mathbf{T}_4)dx\hat{}dy\hat{}dz\hat{}dt = 0, \qquad (5.21)$$

the 3-form of Torsion is a local invariant whenever the second Poincare invaraint vanishes; $\mathbf{E} \circ \mathbf{B} \Rightarrow 0$. In otherwords, $F\hat{}F \neq 0$ is a local necessary condition for topological change. It is also a remarkable fact that any evolution in the direction of the Torsion vector leaves the Action 1-form conformally invariant, in the sense that:

$$L_{(\gamma \mathbf{T}_4)}A = i(\gamma \mathbf{T}_4)dA + di(\gamma \mathbf{T}_4)A = \gamma(\mathbf{E} \circ \mathbf{B})A + 0. \qquad (5.22)$$

The torsion vector on a domain of 4 variables is transverse to the 1-form of Action, as $A\hat{}(A\hat{}F) = 0$. Evolution in the direction of the Torsion vector in not Hamiltonian, unless the second Poincare invariant vanishes. In section 6 below this idea will be related to thermodynamic irreversibility.

5.7 Evolution of the lines of Spin Current with respect to plasma currents.

Consider the evolution of the lines of Spin current

$$L_{(\rho V)}A\hat{}G = i(\rho V)d(A\hat{}G) + d(i(\rho V)A\hat{}G) = i(\rho V)d(A\hat{}G) + d\{(i(\rho V)A)\hat{}G - A\hat{}i(\rho V)G\} \qquad (5.23)$$

First consider those systems where the first Poincare invariant vanishes,

$F$^$G - A$^$J = 0$. The lines in space time which are tangent to the 3-form $A$^$G$ then have zero divergence. The lines can only start and stop on boundary points, or they are closed on themselves. The Spin lines are either parallel to the plasma current or they are orthogonal to the plasma current. As the electromagnetic current is exact, any three dimensional domain of support for a finite plasma current cannot be compact without a boundary. If the lines of plasma current do not stop or start on boundary points (current loops), then the Spin lines which terminate on boundary points can be linked by the current loops.

The concept of the spin vector depends on the existence of $G$, but not on the concept of $J = dG$. That is, the Spin vector can be associated with separated domains of charges, which can be compact domains without boundary that are compliments of the domain of finite charge current densities, which are domains that can not be compact without boundary.

## 6. Thermodynamics

6.1 Topological Thermodynamics and Irreversibility

The basic tool for studying topological evolution is Cartan's magic formula in which it is presumed that a physical (hydrodynamic) system can be described adequately by a 1-form of Action, $A$, and that a physical process can be represented by a contravariant vector field, $\mathbf{V}$, which can be used to represent a dynamical system or a flow:

$$L_{(\mathbf{V})} \int A = \int L_{(\mathbf{V})} A = \int \{ i(\mathbf{V}) dA + d(i(\mathbf{V})A) \} \qquad (6.1a)$$

$$= \int \{ W + d(U) \} \ = \int Q. \qquad (6.1b)$$

The basic idea behind this formalism (which is at the foundation of the Cartan-Hilbert variational principle) is that postulate of potentials is valid: $F - dA = 0$. The base manifold will be the 4-dimensional variety $\{x, y, z, t\}$ of engineering practice, but no metrical features are presumed a priori. If relative to the process, $V$, the RHS of equation 6 is zero, $\int Q. \Rightarrow 0$, then $\int A$ is said to be an integral invariant of the evolution generated by $\mathbf{V}$. In thermodynamics such processes are said to be adiabatic.

From the point of view of differential topology, the key idea is that the Pfaff dimension, or class [23], of the 1-form of Action specifies topological properties of the system. Given the Action 1-form, $A$, the Pfaff sequence, $\{A, dA, A\hat{}dA, dA\hat{}dA, ...\}$ will terminate at an integer number of terms ≤ the number of dimensions of the domain of definition. On a 2n+2=4 dimensional domain, the top Pfaffian, $dA\hat{}dA$, will define a volume element with a density function whose singular zero set (if it exists) reduces the symplectic domain to a contact manifold of dimension 2n+1=3. This (defect) contact manifold supports a unique extremal field that leaves the Action integral ”stationary”, and leads to the Hamiltonian conservative representation for the Euler flow in hydrodynamics. The irreversible regime will be on an irreducible symplectic manifold of Pfaff dimension 4, where $dA\hat{}dA \neq 0$. Topological defects (or coherent structures) appear as singularities of lesser Pfaff (topological) dimension, $dA\hat{}dA = 0$.

