

```

> restart;with(linalg):with(liesymm):with(diffforms):defform(L=0,f=0,u=0,x=0,y=0,z=
0,t=0,lambda=0,C=const,B=const,Phi=0,FF=0,phi=0):
Warning, the protected names norm and trace have been redefined and unprotected
Warning, the protected name close has been redefined and unprotected
Warning, the names &, d and wdegree have been redefined

```

```

>

```

## Curvatures of 3 Dimensional Implicit Surfaces by various methods.

November 18,2001

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First example,

The Monkey saddle as an implicit surface  $\phi(x,y,z)$  mapped to zero.

```

> Phi:=phi(x,y,z);

```

$$\Phi := \phi(x, y, z)$$

```

> Phi:=z-x^3+3*x*y^2;

```

$$\Phi := z - x^3 + 3 x y^2$$

### Method 1: The Jacobian matrix method for an implicit surfact.

- Compute the gradient field of the implicit surface function, Phi.
- Make the 3D vector field homogeneous of degree zero in its component functions, by division by a Holder Norm of degree 1. Assume isotropy ( $a=b=c=1$ ) and degree  $n=1$ . Choose quadratic index  $p=2$  to agree with the Gauss map.  
(The vector need not be homogeneous in its arguments.)
- Compute the Jacobian matrix of the renormalized field.
- The Jacobian matrix has  $\det = 0$ , and its other similarity invariants are the Mean and the Gauss Curvature.

```

> ZZ:=grad(Phi,[x,y,z]);

```

$$ZZ := [-3 x^2 + 3 y^2, 6 x y, 1]$$

```

> a:=1;b:=1;c:=1;n:=1;p:=2;

```

$$a := 1$$

$$b := 1$$

$$c := 1$$

$$n := 1$$

$$p := 2$$

```

> Holder3D:=(a*ZZ[1]^p+b*ZZ[2]^p+c*ZZ[3]^p)^(n/p);

```

$$Holder3D := \sqrt{(-3 x^2 + 3 y^2)^2 + 36 x^2 y^2 + 1}$$

```

> JAC3D:=simplify(evalm(jacobian(ZZ/Holder3D,[x,y,z])));DETJAC3D:=simplify(det(JAC
3D),trig);MeanCurvature:=factor(simplify(trace(JAC3D)/2,trig));GaussCurvature:=f
actor(trace(adjoint(JAC3D)));

```

$$\begin{aligned}
 JAC3D &:= \begin{bmatrix} -6 \frac{x(18y^4 + 18x^2y^2 + 1)}{\%1^{(3/2)}} & 6 \frac{y(18x^4 + 18x^2y^2 + 1)}{\%1^{(3/2)}} & 0 \\ -6 \frac{y(9x^4 - 9y^4 - 1)}{\%1^{(3/2)}} & 6 \frac{x(-9y^4 + 9x^4 + 1)}{\%1^{(3/2)}} & 0 \\ -18 \frac{x(x^2 + y^2)}{\%1^{(3/2)}} & -18 \frac{y(x^2 + y^2)}{\%1^{(3/2)}} & 0 \end{bmatrix} \\
 \%1 &:= 9x^4 + 18x^2y^2 + 9y^4 + 1 \\
 DETJAC3D &:= 0 \\
 MeanCurvature &:= 27 \frac{x(-3y^2 + x^2)(x^2 + y^2)}{(9x^4 + 18x^2y^2 + 9y^4 + 1)^{(3/2)}} \\
 GaussCurvature &:= -36 \frac{x^2 + y^2}{(9x^4 + 18x^2y^2 + 9y^4 + 1)^2}
 \end{aligned}$$

The Monkey Saddle also admits a parametrization as the 3D Position vector  $x,y \Rightarrow R = [x,y,x^2 - 3y^2]$ . Parametric methods yield the same values as above for the Mean and Gauss Curvature.

