```
> restart;
```

Implicit Surfaces

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A Maple program that first uses the Jacobian matrix of the unit normal field to define the Gauss and Mean curvatures of the surface -- in terms of the similarity invariants of the Shape matrix. Then, the implicit surface is projectively embedded and then the Cartan repere mobile method is used.

See notes at http://www.uh.edu/~rkiehn/pdf/defects2.pdf

From the method of parametric surfaces applied to the Monge surface

see http://www.uh.edu/~rkiehn/pdf/parametric.pdf

it is discovered that there is an equivalence between the shape matrix, the Repere Mobile, and the Jacobian matrix of the surface normal field (the gradient of the implicit function) when the normal field is scaled by the euclidean norm (the Gauss map). The scaling of the normal field can also be done using the generalization of the euclidean distance given by the Holder norm. The key idea is that any form of the Holder norm which is homogeneous of degree 1 in the components of the normal field leads to a Jacobian matrix with zero determinant. This is the condition that yields a global N-1 surface.

The method will be applied to simple surfaces that are defined by zero sets of a function on a space of variables. Then the method will be applied to Cartan surfaces in terms of the 1-form constraint on the differentials of the domain. Such surfaces do not admit a unique functional form.

```
>  with (linalg): with(plots):setup[x,y,z]:
> DR:=([dx,dy,dz]);
Warning, new definition for norm
Warning, new definition for trace
```

$$DR := [dx, dy, dz]$$

```
> defform(x=0,y=0,qz=0,v=0,u=0,vz=0,px=0,py=0,pz=0,t=0,s=0,a=const,b=const,c=const
  ,k=const,mu=const,m=const);
```

defform($x = 0, y = 0, qz = 0, v = 0, u = 0, vz = 0, px = 0, py = 0, pz = 0, t = 0, s = 0, a = const, b = const, c = const, k = const, \mu = const, m = const$)

The Implicit Sphere

Define an implicit surface function: The first example is the sphere:

```
> Phi:=(x^2 + y^2+z^2 -1);
```

$$\Phi := x^2 + y^2 + z^2 - 1$$

```
>
```

Compute the Gradient field of the function and then scale the normal field by dividing by a Holder norm. Then compute the Jacobian matrix of the scaled normal field, and its similarity invariants.

(The normal field could also be specified as the coefficients of a non-integrable 1-form.)

In the example below, a Holder type norm is used to define the "magnitude".

Various choices for the Holder exponents will force the similarity invariants of the Jacobian matrix to vanish. When the norm is homogeneous of degree 1, the determinant of the Jacobian of the "normalized" vector field vanishes.

The norm does not have to be isotropic (the constants a, b, c, can be arbitrary). Therefor the norm can have any signature, and the determinant still vanishes!!

The classic case is when the Holder norm exponents are p=2 and n = 1, and the anisotropic coefficients are a=b=c=1 (the quadratic form). This case is called the Gauss Map.

Set a=b=1, c=-1 to get a 1-sheeted Hyperbolic Map. This can be done in 4 variables and will be related to the Minkowski space of Special Relativity.

```
> NN:=eval(([diff(Phi,x),diff(Phi,y),diff(Phi,z)]));
> magn:=(a*NN[1]^p+b*NN[2]^p+c*NN[3]^p)^(n/p);
```

$$NN := [2\,x, 2\,y, 2\,z]$$

$$magn := (a\,(2\,x)^p + b\,(2\,y)^p + c\,(2\,z)^p)^{\left(\frac{n}{p}\right)}$$

(Experiment with other values for the Holder exponents, p and n, and other signatures.)
Recall that in the parametric surface theory when the euclidean norm was used on the adjoint (normal) field, Big Omega of the Cartan matrix vanished. Could not one of the other norms be used to study deformable media??
The example below is for the Gauss Map.

```
> p:=2;n:=1;
> a:=1;b:=1;c:=1;
```

$$p := 2$$
$$n := 1$$
$$a := 1$$
$$b := 1$$
$$c := 1$$

```
> unitnormal:=evalm(NN/magn);
```

$$unitnormal := \left[2\,\frac{x}{\sqrt{4\,x^2+4\,y^2+4\,z^2}}, 2\,\frac{y}{\sqrt{4\,x^2+4\,y^2+4\,z^2}}, 2\,\frac{z}{\sqrt{4\,x^2+4\,y^2+4\,z^2}}\right]$$

