Fractals and Homogeneous Differential Forms

R. M. Kiehn

Emeritus, Physics Dept., Univ. Houston
03/13/2000 - updated 10/07/2002
http://www.cartan.pair.com
rkiehn2352@aol.com

Abstract

A remarkable relationship exists between homogeneous differential forms, period integrals, fractals, and thermodynamic irreversibility.

1. INTRODUCTION

1.1. Homogeneous functions

The concept of a homogeneous function is encoded by the equation:

$$f(\beta \xi^k) = \beta^D f(\xi^k). \tag{1.1}$$

Such functions of $0 < k \leq n$ variables ξ^k are said to be homogeneous of degree D. Changing the scale of each of the independent variables by the same factor replicates the same homogeneous function, times a factor. The homogeneous function is self similar upon change of scale, β.

A homogeneous function, $f(\xi^k)$, satisfies Euler's equation

$$\xi^k \partial f(\xi^k)/\partial \xi^k = D \cdot f(\xi^k). \tag{1.2}$$

Usually the homogeneity degree D has been interpreted as an integer. However, since the popularization of fractals it is to be recognized that D does not need to be an integer.

1. The factor β can be interpreted as a scaling parameter. It need not be a constant!.

2. The factor D can be interpreted as the homogeneity degree, or the fractal dimension.

3. The factor β^D could be interpreted as the number of copies, N (of $f(\xi^k)$).

These interpretations lead to the formula

$$N = \beta^D, \tag{1.3}$$

and the "Hausdorff" dimension,

$$D = \ln N / \ln \beta, \tag{1.4}$$

A useful homogeneous function is given by the Holder norm λ which is homogeneous of degree D for any polynomial exponent, σ, and any set of constants a_k:

$$\text{Holder Norm} \quad : \quad \lambda(\xi^k) = \{a_1(\xi^1)^\sigma + a_2(\xi^2)^\sigma + ... + a_n(\xi^n)^\sigma\}^{D/\sigma} \tag{1.5}$$
$$\lambda(\beta\xi^k) \;=\; \beta^D \lambda(\xi^k). \tag{1.6}$$

The a_k are usually treated as constants that determine the isotropy and signature of the Norm. Note that the length of a vector defined as the square root of a quadratic (Euclidean) inner product is a homogeneous function of degree $D = 1$, isotropic in its components, and of signature zero (no minus signs). In terms of the vector components, V^k,

$$\lambda(V^k) \;=\; \langle \mathbf{V} | \circ | \mathbf{V} \rangle = \{(V^1)^2 + (V^2)^2 + (V^3)^2\}^{1/2}, \tag{1.7}$$
$$\sigma \;=\; 2, \; D = 1, \; a1 = a2 = ... = ak = 1. \tag{1.8}$$

1.2. Homogeneous differential forms

In general, a homogeneous p-form, $\omega(A_k(\xi^k), d\xi^k)$, will satisfy the equation[1]

$$L_{(V^a)}\omega = k\,\omega. \tag{1.9}$$

Consider those situations where there exists a map from a set of a parameters η^a to the n independent (coordinate) variables, ξ^k, and therefor another map (obtained by functional substitution) from the η^a to the coefficient functions, $A_k(\xi^k)$.

$$\eta^m \quad \Rightarrow \quad \xi^k(\eta^m), \tag{1.10}$$

$$\eta^m \quad \Rightarrow \quad \widehat{A}_k(\eta^m) = A_k(\xi^k(\eta^a)). \tag{1.11}$$

Assume that the ξ^k are homogeneous of degree D, relative to the parameters, η^a :

$$\eta^a \partial \xi^k / \partial \eta^a = D(\xi)\ \xi^k. \tag{1.12}$$

Note that the Lie differential, with respect to η^a, of the elements $d\xi^k$ is

$$
\begin{aligned}
L_{(\eta^a)}(d\xi^k) \quad &= \quad d(i(\eta^a)d\xi^k) + i(\eta^a)dd\xi^k \tag{1.13}\\
&= \quad d(i(\eta^a)(\partial\xi^k/\partial\eta^a)d\eta^a) + 0 \tag{1.14}\\
&= \quad d(\eta^a\partial\xi^k/\partial\eta^a) = d(D(\xi)\ \xi^k) \tag{1.15}\\
&= \quad D(\xi)\ d\xi^k + (\xi^k\partial D/\partial\xi^k)d\xi^k \tag{1.16}\\
&= \quad D(\xi)(1 + B(\xi))d\xi^k \tag{1.17}\\
\text{where} \ \ (\xi^k\partial D/\partial\xi^k) \quad &= \quad B(\xi)D(\xi). \tag{1.18}
\end{aligned}
$$