>

### Method 2: The Frame matrix method for an implicit surfact by embedding into 4D.

The implicit function can be viewed as a family of Monge surfaces in 4D  $[x,y,z,u]$  with  $u = \text{Phi}(x,y,z)$

- Compute the gradient field of the implicit surface function, Phi
- Make the 3D vector field homogeneous of degree zero in its component functions, by division by a Holder Norm of degree 1. Assume isotropy ( $a=b=c=1$ ) and degree  $n=1$ . Choose quadratic index  $p=2$  to agree with the Gauss map.
- Construct the homogeneous degree zero covariant field NN of 4 components with respect to  $[x,y,z,u]$  whose first 3 components are as above, and with a 4th component equal to 1.
- Algebraically construct 3 independent 4D vectors orthogonal to the this 4D normal field.
- Construct a Frame matrix whose columns are the the 3 orthogonal vectors and with the 4th column the 4D normal field.

> `NN3D:=evalm(ZZ/Holder3D):NN:=[NN3D[1],NN3D[2],NN3D[3],1];`

$$NN := \left[ \frac{\%1}{\sqrt{\%1^2 + 36x^2y^2 + 1}}, 6 \frac{xy}{\sqrt{\%1^2 + 36x^2y^2 + 1}}, \frac{1}{\sqrt{\%1^2 + 36x^2y^2 + 1}}, 1 \right]$$

$\%1 := -3x^2 + 3y^2$

> `FT:=array([[1,0,0,NN[1]], [0,1,0,NN[2]], [0,0,1,NN[3]], [-NN[1],-NN[2],-NN[3],1]]):`

The Cartan Frame matrix of columns FF:

> `FF:=transpose(FT);gmn:=innerprod(FT,FF);`

$$FF := \begin{bmatrix} 1 & 0 & 0 & -\frac{\%1}{\sqrt{\%2}} \\ 0 & 1 & 0 & -6 \frac{xy}{\sqrt{\%2}} \\ 0 & 0 & 1 & -\frac{1}{\sqrt{\%2}} \\ \frac{\%1}{\sqrt{\%2}} & 6 \frac{xy}{\sqrt{\%2}} & \frac{1}{\sqrt{\%2}} & 1 \end{bmatrix}$$

$$gmn := \begin{bmatrix} \frac{18x^4 + 18y^4 + 1}{\%1} & -18 \frac{(x^2 - y^2)xy}{\%1} & -3 \frac{x^2 - y^2}{\%1} & 0 \\ -18 \frac{(x^2 - y^2)xy}{\%1} & \frac{9x^4 + 54x^2y^2 + 9y^4 + 1}{\%1} & 6 \frac{xy}{\%1} & 0 \\ -3 \frac{x^2 - y^2}{\%1} & 6 \frac{xy}{\%1} & \frac{9x^4 + 18x^2y^2 + 9y^4 + 2}{\%1} & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

$$\%1 := -3x^2 + 3y^2$$

$$\%2 := \%1^2 + 36x^2y^2 + 1$$

$$\%1 := 9x^4 + 18x^2y^2 + 9y^4 + 1$$

> `DET:=det(FF);FFINV:=inverse(FF):(FF[1,4]);d(Holder3D);d(Phi);`

>

>

$$DET := 2$$

$$-\frac{-3x^2 + 3y^2}{\sqrt{(-3x^2 + 3y^2)^2 + 36x^2y^2 + 1}}$$

$$18 \frac{y(x^2 + y^2)d(y)}{\sqrt{9x^4 + 18x^2y^2 + 9y^4 + 1}} + \frac{18x(x^2 + y^2)d(x)}{\sqrt{9x^4 + 18x^2y^2 + 9y^4 + 1}}$$

$$6xy d(y) + (-3x^2 + 3y^2)d(x) + d(z)$$

Compute the 1-form basis elements based upon the Frame matrix.