Classical hydrodynamic processes can be represented by certain nested categories of vector fields, **V**. Recall that in order to be Extremal, the process, **V**, must satisfy the equation

$$Extremal --(unique\ Hamiltonian): \qquad i(\mathbf{V})dA = 0; \tag{6.2a}$$

in order to be Hamiltonian the process must satisfy the equation

$$Bernouilli--Casimir--Hamiltonian: \qquad i(\mathbf{V})dA = d\Theta; \tag{6.2b}$$

in order to be Symplectic, the process must satisfy the equation

$$Helmholtz--Symplectic: \qquad di(\mathbf{V})dA = 0. \tag{6.2c}$$

Extremal processes cannot exist on the non-singular symplectic domain, because a non-degenerate anti-symmetric matrix (the coefficients of the 2-form $dA$) does not have null eigenvectors on space of even dimensions . Although unique extremal stationary states do not exist on the domain of Pffaf dimension 4, there can exist evolutionary invariant Bernoulli-Casimir functions, Θ, that generate non-extremal, ”stationary” states. Such Bernoulli processes can correspond to energy dissipative symplectic processes, but they, as well as all symplectic processes, are reversible in the thermodynamic sense described below. The mechanical energy need not be constant, but the Bernoulli-Casimir function(s), Θ, are evolutionary invariant(s), and may be used to describe non-unique stationary state(s).

The equations, above, that define several familiar categories of processes, are in effect constraints on the topological evolution of any physical system represented by an Action 1-form, $A$. The Pfaff dimension of the 1-form of virtual work, $W = i(\mathbf{V})dA$, is 2 or less for the three categories. The extremal constraint of equation 6.2a can be used to generate the Euler equations of hydrodynamics for a incompressible fluid. The Bernoulli-Casimir constraint of equation 6.2b can be used to generate the equations for a barotropic compressible fluid. The Helmholtz constraint of equation 6.2c can be used to generate the equations for a Stokes flow. All such processes are thermodynamically reversible. None of these constraints above will generate the Navier-Stokes equations, which require that the topological dimension of the 1-form of virtual work must be greater than 2.

A crucial idea is the recognition that irreversible processes must on domains of Pfaff dimension which support Topological Torsion, $A\hat{}dA \neq 0$, with its attendant properties of non-uniqueness, envelopes, regressions, and projectivized tangent bundles. Such domains are of Pfaff dimension 3 or greater. Moreover, as described below, it would appear that thermodynamic irreversibility must support a non-zero Topological Parity 4-form, $dA\hat{}dA \neq 0$. Such domains are of Pfaff dimension 4 or greater.

Although there does not exist a unique gauge independent stationary state on the symplectic manifold of Pfaff dimension 4, remarkably there does exist a unique vector field on the symplectic domain, with components that are generated by the 3-form $A\hat{}dA$. This unique (to within a factor) vector field is defined as the Torsion Current, $\mathbf{T}_4$, and satisfies (on the 2n+2=4 dimensional manifold) the equation,

$$i(\mathbf{T}_4)dx\hat{}dy\hat{}dz\hat{}dt = A\hat{}dA \tag{6.3}$$

This (four component) vector field, $\mathbf{T}_4$, has a non-zero divergence almost everywhere, for if the divergence is zero, then the 4-form $dA\hat{}dA$ vanishes, and the domain is no longer a symplectic manifold! The Torsion vector, $\mathbf{T}_4$, can be used to generate a dynamical system that will decay to the stationary states ($div_4(\mathbf{T}_4) \Rightarrow 0$) starting from arbitrary initial conditions. These processes are irreversible in the thermodynamic sense. It is remarkable that this unique evolutionary vector field, $\mathbf{T}_4$, is completely determined (to within a factor) by the physical system itself; e.g., the components of the

1-form, $A$, determine the components of the Torsion vector.