In the theory of Finsler spaces it is assumed that the differentials $d\xi^k$ are homogeneous of degree 1. When $D(\xi)(1 + B(\xi)) = 1$, Chern has defined the system as being "projectivized"[2]. A simple case is where $D(\xi) = 1$, and $B(\xi) = 0$

Euler's equation for homogeneous functions can be replicated for exterior differential forms in terms of Cartan's magic formula for the Lie differential. Consider the Lie differential of a set of functions homogeneous functions, $A_k(\xi^k)$, homogeneous of degree H relative to the variables, ξ^k. Hence the assumption implies that

$$\xi^m \partial A_k(\xi^k)/\partial\xi^m = H(\xi)\ A_k(\xi^m). \tag{1.19}$$

Note that the Lie differential of the function set $A_k(\xi^k)$

$$
\begin{aligned}
L_{(\eta^a)}A_k(\xi^k) \quad &= \quad d(i(\eta^a)A_k(\xi^k)) + i(\eta^a)dA_k(\xi^k) \tag{1.20}\\
&= \quad 0 + i(\eta^a)(\partial A_k(\xi^k)/\partial\xi^m)(\partial\xi^m/\partial\eta^a)d\eta^a \tag{1.21}\\
&= \quad (\partial A_k(\xi^k)/\partial\xi^m)(\eta^a\partial\xi^m/\partial\eta^a) \tag{1.22}\\
&\Rightarrow \quad D(\xi)\ (\partial A_k(\xi^k)/\partial\xi^m)(\xi^m) \tag{1.23}\\
&= \quad D(\xi)\cdot H(\xi)\ A_k(\xi^k). \tag{1.24}
\end{aligned}
$$

Combining these results for a 1-form yields the equation

$$
\begin{aligned}
L_{(\eta^a)}(A_k(\xi^k)dx^k) &= \{L_{(\eta^a)}A_k(\xi^k)\}d\xi^k + A_k(\xi^k)L_{(\eta^a)}(d\xi^k) && (1.25) \\
&= D(\xi)\,H(\xi)\,A_k(\xi^k)d\xi^k + D(\xi)\,(1 + B(\xi))\,A_k(\xi^k)d\xi^k && (1.26) \\
&= (H(\xi) + 1 + B(\xi))\,D(\xi)\,A_k(\xi^k)d\xi^k. && (1.27)
\end{aligned}
$$

This formula generalizes for p-forms to read

$$
\begin{aligned}
L_{(\eta^a)}\omega^p &= (H(\xi) + p(1 + B(\xi)))D(\xi)\,\omega^p = k\,\omega^p, && (1.28) \\
\text{where}\quad \xi^m \partial A_k/\partial \xi^m &= H(\xi)\,A_k, && (1.29) \\
\eta^a \partial \xi^k/\partial \eta^a &= D(\xi)\,\xi^k, && (1.30) \\
\text{and}\quad \xi^k \partial D/\partial \xi^k) &= B(\xi)\,D(\xi). && (1.31)
\end{aligned}
$$

A special class of p-forms are those which are closed; i.e., the exterior derivative of the p-form vanishes. From the definition of a homogeneous form, if the homogeneity index is not zero, then the closed *homogeneous* p-form is exact:

$$
\begin{aligned}
L_{(\eta^a)}\omega^p &= d(i(\eta^a)\omega^p) + i(\eta^a)d\omega^p && (1.32) \\
\text{for close homogeneous p-forms} &= d(i(\eta^a)\omega^p) + 0 = k\,\omega^p && (1.33)
\end{aligned}
$$

Special interest is placed upon those closed p-forms which are not exact, for they lead to topological period integrals. If these closed homogeneous p-forms are not exact, they must be homogeneous of degree zero.

Using this theorem, an algorithm will be constructed below to generate closed, but not exact p-forms, in terms of homogeneous of degree zero p-forms. There are two ways to generate homogeneous of degree zero p-forms: either $D(\xi) = 0$, or $H(\xi) = -p$. The choice $D(\xi) = 0$ is not acceptable, for that would imply zero displacements[1]. The choice $H(\xi) = -p$ appears to work well for the construction of period integrals.