> `DR:=innerprod(FF,[d(x),d(y),d(z),d(u)]):sigma1:=(wcollect(factor(DR[1])));sigma2:=wcollect(DR[2]);sigma3:=wcollect(DR[3]);omega:=wcollect(DR[4]);`

$$\sigma_1 := d(x) + \frac{(3x^2 - 3y^2)d(u)}{\sqrt{9x^4 + 18x^2y^2 + 9y^4 + 1}}$$

$$\sigma_2 := d(y) - \frac{6xy d(u)}{\sqrt{9x^4 + 18x^2y^2 + 9y^4 + 1}}$$

$$\sigma_3 := d(z) - \frac{d(u)}{\sqrt{9x^4 + 18x^2y^2 + 9y^4 + 1}}$$

$$\omega := \frac{(-3x^2 + 3y^2)d(x)}{\sqrt{\%1}} + \frac{6xy d(y)}{\sqrt{\%1}} + \frac{d(z)}{\sqrt{\%1}} + d(u)$$

$$\%1 := 9x^4 + 18x^2y^2 + 9y^4 + 1$$

The forms sigma and omega are the 1-forms relative to the basis frame FF. Note that little omega is proportional to the ACTION 1-form in 4D composed of the homogeneous components of the "gradient" field for the first 3 components, and 1 for the 4th component.

> `Field:=d(omega);Torsion:=omega&^Field;Parity:=Field&^Field;Bx:=getcoeff(Field&^d(x));By:=getcoeff(Field&^d(y));Bz:=getcoeff(Field&^d(z));Helicity:=wcollect(omega&^Torsion);`

$$Field := 54 \frac{y(3x^4 + 2x^2y^2 - y^4)(d(y) \&\wedge d(x))}{\%1^{(3/2)}} - \frac{18y(x^2 + y^2)(d(y) \&\wedge d(z))}{\%1^{(3/2)}} - \frac{18x(x^2 + y^2)(d(x) \&\wedge d(z))}{\%1^{(3/2)}}$$

$$\%1 := 9x^4 + 18x^2y^2 + 9y^4 + 1$$

$$Torsion := 54 \frac{y(3x^4 + 2x^2y^2 - y^4) \&\wedge (d(u), d(y), d(x))}{\%1^{(3/2)}} - \frac{18y(x^2 + y^2) \&\wedge (d(u), d(y), d(z))}{\%1^{(3/2)}}$$

$$\frac{18 x (x^2 + y^2) \wedge (d(u), d(x), d(z))}{\%1^{(3/2)}}$$

$$\%1 := 9 x^4 + 18 x^2 y^2 + 9 y^4 + 1$$

Parity := 0

$$B_x := -18 \frac{y (x^2 + y^2)}{(9 x^4 + 18 x^2 y^2 + 9 y^4 + 1)^{(3/2)}}$$

$$B_y := -18 \frac{x (x^2 + y^2)}{(9 x^4 + 18 x^2 y^2 + 9 y^4 + 1)^{(3/2)}}$$

$$B_z := 54 \frac{y (3 x^4 + 2 x^2 y^2 - y^4)}{(9 x^4 + 18 x^2 y^2 + 9 y^4 + 1)^{(3/2)}}$$

Helicity := 0

The Torsion current (of omega) has no 4th component (the Helicity is zero) and the the space parts are proportional to the vorticity, (or the B field) (E is zero, phi =1)

> **VV:=factor(simplify(det(FF), trig)):**

> **dFF:=array([[d(FF[1,1]),d(FF[1,2]),d(FF[1,3]),d(FF[1,4])],[d(FF[2,1]),d(FF[2,2]),d(FF[2,3]),d(FF[2,4])],[d(FF[3,1]),d(FF[3,2]),d(FF[3,3]),d(FF[3,4])],[d(FF[4,1]),d(FF[4,2]),d(FF[4,3]),d(FF[4,4])])]:**

Evaluate the Cartan matrix of 1-forms based upon the Frame matrix

> **cartan:=evalm(FFINV\*dFF):**

> **gamma11:=(wcollect(cartan[1,1])):**

$$\gamma_{11} := -9 \frac{(x^2 - y^2) y (18 x^4 + 18 x^2 y^2 + 1) d(y)}{(9 x^4 + 18 x^2 y^2 + 9 y^4 + 1)^2} + \frac{9 (x^2 - y^2) x (18 y^4 + 18 x^2 y^2 + 1) d(x)}{(9 x^4 + 18 x^2 y^2 + 9 y^4 + 1)^2}$$