Compute the Jacobian matrix of the unit normal field:

```
> JACNN:=jacobian(unitnormal,[x,y,z]);
```

$$JACNN := \begin{bmatrix} -8\,\frac{x^2}{\%1^{3/2}} + \frac{2}{\sqrt{\%1}} & -8\,\frac{x\,y}{\%1^{3/2}} & -8\,\frac{x\,z}{\%1^{3/2}} \\ -8\,\frac{x\,y}{\%1^{3/2}} & -8\,\frac{y^2}{\%1^{3/2}} + \frac{2}{\sqrt{\%1}} & -8\,\frac{y\,z}{\%1^{3/2}} \\ -8\,\frac{x\,z}{\%1^{3/2}} & -8\,\frac{y\,z}{\%1^{3/2}} & -8\,\frac{z^2}{\%1^{3/2}} + \frac{2}{\sqrt{\%1}} \end{bmatrix}$$

$$\%1 := 4\,x^2 + 4\,y^2 + 4\,z^2$$

```
> DETJAC:=det(JACNN);
```

$$DETJAC := 0$$

The determinant of the matrix is everywhere zero, (by the choice of the norm which is homogeneous of degree 1 and of any signature). This result implies that the system forms a global 2D surface.
Recall that the Projectivized Tangent to the Fibers in Finsler spaces is essentially a normalized field of the type described above. Question: what are the physical consequences of assuming other exponents and signatures in Finsler geometries?

Now compute the other similarity invariants: the trace of the Jacobian and the trace of the Jacobian adjoint (which exists even though the inverse matrix does not.)

```
> MEAN_CURVATURE:=factor(simplify(trace(JACNN)/2));
```

$$MEAN\_CURVATURE := \frac{1}{\sqrt{x^2+y^2+z^2}}$$

```
> ADJOINTJACNN:=(adj(JACNN));
```

$$ADJOINTJACNN := \begin{bmatrix} \frac{x^2}{(x^2+y^2+z^2)^2} & \frac{y\,x}{(x^2+y^2+z^2)^2} & \frac{x\,z}{(x^2+y^2+z^2)^2} \\ \frac{y\,x}{(x^2+y^2+z^2)^2} & \frac{y^2}{(x^2+y^2+z^2)^2} & \frac{z\,y}{(x^2+y^2+z^2)^2} \\ \frac{x\,z}{(x^2+y^2+z^2)^2} & \frac{z\,y}{(x^2+y^2+z^2)^2} & \frac{z^2}{(x^2+y^2+z^2)^2} \end{bmatrix}$$

```
> GAUSS_CURVATURE:=simplify(trace(ADJOINTJACNN));
```

$$GAUSS\_CURVATURE := \frac{1}{x^2+y^2+z^2}$$

Besides the similarity invariants above there are three other invariants that are useful for describing the 3x3 Jacobian matrix. These are called the Brand invariants and are discussed in L. Brand's book on Vector and Tensor Analysis, Wiley 1962, page 173.
These three other invariants are related to the anti-symmetric parts of the Jacobian matrix (a dyadic in the Brand-Gibbs notation). For a 3x3 matrix the anti-symmetric matrix elements are related to the "curl" of the unitnormal field, which will be defined herein as the VORTICITY.
The adjoint matrix can also have antisymmetric parts. In 3 dimensions these antisymmetric parts form another "curl" type vector field. This second curl-type vector field can be constructed by multiplying the curl of the unit normal field by the Jacobian matrix itself. This second anti-symmetric construction will be defined as the "BRAND" vector.
In Hydrodynamics, the "inner" product of the VORTICITY with itself is called the "enstrophy" of the flow. The innerproduct of the VORTICITY and the BRAND vector has been associated with "vortex stretching. The innerproduct of the BRAND vector with itself has never been defined.
Yet all three are invariant scalars under similarity transformations.