1.3. Fractal differential forms

From the arguments presented above, it would appear that if the coefficients of a projectivized differential form are of homogeneity index $H(\xi)$ not an integer, then the exterior differential form is a fractal exterior differential form. The evolution of a fractal differential form can be studied in terms of Cartan's Magic formula.

(More to come)

---

[1] In fact, a popular choice is that $D = 1$, a choice that is called the projectivized p-form, and which is the choice utilized to define Finsler geometries.

2. Applications of Homogeneous differential forms

2.1. The Torsion Vector

The Lie derivative of a 1-form A is said to be homogeneous of degree k if

$$L_{(\mathbf{V})}A = kA \qquad (2.1)$$

In even dimensional spaces, where 2N is the maximal rank of the 2-form dA, there is a remarkable vector direction field, completely determined to within a factor by the coefficients of the 1-form, A. This unique direction field, $\mathbf{T}$, in even dimensional spaces satisfies the homogeneity equation:

$$L_{(\mathbf{T})}A = \Gamma A \qquad (2.2)$$

This vector, defined as the Topological Torsion Vector, $\mathbf{T}(\xi^k)$, is uniquely determined to within a factor by the equations,

$$
\begin{aligned}
i(\mathbf{T})\Omega &= A\hat{\,}dA...\hat{\,}dA, & (2.3) \\
\text{for a volume element } \Omega &= d\xi^1\hat{\,}d\xi^2...\hat{\,}d\xi^{2N}. & (2.4)
\end{aligned}
$$

The Topological Torsion vector[2] has the properties:

$$
\begin{aligned}
i(\mathbf{T})A &= 0, & (2.5) \\
i(\mathbf{T})dA &= \Gamma A, & (2.6) \\
L_{(\mathbf{T})}A &= \Gamma A. & (2.7)
\end{aligned}
$$

The homogeneity index for the torsion vector is not necessarily unity, $\Gamma \neq 1$. In an even dimensional space of 4 independent variables, with the orientation

$$\Omega = dx\hat{\,}dy\hat{\,}d\xi\hat{\,}dt \qquad (2.8)$$

the torsion vector direction field, $\mathbf{T}_4$, is defined by the components of the 3-form, $A\hat{\,}dA$. Then, in the symbol format of electromagnetism,

---

[2] On a space of 4 dimensions, the Topological Torsion vector has the properties of the Liouville vector[1], except that the homogeneity degree is not unity.

$$A = \mathbf{A} \circ d\mathbf{r} - \phi dt, \tag{2.9}$$

$$\mathbf{T}_4 = -[\mathbf{E} \times \mathbf{A} + \mathbf{B}\phi, \mathbf{A} \circ \mathbf{B}], \tag{2.10}$$

$$\text{and} \quad L_{(\mathbf{T}_4)}A = -(\mathbf{E} \circ \mathbf{B})A = \Gamma A. \tag{2.11}$$

It is also true that

$$dA\,\hat{}\,dA = F\,\hat{}\,F = 2(\mathbf{E} \circ \mathbf{B})\Omega. \tag{2.12}$$

For 1-forms, as

$$k = \Gamma = -(\mathbf{E} \circ \mathbf{B}) = (H(\xi) + 1 + B(\xi))\, D(\xi), \tag{2.13}$$

the Cartan Magic formula implies that when the coefficient $\Gamma = -(\mathbf{E} \circ \mathbf{B})$ is not an integer, the topological torsion vector generates a fractal set for any 1-form in 4 dimensional space of Pfaff dimension 4.

For 1-forms that satisfy the Frobenius constraint, $A\,\hat{}\,F = 0$, and are of Pfaff dimension 2, the Topological Torsion vector $\mathbf{T}$ vanishes. It follows $\Gamma = -(\mathbf{E} \circ \mathbf{B}) = 0$. For homogeneous 1-forms that are of Pfaff dimension 3, $A\,\hat{}\,F \neq 0$, but $F\,\hat{}\,F = 2(\mathbf{E} \circ \mathbf{B})\Omega = 0$. Hence, the Topological Torsion vector acting on a homogeneous 1-form of Pfaff dimension 3 by means of Cartan's Magic formula describes a homogeneous 1-form of degree zero. It is also possible that the 1-form A is not homogeneous, but the 3-form $A\,\hat{}\,F$ can be made homogeneous of degree zero, by multiplication of a suitable integrating factor.