> **gamma12:=(wcollect(cartan[1,2])):**

> **gamma13:=(wcollect(cartan[1,3])):**

> **gamma21:=(wcollect(cartan[2,1])):**

> **gamma22:=(wcollect(cartan[2,2])):**

> **gamma23:=(wcollect(cartan[2,3])):**

> **gamma31:=(wcollect(cartan[3,1])):**

> **gamma32:=(wcollect(cartan[3,2])):**

> **gamma33:=(wcollect(cartan[3,3])):**

> **# second fundamental form and shape matrix**

> **hh1:=factor(wcollect(cartan[4,1])):**

>

$$hh1 := 3 \frac{18 y d(y) x^4 + 18 y^3 d(y) x^2 + y d(y) - 18 x d(x) y^4 - 18 x^3 d(x) y^2 - x d(x)}{(9 x^4 + 18 x^2 y^2 + 9 y^4 + 1)^{(3/2)}}$$

> **gg1:=wcollect(simplify(factor(wcollect(cartan[1,4])), trig)):**

$$gg1 := -6 \frac{(18 y x^4 + 18 y^3 x^2 + y) d(y)}{(9 x^4 + 18 x^2 y^2 + 9 y^4 + 1)^{(3/2)}} - \frac{6 (-18 x^3 y^2 - x - 18 x y^4) d(x)}{(9 x^4 + 18 x^2 y^2 + 9 y^4 + 1)^{(3/2)}}$$

> **hh2:=wcollect(cartan[4,2]):**

$$hh2 := 3 \frac{x (-9 y^4 + 9 x^4 + 1) d(y)}{(9 x^4 + 18 x^2 y^2 + 9 y^4 + 1)^{(3/2)}} - \frac{3 y (9 x^4 - 9 y^4 - 1) d(x)}{(9 x^4 + 18 x^2 y^2 + 9 y^4 + 1)^{(3/2)}}$$

> **gg2:=(wcollect(cartan[2,4])):**

$$gg2 := \left( -54 \frac{(x^2 - y^2) x y^2 (18 x^4 + 18 x^2 y^2 + 1)}{\%1^{(5/2)}} - \frac{6(9 x^4 + 9 y^4 + 1) x (-9 y^4 + 9 x^4 + 1)}{\%1^{(5/2)}} - \frac{54 x y^2 (x^2 + y^2)}{\%1^{(5/2)}} \right) d(y) \\ + \left( 54 \frac{(x^2 - y^2) x^2 y (18 y^4 + 18 x^2 y^2 + 1)}{\%1^{(5/2)}} + \frac{6(9 x^4 + 9 y^4 + 1) y (9 x^4 - 9 y^4 - 1)}{\%1^{(5/2)}} - \frac{54 x^2 y (x^2 + y^2)}{\%1^{(5/2)}} \right) d(x) \\ \%1 := 9 x^4 + 18 x^2 y^2 + 9 y^4 + 1$$

> `hh3:=wcollect(cartan[4,3]);`

$$hh3 := -9 \frac{y (x^2 + y^2) d(y)}{(9 x^4 + 18 x^2 y^2 + 9 y^4 + 1)^{(3/2)}} - \frac{9 x (x^2 + y^2) d(x)}{(9 x^4 + 18 x^2 y^2 + 9 y^4 + 1)^{(3/2)}}$$

> `gg3:=(wcollect(cartan[3,4]));`

`gg3 :=`

$$\left( -9 \frac{(x^2 - y^2) y (18 x^4 + 18 x^2 y^2 + 1)}{\%1^{(5/2)}} + \frac{18 x^2 y (-9 y^4 + 9 x^4 + 1)}{\%1^{(5/2)}} + \frac{9 (18 x^4 + 36 x^2 y^2 + 18 y^4 + 1) y (x^2 + y^2)}{\%1^{(5/2)}} \right) d(y) \\ + \left( 9 \frac{(x^2 - y^2) x (18 y^4 + 18 x^2 y^2 + 1)}{\%1^{(5/2)}} - \frac{18 x y^2 (9 x^4 - 9 y^4 - 1)}{\%1^{(5/2)}} + \frac{9 (18 x^4 + 36 x^2 y^2 + 18 y^4 + 1) x (x^2 + y^2)}{\%1^{(5/2)}} \right) d(x) \\ \%1 := 9 x^4 + 18 x^2 y^2 + 9 y^4 + 1$$