The unit normal field can have vorticity, but no helicity if generated from a implicit function. If the unit normal field is generated from a 1-form then the system may not be integrable and helicity will be non-zero.

```
>
>
```

For the Implicit sphere, the vorticity vanishes.

```
> VORTICITY:=evalm(curl(unitnormal,[x,y,z]));
```

$$VORTICITY := [0, 0, 0]$$

```
> BRAND:=innerprod(JACNN,VORTICITY);
```

$$BRAND := [0, 0, 0]$$

```
> HELICITY:=simplify(innerprod(unitnormal,VORTICITY));
```

$$HELICITY := 0$$

```
>
> MAGNUS_FORCE:=(crossprod(unitnormal,VORTICITY));
```

$$MAGNUS\_FORCE := [0, 0, 0]$$

```
> # Brandt invariants
> enstrophy:=simplify(innerprod(VORTICITY,VORTICITY));
```

$$enstrophy := 0$$

```
>
> stretch:=simplify(innerprod(BRAND,VORTICITY));
```

$$stretch := 0$$

```
> brand_invariant:=simplify(innerprod(BRAND,BRAND));
```

$$brand\_invariant := 0$$

```
>
> restart;
```

SECOND EXAMPLE

The Catenoid

```
>  with (linalg): with(plots):setup[x,y,z]:
> DR:=([dx,dy,dz]);
Warning, new definition for norm
Warning, new definition for trace
```

$$DR := [dx, dy, dz]$$

```
> defform(x=0,y=0,qz=0,v=0,u=0,vz=0,px=0,py=0,pz=0,t=0,s=0,a=const,b=const,c=const
  ,k=const,mu=const,m=const);
```

$$\mathrm{defform}(x = 0, y = 0, qz = 0, v = 0, u = 0, vz = 0, px = 0, py = 0, pz = 0, t = 0, s = 0, a = const, b = const, c = const, k = const, \mu = const, m = const)$$

Define an implicit surface function: The second example is the catenoid

```
> Phi:=(x^2+y^2-cosh(z)^2);
```

$$\Phi := x^2 + y^2 - \cosh(z)^2$$

```
>
> NN:=eval(([diff(Phi,x),diff(Phi,y),diff(Phi,z)]));
> magn:=(a*NN[1]^p+b*NN[2]^p+c*NN[3]^p)^(n/p);
```

$$NN := [2\,x, 2\,y, -2\cosh(z)\sinh(z)]$$

$$magn := (a\,(2\,x)^p + b\,(2\,y)^p + c\,(-2\cosh(z)\sinh(z))^p)^{\left(\frac{n}{p}\right)}$$

```
> p:=2;n:=1;
> a:=1;b:=1;c:=1;
```

$$p := 2$$
$$n := 1$$
$$a := 1$$
$$b := 1$$
$$c := 1$$

```
> unitnormal:=evalm(NN/magn);
```

$$unitnormal := \left[2\frac{x}{\sqrt{\%1}}, 2\frac{y}{\sqrt{\%1}}, -2\frac{\cosh(z)\sinh(z)}{\sqrt{\%1}}\right]$$

$$\%1 := 4\,x^2 + 4\,y^2 + 4\cosh(z)^2\sinh(z)^2$$

Compute the Jacobian matrix of the unit normal field:

```
> JACNN:=jacobian(unitnormal,[x,y,z]);
```

$$JACNN := \begin{bmatrix} -8\frac{x^2}{\%1^{3/2}} + \frac{2}{\sqrt{\%1}} & -8\frac{x\,y}{\%1^{3/2}} & -\frac{x\,\%2}{\%1^{3/2}} \\ -8\frac{x\,y}{\%1^{3/2}} & -8\frac{y^2}{\%1^{3/2}} + \frac{2}{\sqrt{\%1}} & -\frac{y\,\%2}{\%1^{3/2}} \\ 8\frac{\cosh(z)\sinh(z)\,x}{\%1^{3/2}} & 8\frac{\cosh(z)\sinh(z)\,y}{\%1^{3/2}} & \frac{\cosh(z)\sinh(z)\,\%2}{\%1^{3/2}} - 2\frac{\sinh(z)^2}{\sqrt{\%1}} - 2\frac{\cosh(z)^2}{\sqrt{\%1}} \end{bmatrix}$$