For 1-forms that are of Pfaff dimension 4, the Topological Torsion vector is not zero, but generates fractal 1-forms (self similar homogeneous with $k \neq$ integer). For projectivized systems ($D(\xi) = 1, B(\xi) = 0$) the fractal index is determined by the formula,

$$k = \Gamma = -(\mathbf{E} \circ \mathbf{B}) = (H(\xi) + 1). \tag{2.14}$$

The relative orientation of the $\mathbf{E}$ and $\mathbf{B}$ fields will determine the sign of the homogeneity index.

For a thermodynamic process in the direction of the Torsion Vector, $\mathbf{T}_4$, it follows from the general theory of continuous topological evolution[3] that the heat 1-form

$$Q = L_{(\mathbf{T}_4)}A = -(\mathbf{E} \circ \mathbf{B})A = \Gamma A, \tag{2.15}$$

is such that

$$Q\hat{\ }dQ \ = \ (H+1)^2 A\hat{\ }F \tag{2.16}$$
$$= \ \Gamma^2 A\hat{\ }F, \tag{2.17}$$
$$= \ (\mathbf{E} \circ \mathbf{B})^2 A\hat{\ }F. \tag{2.18}$$

If $\mathbf{E} \circ \mathbf{B} \neq 0$, then $Q\hat{\ }dQ \neq 0$, and the process represented by Torsion Vector, $\mathbf{T}_4$, is thermodynamically *irreversible*. Note that the result is independent of the orientation of the $\mathbf{E}$ and $\mathbf{B}$ fields.

2.2. Period integrals

Many (if not all[3]) P-1 dimensional period integrals (topological deformation invariants) can be constructed from those P-1 dimensional currents which are homogeneous of degree zero. Given a vector $\mathbf{V}^a(\xi^k)$ of an ordered array of P functions of m variables, construct the P dimensional volume element Ω and the P-1 current, J, as

$$J \ = \ i(\mathbf{V})\Omega/\lambda = i(\mathbf{V})\{dV^1\hat{\ }dV^2\hat{\ }...\hat{\ }dV^P\}/\lambda, \tag{2.19}$$
$$\lambda \ = \ \lambda(V^a) = \{a_1(V^1)^\sigma + a_2(V^1)^\sigma + ... + a_n(V^P)^\sigma\}^{D/\sigma} \tag{2.20}$$

The scaling, or renormalization factor, λ, is the Holder norm with the Holder homogeneity index fixed to the value P. The resulting renormalized P-1 form, J, is homogeneous of degree zero. As the homogeneous P-1 form is of homogeneity degree zero, the current has the remarkable property of being closed, but not exact. (If the homogeneity index was not zero, then the P-1 form would be exact. The classic example is the 1-form (for P=2):

$$\gamma(\eta^m) \ = \ (V^1 dV^2 - V^2 dV^1)/\lambda, \tag{2.21}$$
$$\lambda \ = \ (\{a_1(V^1)^\sigma + a_2(V^2)^\sigma\}^{P/\sigma}. \tag{2.22}$$

The P-1 = 2-1 = 1-form $\gamma(\eta^m)$ is homogeneous of degree zero, and is closed, but the form has finite circulation for those closed integration cycles that encircle the obstructions generated by the zeros of the Holder norm.

---

[3]It is a conjecture that all period integrals can be constructed from p-1 forms that are homogeneous of degree zero.

For a simple vector, $\mathbf{V} = [V^1, V^2] = [x, y]$, the closed 1-form has a possible representation

$$\gamma = (ydx - xdy)/(x^2 + y^2), \qquad (2.23)$$

where the Holder norm has been chosen such that $a = 1$, $b = 1$, $\sigma = 2$, $P = 2$. The obstruction is the origin point, x=0, y=0.

For the $[x, y]$ representation,

$$d\gamma = 0, \qquad (2.24)$$

$$\oint_{\text{cycle enclosing origin}} \gamma = 2\pi. \qquad (2.25)$$

Such is the case for all homogeneous closed p-forms that are not exact. They all must be homogeneous of degree zero. They all lead to period integrals which are deformation invariants. Hence these objects constructed from closed integrals of closed, but not exact, differential forms are topological objects independent from metric constraints. Bottom line: the period integrals are p-dimensional obstruction, or hole, counters.