> `Omega:=simplify(wcollect(cartan[4,4]));`

$$\Omega := 0$$

Big Omega vanishes for the specific 4d euclidean norm chosen

> `shape11:=-factor(gg1^(d(z)&d(y))/d(x)&d(z)&d(y));`

$$shape11 := -6 \frac{x (18 y^4 + 18 x^2 y^2 + 1)}{(9 x^4 + 18 x^2 y^2 + 9 y^4 + 1)^{(3/2)}}$$

> `shape12:=-factor(gg1^(d(x)&d(z))/d(y)&d(x)&d(z));`

$$shape12 := 6 \frac{y (18 x^4 + 18 x^2 y^2 + 1)}{(9 x^4 + 18 x^2 y^2 + 9 y^4 + 1)^{(3/2)}}$$

> `shape13:=-factor(gg1^(d(y)&d(x))/d(z)&d(y)&d(x));`

$$shape13 := 0$$

> `shape21:=-factor(gg2^(d(z)&d(y))/d(x)&d(z)&d(y));`

$$shape21 := -6 \frac{y (9 x^4 - 9 y^4 - 1)}{(9 x^4 + 18 x^2 y^2 + 9 y^4 + 1)^{(3/2)}}$$

> `shape22:=-factor(gg2^(d(x)&d(z))/d(y)&d(x)&d(z));ss:=shape22*VV^(2/2)*Holder 3D^(3);VV;`

$$shape22 := 6 \frac{x (-9 y^4 + 9 x^4 + 1)}{(9 x^4 + 18 x^2 y^2 + 9 y^4 + 1)^{(3/2)}}$$

$$ss := 12 \frac{x (-9 y^4 + 9 x^4 + 1) ((-3 x^2 + 3 y^2)^2 + 36 x^2 y^2 + 1)}{(9 x^4 + 18 x^2 y^2 + 9 y^4 + 1)^{(3/2)}} \\ 2$$

> `shape23:=-factor(gg2^(d(y)&d(x))/d(z)&d(y)&d(x));`

$$shape23 := 0$$

> `shape31:=-factor(gg3^(d(z)&d(y))/d(x)&d(z)&d(y));`

$$shape31 := -18 \frac{x (x^2 + y^2)}{(9 x^4 + 18 x^2 y^2 + 9 y^4 + 1)^{(3/2)}}$$

```

> shape32:=-factor(gg3^(d(x)&d(z))/d(y)&d(x)&d(z));
shape32 := -18  $\frac{y(x^2+y^2)}{(9x^4+18x^2y^2+9y^4+1)^{(3/2)}}$ 
> shape33:=-factor(gg3^(d(y)&d(x))/d(z)&d(y)&d(x));
shape33 := 0
> SHAPE:=evalm(array([[shape11,shape12,shape13],[shape21,shape22,shape23],[shape31,shape32,shape33]]));Jacobian:=evalm(JAC3D);

```

$$SHAPE := \begin{bmatrix} -6 \frac{x(18y^4+18x^2y^2+1)}{\%1^{(3/2)}} & 6 \frac{y(18x^4+18x^2y^2+1)}{\%1^{(3/2)}} & 0 \\ -6 \frac{y(9x^4-9y^4-1)}{\%1^{(3/2)}} & 6 \frac{x(-9y^4+9x^4+1)}{\%1^{(3/2)}} & 0 \\ -18 \frac{x(x^2+y^2)}{\%1^{(3/2)}} & -18 \frac{y(x^2+y^2)}{\%1^{(3/2)}} & 0 \end{bmatrix}$$