$$\%1 := 4\,x^2 + 4\,y^2 + 4\cosh(z)^2\sinh(z)^2$$

$$\%2 := 8\cosh(z)\sinh(z)^3 + 8\cosh(z)^3\sinh(z)$$

```
> DETJAC:=det(JACNN);
```

$$DETJAC := 0$$

```
> MEAN_CURVATURE:=factor(simplify(trace(JACNN)/2,trig));
```

$$MEAN\_CURVATURE := -\frac{(\cosh(z)-1)(\cosh(z)+1)(x^2+y^2-\cosh(z)^2)}{(x^2+y^2+\cosh(z)^4-\cosh(z)^2)^{3/2}}$$

Note that on the zero set of Phi (the surface) the Mean Curvature vanishes!!
Hence the catenoid is a Minimal surface.

```
> ADJOINTJACNN:=(adj(JACNN));
```

$$ADJOINTJACNN := \begin{bmatrix} -\frac{x^2\,\%2}{\%1^2} & -\frac{x\,y\,\%2}{\%1^2} & \frac{x\cosh(z)\sinh(z)\,\%2}{\%1^2} \\ -\frac{x\,y\,\%2}{\%1^2} & -\frac{y^2\,\%2}{\%1^2} & \frac{y\cosh(z)\sinh(z)\,\%2}{\%1^2} \\ -\frac{\sinh(z)\cosh(z)\,x}{\%1^2} & -\frac{\cosh(z)\sinh(z)\,y}{\%1^2} & \frac{\cosh(z)^2\sinh(z)^2}{\%1^2} \end{bmatrix}$$

$$\%1 := x^2+y^2+\cosh(z)^2\sinh(z)^2$$

$$\%2 := \sinh(z)^2+\cosh(z)^2$$

```
> GAUSS_CURVATURE:=factor(trace(ADJOINTJACNN));
```

$$GAUSS\_CURVATURE := -\frac{x^2\sinh(z)^2+x^2\cosh(z)^2+y^2\sinh(z)^2+y^2\cosh(z)^2-\cosh(z)^2\sinh(z)^2}{(x^2+y^2+\cosh(z)^2\sinh(z)^2)^2}$$

```
> VORTICITY:=evalm(curl(unitnormal,[x,y,z])):CURLx:=simplify(VORTICITY[1]);CURLy:=
  simplify(VORTICITY[2]);CURLz:=simplify(VORTICITY[3]);
```

$$CURLx := 2\,\frac{\cosh(z)^3\sinh(z)\,y}{(x^2+y^2+\cosh(z)^4-\cosh(z)^2)^{3/2}}$$

$$CURLy := -2\,\frac{x\cosh(z)^3\sinh(z)}{(x^2+y^2+\cosh(z)^4-\cosh(z)^2)^{3/2}}$$

$$CURLz := 0$$

```
> BRAND:=innerprod(JACNN,VORTICITY);
```

$$BRAND := \left[\frac{\cosh(z)\sinh(z)\,y\,(1+\sinh(z)^2+\cosh(z)^2)}{(x^2+y^2+\cosh(z)^2\sinh(z)^2)^2}, -\frac{x\cosh(z)\sinh(z)\,(1+\sinh(z)^2+\cosh(z)^2)}{(x^2+y^2+\cosh(z)^2\sinh(z)^2)^2}, 0\right]$$

```
> HELICITY:=simplify(innerprod(unitnormal,VORTICITY));
```

$$HELICITY := 0$$

```
>
> MAGNUS_FORCE:=(crossprod(unitnormal,VORTICITY));
```

$$MAGNUS\_FORCE := \left[2\,\frac{\cosh(z)\sinh(z)\left(-\frac{x\,\%1}{\%2^{3/2}}-8\,\frac{\cosh(z)\sinh(z)\,x}{\%2^{3/2}}\right)}{\sqrt{\%2}},\right.$$

$$-2\,\frac{\cosh(z)\sinh(z)\left(8\,\frac{\cosh(z)\sinh(z)\,y}{\%2^{3/2}}+\frac{y\,\%1}{\%2^{3/2}}\right)}{\sqrt{\%2}},$$

$$\left.2\,\frac{x\left(-\frac{x\,\%1}{\%2^{3/2}}-8\,\frac{\cosh(z)\sinh(z)\,x}{\%2^{3/2}}\right)}{\sqrt{\%2}}-2\,\frac{y\left(8\,\frac{\cosh(z)\sinh(z)\,y}{\%2^{3/2}}+\frac{y\,\%1}{\%2^{3/2}}\right)}{\sqrt{\%2}}\right]$$