To understand what is meant by thermodynamic irreversibility, realize that Cartan's magic formula of topological evolution is equivalent to the first law of thermodynamics.

$$L_{(\mathbf{v})}A = i(\mathbf{V})dA + d(i(\mathbf{V})A) = W + dU = Q. \qquad (6.4)$$

$A$ is the "Action" 1-form that describes the hydrodynamic system. **V** is the vector field that defines the evolutionary process. $W$ is the 1-form of (virtual) work. $Q$ is the 1-form of heat. From classical thermodynamics, a process is irreversible when the heat 1-form $Q$ does not admit an integrating factor. From the Frobenius theorem, the lack of an integrating factor implies that $Q$^$dQ \neq 0$. Hence a simple test may be made for any process, **V**, relative to a physical system described by an Action 1-form, $A$:

$$\textit{If}\ \ L_{(\mathbf{v})}A\text{\^{}}L_{(\mathbf{v})}dA \neq 0 \ \textit{then the process is irreversible}. \qquad (6.5)$$

This topological definition implies that the three categories (above) of symplectic, Hamiltonian or extremal processes, $\subset \mathbf{S}$, are reversible (*as* $L_{(\mathbf{S})}dA = dQ = 0$). However, for evolution in the direction of the Torsion vector, $\mathbf{T}_4$, direct computation demonstrates that the fundamental equations lead to a conformal evolutionary process, a process which is thermodynamically irreversible:

$$L_{(\mathbf{T}_4)}A = \sigma A \quad \textit{and} \quad i(\mathbf{T}_4)A = 0, \qquad (6.6)$$

such that

$$L_{(\mathbf{T}_4)}A\text{\^{}}L_{(\mathbf{T}_4)}dA = Q\text{\^{}}dQ = \sigma^2 A\text{\^{}}dA \neq 0. \qquad (6.7)$$

6.2 Applications to Electromagnetism

All of the development of section 6.1 will carry over to the electromagnetic system, which also subsumes the postulate of potentials. The topological torsion 3-form, $A$^$dA$, induces the torsion current

$$\mathbf{T}_4 = \{(\mathbf{E} \times \mathbf{A} + \mathbf{B}\phi); \mathbf{A} \circ \mathbf{B}\} \equiv \{\mathbf{S}, h\}. \qquad (6.8)$$

If $div_4\mathbf{T} = -2\,\mathbf{E} \circ \mathbf{B} \neq 0$, the electromagnetic 1-form, $A$, defines a domain of Pfaff dimension 4. Such domains cannot support topologically transverse magnetic waves (*as* $A$^$F \neq 0$). Evolutionary processes (including plasma currents) that are proportional to the Torsion current are thermodynamically irreversible, if $\sigma = \mathbf{E} \circ \mathbf{B} \neq 0$. However, the conformal properties of evolution in the direction of the Torsion current lead to extraordinary properties when the plasma current is in the direction of the Torsion vector. From the thermodynamic arguments in section 6.1 based on the postulate of potentials, but using the notation of an electromagnetic system

$$L_{(\mathbf{T}_4)}A = \sigma A = (\mathbf{E} \circ \mathbf{B})A \tag{6.9}$$

and

$$L_{(\mathbf{T}_4)}(A\char`^F) = 2\sigma A = 2(\mathbf{E} \circ \mathbf{B})A\char`^F. \tag{6.10}$$

Hence, it follows that motion along the direction of the torsion vector freezes-in the lines of the torsion vector in space time, but the process is irreversible unless the second Poincare invariant is zero.

Recall that the definition of a plasma current, $J$, is equivalent to an evolutionary process such that

$$\text{Definition of a plasma Current } J: \quad L_{(J)}G = 0. \tag{6.11}$$

Hence consider a plasma current which is also in the direction of the Torsion vector. Then

$$L_{(J)}A\char`^G = (L_{(J)}A)\char`^G + A\char`^L_{(J)}G = (L_{(\gamma\mathbf{T}_4)}A)\char`^G + A\char`^L_{(J)}G = \gamma \bullet (\mathbf{E} \circ \mathbf{B})\ A\char`^G + 0 \tag{6.12}$$

Hence for plasma motions in the direction of the (possibly dissipative) torsion vector, both the ”lines” of the Spin vector are ”frozen in” and the lines of the Torsion vector are ”frozen in”. Such ”frozen in” objects can be used to give a topological definition of deformable coherent structures in a plasma. Moreover, as the evolutionary process causes the frozen in structures to deform and decay, it is conceivable that evolution could proceed to form stationary (not stagnant) states (where $\mathbf{E} \circ \mathbf{B} \Rightarrow 0$), such that the frozen in field line structures become local deformation invariants, or topological defects. Electromagnetic coherent structures are evolutionary deformable (and perhaps decaying) domains of Pfaff dimension 4, which form stationary states of topological defects (including the null state) in regions of Pfaff dimension 3, where $\mathbf{E} \circ \mathbf{B} = 0$.

Note that all semi-ideal (see section 5) plasma current processes are reversible in a thermodynamic sense.

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