%1 := 9x<sup>4</sup> + 18x<sup>2</sup>y<sup>2</sup> + 9y<sup>4</sup> + 1

$$Jacobian := \begin{bmatrix} -6 \frac{x(18y^4+18x^2y^2+1)}{\%1^{(3/2)}} & 6 \frac{y(18x^4+18x^2y^2+1)}{\%1^{(3/2)}} & 0 \\ -6 \frac{y(9x^4-9y^4-1)}{\%1^{(3/2)}} & 6 \frac{x(-9y^4+9x^4+1)}{\%1^{(3/2)}} & 0 \\ -18 \frac{x(x^2+y^2)}{\%1^{(3/2)}} & -18 \frac{y(x^2+y^2)}{\%1^{(3/2)}} & 0 \end{bmatrix}$$

%1 := 9x<sup>4</sup> + 18x<sup>2</sup>y<sup>2</sup> + 9y<sup>4</sup> + 1

The Shape matrix computed from the Cartan matrix of 1-forms (for the 3D euclidean norm) is precisely the same as the Jacobian of the 3D normalized gradient field. (The 3D gauss map). Hence the similarity invariants are the same.

The mean curvature is the Trace of the Shape matrix (a 3D matrix). The determinant vanishes, and the Gauss curvature is the trace of the adjoint of the Shape matrix. All of these quantities are similarity invariants of the shape matrix.

```

>
> HHT:=factor(wcollect(trace(SHAPE)/2,trig));print(`Mean Curvature is `,HHT);
The difference between the two mean curvatures is zero.

```

$$HHT := 27 \frac{x(-3y^2+x^2)(x^2+y^2)}{(9x^4+18x^2y^2+9y^4+1)^{(3/2)}}$$

Mean Curvature is , 27  $\frac{x(-3y^2+x^2)(x^2+y^2)}{(9x^4+18x^2y^2+9y^4+1)^{(3/2)}}$

```

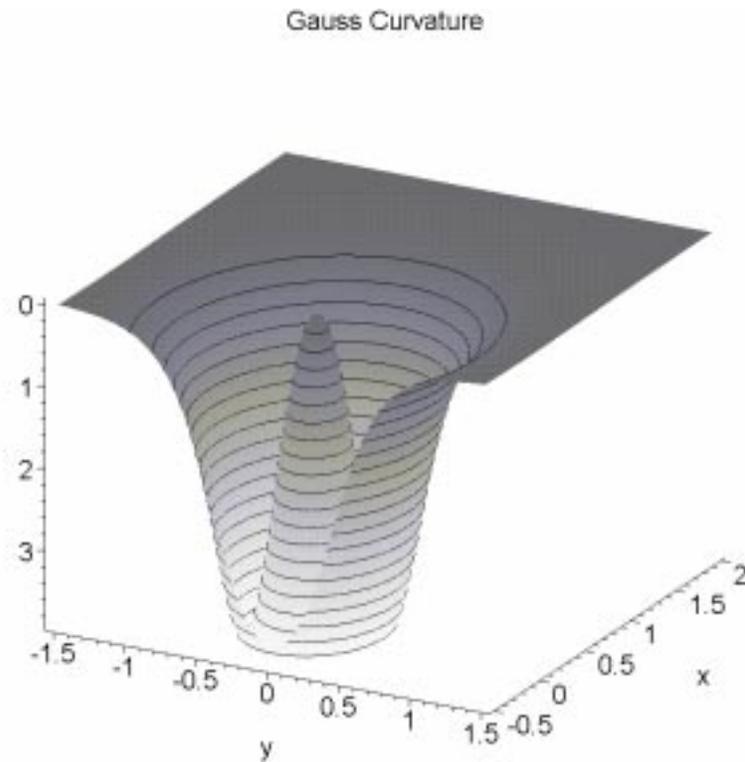
>
> ADJSH:=adjoint(SHAPE):
> KK:=factor(simplify(trace(ADJSH),trig)):
> print(`Gauss Curvature is `,KK);

```

$$Gauss\ Curvature\ is\ ,\ -36 \frac{x^2+y^2}{(9x^4+18x^2y^2+9y^4+1)^2}$$

For example, a plot of the Gauss Curvature over the plane of surface variables:

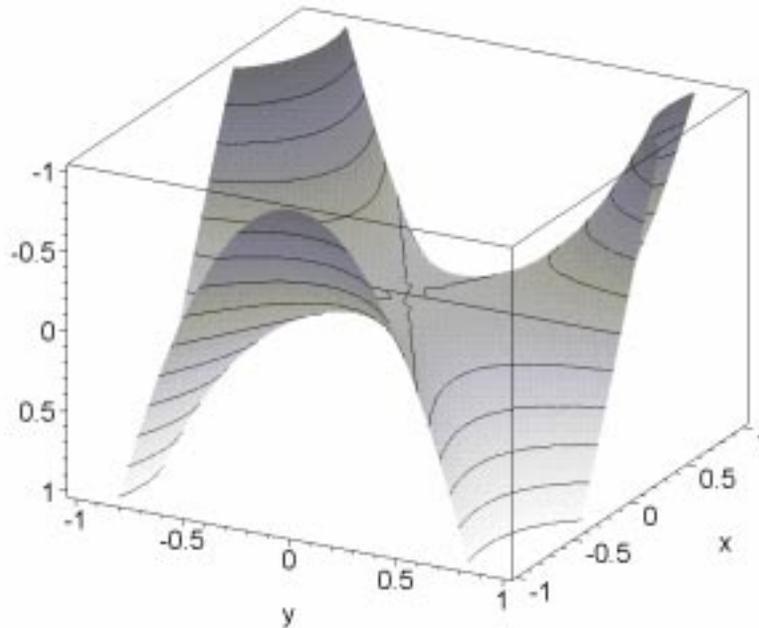
```
> plot3d(-KK,x=-0.5..2,y=-1.5..1.5,numpoints=5000,title=`Gauss  
Curvature`,axes=FRAMED,shading=ZGREYSCALE,style=PATCHCONTOUR,orientation=[-28,-1  
18]);
```



```
>  
> with(plots):implicitplot3d( Phi=0,x=-1..1,y=-1..1,z=-1..1,  
title=`Monkey  
Saddle`,numpoints=9000,shading=ZGREYSCALE,style=PATCHCONTOUR,orientation=[-28,-1  
18],axes=BOXED);
```

Warning, the name changecoords has been redefined

Monkey Saddle



Note that the Gauss curvature is axially symmetric, but the Monkey saddle is not.

### Method 3: The induced metric method.

- Compute the gradient field of the implicit surface function.
- Make the 3D vector field homogeneous of degree zero in its component functions, by division by a Holder Norm of degree 1. Assume isotropy ( $a=b=c=1$ ) and degree  $n=1$ . Choose quadratic index  $p=2$  to agree with the Gauss map.

The vector need not be homogeneous in the arguments of its component functions.

- Compute the JACobian matrix of the renormalized field.
- The JACobian matrix has  $\det = 0$
- Compute the pullback metric on the 2D surface is  $\text{Transpose}(\text{JACobian}) \cdot \text{JACobian}$
- Compute the Christoffel connection coefficients on the the 2D surface

> `metric3d:=innerprod(transpose(JAC3D),JAC3D);`

$$metric3d := \begin{bmatrix} 36 \frac{9x^4y^2 + x^2 + 18x^2y^4 + 9y^6 + y^2}{\%1^2} & -324 \frac{xy(x^4 + 2x^2y^2 + y^4)}{\%1^2} & 0 \\ -324 \frac{xy(x^4 + 2x^2y^2 + y^4)}{\%1^2} & 36 \frac{9x^6 + 18x^4y^2 + 9x^2y^4 + x^2 + y^2}{\%1^2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\%1 := 9x^4 + 18x^2y^2 + 9y^4 + 1$

The induced 2D metric is:

> `metric:=( [[metric3d[1,1],metric3d[1,2]], [metric3d[2,1],metric3d[2,2]] ] );`

$$metric := \left[ \left[ 36 \frac{9x^4y^2 + x^2 + 18x^2y^4 + 9y^6 + y^2}{\%1^2}, -324 \frac{xy(x^4 + 2x^2y^2 + y^4)}{\%1^2} \right], \left[ -324 \frac{xy(x^4 + 2x^2y^2 + y^4)}{\%1^2}, 36 \frac{9x^6 + 18x^4y^2 + 9x^2y^4 + x^2 + y^2}{\%1^2} \right] \right]$$

```
[ %1 := 9 x^4 + 18 x^2 y^2 + 9 y^4 + 1
```

```
> dim:=2;
```

```
dim := 2
```

```
> metricinverse:=inverse(metric);coord:=[x,y];
```

$$metricinverse := \begin{bmatrix} \frac{1}{36} \frac{\%1 (9 x^4 + 9 x^2 y^2 + 1)}{x^2 + y^2} & \frac{1}{4} x y \%1 \\ \frac{1}{4} x y \%1 & \frac{1}{36} \frac{\%1 (9 x^2 y^2 + 9 y^4 + 1)}{x^2 + y^2} \end{bmatrix}$$

```
%1 := 9 x^4 + 18 x^2 y^2 + 9 y^4 + 1
```

```
coord := [x, y]
```

```
[ Compute the metric induced Christoffel symbols.
```

```
> for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do  
d1gun[i,j,k] := (diff(metric[i,j],coord[k])) od od od:
```

```
> #for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do if  
d1gun[i,j,k]=0 then else print(`dgun^(i,j,k)=d1gun[i,j,k]`) fi od od od:
```

```
> for i from 1 to dim do for j from i to dim do for k from 1 to dim do C1S[i,j,k]  
:= 0 od od od; for i from 1 to dim do for j from 1 to dim do for k from 1 to  
dim do C1S[i,j,k] := 1/2*d1gun[i,k,j]+1/2*d1gun[j,k,i]-1/2*d1gun[i,j,k] od od  
od;
```

```
> for k from 1 to dim do for i from 1 to dim do for j from 1 to dim do s := 0;  
for m to dim do s := s+metricinverse[k,m]*C1S[i,j,m] od; C2S[k,i,j] :=  
simplify(factor(s),trig) od od od;
```

```
> for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do if  
C2S[i,j,k]=0 then else print(`ChristoffelGamma2^(i,j,k)=C2S[i,j,k]`) fi od od  
od;
```

$$ChristoffelGamma2(1, 1, 1) = -\frac{(27 x^4 + 54 x^2 y^2 + 27 y^4 - 1) x}{(9 x^4 + 18 x^2 y^2 + 9 y^4 + 1) (x^2 + y^2)}$$

$$ChristoffelGamma2(1, 1, 2) = -\frac{(3 x^2 - 1 + 3 y^2) (3 x^2 + 1 + 3 y^2) y}{(9 x^4 + 18 x^2 y^2 + 9 y^4 + 1) (x^2 + y^2)}$$

$$ChristoffelGamma2(1, 2, 1) = -\frac{(3 x^2 - 1 + 3 y^2) (3 x^2 + 1 + 3 y^2) y}{(9 x^4 + 18 x^2 y^2 + 9 y^4 + 1) (x^2 + y^2)}$$

$$ChristoffelGamma2(1, 2, 2) = -\frac{x}{x^2 + y^2}$$

$$ChristoffelGamma2(2, 1, 1) = -\frac{y}{x^2 + y^2}$$

$$ChristoffelGamma2(2, 1, 2) = -\frac{(3 x^2 - 1 + 3 y^2) (3 x^2 + 1 + 3 y^2) x}{(9 x^4 + 18 x^2 y^2 + 9 y^4 + 1) (x^2 + y^2)}$$

$$ChristoffelGamma2(2, 2, 1) = -\frac{(3 x^2 - 1 + 3 y^2) (3 x^2 + 1 + 3 y^2) x}{(9 x^4 + 18 x^2 y^2 + 9 y^4 + 1) (x^2 + y^2)}$$

$$ChristoffelGamma2(2, 2, 2) = -\frac{(27 x^4 + 54 x^2 y^2 + 27 y^4 - 1) y}{(9 x^4 + 18 x^2 y^2 + 9 y^4 + 1) (x^2 + y^2)}$$

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