$\%1 := 8\cosh(z)\sinh(z)^3 + 8\cosh(z)^3\sinh(z)$

$\%2 := 4x^2 + 4y^2 + 4\cosh(z)^2\sinh(z)^2$

```
> # Brandt invariants
> enstrophy:=simplify(innerprod(VORTICITY,VORTICITY),trig);
```

$$enstrophy := 4\cosh(z)^6(-y^2 - x^2 + y^2\cosh(z)^2 + x^2\cosh(z)^2) \Big/ (x^6 + 3x^4y^2 + 3x^4\cosh(z)^4 - 3x^4\cosh(z)^2 + 3x^2y^4 + 6x^2\cosh(z)^4y^2 - 6x^2\cosh(z)^2y^2 + 3\cosh(z)^8x^2 - 6\cosh(z)^6x^2 + 3\cosh(z)^4x^2 + y^6 + 3y^4\cosh(z)^4 - 3y^4\cosh(z)^2 + 3\cosh(z)^8y^2 - 6\cosh(z)^6y^2 + 3\cosh(z)^4y^2 + \cosh(z)^{12} - 3\cosh(z)^{10} + 3\cosh(z)^8 - \cosh(z)^6)$$

```
>
> stretch:=simplify(innerprod(BRAND,VORTICITY),trig);
```

$$stretch := 4\frac{\cosh(z)^6(-y^2 - x^2 + y^2\cosh(z)^2 + x^2\cosh(z)^2)}{(x^2 + y^2 + \cosh(z)^4 - \cosh(z)^2)^{7/2}}$$

```
> brand_invariant:=simplify(innerprod(BRAND,BRAND));
```

$$brand\_invariant := 4\cosh(z)^6(-y^2 - x^2 + y^2\cosh(z)^2 + x^2\cosh(z)^2) \Big/ (6\cosh(z)^{12} - 4\cosh(z)^{10} + 12x^4\cosh(z)^4y^2 + 6y^4\cosh(z)^4 - 4\cosh(z)^6y^2 + 12x^2y^4\cosh(z)^4 + 12x^2\cosh(z)^4y^2 + \cosh(z)^8 + x^8 + y^8 + 4x^6y^2 + 6x^4y^4 + 4x^2y^6 + 4\cosh(z)^{12}y^2 - 12\cosh(z)^{10}y^2 + 6y^4\cosh(z)^8 - 12y^4\cosh(z)^6 - 4\cosh(z)^{14} + \cosh(z)^{16} + 6x^4\cosh(z)^8 - 12x^2y^4\cosh(z)^2 - 12x^4\cosh(z)^6 + 4\cosh(z)^{12}x^2 - 12\cosh(z)^{10}x^2 + 12x^2\cosh(z)^8y^2 - 24x^2\cosh(z)^6y^2 + 4x^6\cosh(z)^4 - 4x^6\cosh(z)^2 - 4\cosh(z)^6x^2 + 12\cosh(z)^8y^2 + 6x^4\cosh(z)^4 + 12\cosh(z)^8x^2 + 4y^6\cosh(z)^4 - 4y^6\cosh(z)^2 - 12x^4\cosh(z)^2y^2)$$

```
> factor(stretch/enstrophy);factor(brand_invariant/enstrophy);
```

$$\frac{1}{\sqrt{x^2 + y^2 + \cosh(z)^4 - \cosh(z)^2}}$$

$$\frac{1}{x^2 + y^2 + \cosh(z)^4 - \cosh(z)^2}$$

These ratios look like curvatures of some type? what are they?

```
>
> restart;
```