2.3. The fundamental theorem:

The fundamental theorem utilizes differentiable projections, π, from a base space $\{x^k\}$ of dimension N to vector subspaces $\{V^m(x^k)\}$ of dimension M, with $M \leq N$.

$$\pi : \{x^k\} \Rightarrow \{V^m(x^k)\}, \qquad (2.26)$$

$$d\pi : |dx^k\rangle \Rightarrow |dV^m(x^k)\rangle = [\partial V^m(x^k)/\partial x^j] \circ |dx^j\rangle. \qquad (2.27)$$

A differential "volume" element $\Omega(V^m)$ is constructed on the M dimensional subspace in terms of the differentials of the components of vector direction field, $\mathbf{V} = [V^m(x^k)] = [U, V, W..]$:

$$\Omega(V^m) = dV^1 \char`^ dV^2 \char`^ ... \char`^ dV^M = dU \char`^ dV \char`^ dW... \qquad (2.28)$$

The components of the direction field $[V^m(x^k)]$ can be used to construct a Holder Norm, λ, of the form,

$$\lambda = (a(U)^p + b(V)^p + c(W)^p...)^{M/p} = \left(\sum_{m=1}^{m=M} a_m \cdot (V^m)^p \right)^{n/p}.$$

8

The Holder norm has "signature" and/or "anisotropy" constants, $\{a, b, c, ...\}$, an arbitrary polynomial index, p, and a "homogeneity" index, n. Although the exponent p can be any integer, special focus is given to the quadratic form[4] that is generated when $p = 2$.

The vector direction field $\mathbf{V}$ can be renormalized by division with respect to the Holder norm, and the result can be used to construct an $M - 1$ form, G:

$$G = i(\mathbf{V}/\lambda)\Omega(V^m) = \{U\,dV\hat{}\,dW... - V\,dU\hat{}\,dW... + W\,dU\hat{}\,dV...\}/\lambda(U, V, W).$$
(2.29)

If the homogeneity index is selected such that $n = M$, then the $M - 1$ form G is homogeneous of degree zero. These constructions lead to the fundamental theorem:

Theorem 2.1. *If for $0 < m \le M$ functions, $\mathbf{V}(x^k)$, of N variables, $\{x^k\}$ such that*

$$
\begin{aligned}
G &= i(\mathbf{V}/\lambda)\Omega(V^m), \\
\Omega_M(V^m) &= dV^{1}\hat{}\,dV^{2}\hat{}...\hat{}\,dV^M \\
and\ \lambda &= \{aU^p + bV^p + cW^p...\}^{n/p} \\
&\quad then \quad for\ any\ p\ and\ any\ \{a, b, c...\}\ and \\
n &= M, \\
dG &= 0\ and\ G\ is\ homogeneous\ of\ degree\ 0.
\end{aligned}
$$

Proof: With n not fixed, by direct computation, the "divergence" of G with respect to the "coordinates" $\{U, V, W..\}$ on M is

$$dG = \{(M - n)/\lambda\}\Omega_M$$
(2.30)

Hence, $dG = 0$ for $n = M$, any p, and any signature for the anisotropic constants $\{a, b, c...\}$. In other words, the Holder type divisor, λ, acts as an integrating factor for the vector field, when $n = M$, any p, any $a, b, c...$ The "excluded" points are the zero sets of λ. The fact that G is homogeneous of degree zero implies that although G is closed, it is not exact. Hence as an integrand of a period integral it will have finite values for cycles which are not boundaries, and the values of these integrals will have rational ratios. In short, these integrals are "quantized".

---

[4]When the signature constants are unity, the polynomial index p=1, and the homogeneity index n=1, the Holder renormalization yields the Gauss map, such that $\langle \mathbf{V}/\lambda| \circ |\mathbf{V}/\lambda \rangle = 1$.

The projective mapping can be used by functional substitution to pull back the closed $M-1$ form $G(V^m, dV^m)$ on M to a closed $M-1$ form $G(x^k, dx^k)$ on N.

3. References

References

[1] P. Libermann and C. Marle, *Symplectic Geometry and Analytical Mechanics,* Reidel, 1986. p. 306

[2] S. S. Chern, "On the Euclidean connections in a Finsler space", *Proc. Nat. Acad. Sci.* USA 29 (1943) P. 38

[3] R. M. Kiehn, "Continuous Topological Evolution" arXiv math-ph/0101032