Third Example

A surface generated by a nonintegrable 1-form

```
>  with (linalg): with(plots):setup[x,y,z]:
> DR:=([dx,dy,dz]);
Warning, new definition for norm
Warning, new definition for trace
```

$$DR := [dx, dy, dz]$$

```
> defform(x=0,y=0,qz=0,v=0,u=0,vz=0,px=0,py=0,pz=0,t=0,s=0,a=const,b=const,c=const
  ,k=const,mu=const,m=const);
```

defform($x = 0$, $y = 0$, $qz = 0$, $v = 0$, $u = 0$, $vz = 0$, $px = 0$, $py = 0$, $pz = 0$, $t = 0$, $s = 0$, $a = const$, $b = const$, $c = const$, $k = const$, $\mu = const$, $m = const$)

DEfine the components of a non-integrable 3d 1-form (the Heisenberg group)

```
> NN:=eval(([-y,x,z]));
> magn:=(a*NN[1]^p+b*NN[2]^p+c*NN[3]^p)^(n/p);
```

$$NN := [-y, x, z]$$

$$magn := (a(-y)^p + b\,x^p + c\,z^p)^{\left(\frac{n}{p}\right)}$$

(Experiment with other values for the Holder exponents, p and n, and other signatures.)
The example below is for the Gauss Map.

```
> p:=2;n:=1;
```

```
> a:=1;b:=1;c:=1;
```

$$p := 2$$
$$n := 1$$
$$a := 1$$
$$b := 1$$
$$c := 1$$

```
> unitnormal:=evalm(NN/magn);
```

$$unitnormal := \left[-\frac{y}{\sqrt{y^2+x^2+z^2}}, \frac{x}{\sqrt{y^2+x^2+z^2}}, \frac{z}{\sqrt{y^2+x^2+z^2}}\right]$$

Compute the Jacobian matrix of the unit normal field:

```
> JACNN:=jacobian(unitnormal,[x,y,z]);
```

$$JACNN := \begin{bmatrix} \frac{y\,x}{(y^2+x^2+z^2)^{3/2}} & \frac{y^2}{(y^2+x^2+z^2)^{3/2}} - \frac{1}{\sqrt{y^2+x^2+z^2}} & \frac{y\,z}{(y^2+x^2+z^2)^{3/2}} \\ -\frac{x^2}{(y^2+x^2+z^2)^{3/2}} + \frac{1}{\sqrt{y^2+x^2+z^2}} & -\frac{y\,x}{(y^2+x^2+z^2)^{3/2}} & -\frac{x\,z}{(y^2+x^2+z^2)^{3/2}} \\ -\frac{x\,z}{(y^2+x^2+z^2)^{3/2}} & -\frac{y\,z}{(y^2+x^2+z^2)^{3/2}} & -\frac{z^2}{(y^2+x^2+z^2)^{3/2}} + \frac{1}{\sqrt{y^2+x^2+z^2}} \end{bmatrix}$$

```
> DETJAC:=det(JACNN);
```

$$DETJAC := 0$$

```
> MEAN_CURVATURE:=factor(simplify(trace(JACNN)/2));
```

$$MEAN\_CURVATURE := \frac{1}{2}\frac{y^2+x^2}{(y^2+x^2+z^2)^{3/2}}$$

```
> ADJOINTJACNN:=(adj(JACNN));
```

$$ADJOINTJACNN := \begin{bmatrix} -\frac{y\,x}{(y^2+x^2+z^2)^2} & \frac{x^2}{(y^2+x^2+z^2)^2} & \frac{x\,z}{(y^2+x^2+z^2)^2} \\ -\frac{y^2}{(y^2+x^2+z^2)^2} & \frac{y\,x}{(y^2+x^2+z^2)^2} & \frac{z\,y}{(y^2+x^2+z^2)^2} \\ -\frac{z\,y}{(y^2+x^2+z^2)^2} & \frac{x\,z}{(y^2+x^2+z^2)^2} & \frac{z^2}{(y^2+x^2+z^2)^2} \end{bmatrix}$$

$$ADJOINTJACNN := \begin{bmatrix} -\frac{y\,x}{(y^2+x^2+z^2)^2} & \frac{x^2}{(y^2+x^2+z^2)^2} & \frac{z\,x}{(y^2+x^2+z^2)^2} \\ -\frac{y^2}{(y^2+x^2+z^2)^2} & \frac{y\,x}{(y^2+x^2+z^2)^2} & \frac{y\,z}{(y^2+x^2+z^2)^2} \\ -\frac{y\,z}{(y^2+x^2+z^2)^2} & \frac{z\,x}{(y^2+x^2+z^2)^2} & \frac{z^2}{(y^2+x^2+z^2)^2} \end{bmatrix}$$

```
> GAUSS_CURVATURE:=simplify(trace(ADJOINTJACNN));
```

$$GAUSS\_CURVATURE := \frac{z^2}{(y^2+x^2+z^2)^2}$$

```
> VORTICITY:=(evalm(curl(unitnormal,[x,y,z])));
```

$$VORTICITY := \left[-\frac{y\,z}{(y^2+x^2+z^2)^{3/2}} + \frac{x\,z}{(y^2+x^2+z^2)^{3/2}}, \frac{y\,z}{(y^2+x^2+z^2)^{3/2}} + \frac{x\,z}{(y^2+x^2+z^2)^{3/2}}, -\frac{x^2}{(y^2+x^2+z^2)^{3/2}} + \frac{2}{\sqrt{y^2+x^2+z^2}} - \frac{y^2}{(y^2+x^2+z^2)^{3/2}} \right]$$

```
> BRAND:=innerprod(JACNN,VORTICITY);
```

$$BRAND := \left[\frac{z\,(y-x)}{(y^2+x^2+z^2)^2}, -\frac{z\,(y+x)}{(y^2+x^2+z^2)^2}, \frac{y^2+x^2}{(y^2+x^2+z^2)^2} \right]$$

```
> HELICITY:=simplify(innerprod(unitnormal,VORTICITY));
```

$$HELICITY := 2\,\frac{z}{y^2+x^2+z^2}$$

```
>
> MAGNUS_FORCE:=(crossprod(unitnormal,VORTICITY));
```

$$MAGNUS\_FORCE := \left[\frac{x\left(-\frac{x^2}{(y^2+x^2+z^2)^{3/2}} + \frac{2}{\sqrt{y^2+x^2+z^2}} - \frac{y^2}{(y^2+x^2+z^2)^{3/2}}\right)}{\sqrt{y^2+x^2+z^2}} - \frac{z\left(\frac{y\,z}{(y^2+x^2+z^2)^{3/2}} + \frac{x\,z}{(y^2+x^2+z^2)^{3/2}}\right)}{\sqrt{y^2+x^2+z^2}}, \frac{z\left(-\frac{y\,z}{(y^2+x^2+z^2)^{3/2}} + \frac{x\,z}{(y^2+x^2+z^2)^{3/2}}\right)}{\sqrt{y^2+x^2+z^2}} + \frac{y\left(-\frac{x^2}{(y^2+x^2+z^2)^{3/2}} + \frac{2}{\sqrt{y^2+x^2+z^2}} - \frac{y^2}{(y^2+x^2+z^2)^{3/2}}\right)}{\sqrt{y^2+x^2+z^2}}, -\frac{y\left(\frac{y\,z}{(y^2+x^2+z^2)^{3/2}} + \frac{x\,z}{(y^2+x^2+z^2)^{3/2}}\right)}{\sqrt{y^2+x^2+z^2}} - \frac{x\left(-\frac{y\,z}{(y^2+x^2+z^2)^{3/2}} + \frac{x\,z}{(y^2+x^2+z^2)^{3/2}}\right)}{\sqrt{y^2+x^2+z^2}} \right]$$

```
> # Brandt invariants
> enstrophy:=simplify(innerprod(VORTICITY,VORTICITY));
```

$$enstrophy := \frac{6\,y^2\,z^2 + 6\,x^2\,z^2 + x^4 + 2\,x^2\,y^2 + y^4 + 4\,z^4}{(y^2+x^2+z^2)^3}$$

```
>
> stretch:=simplify(innerprod(BRAND,VORTICITY));
```

$$stretch := \frac{2\,x^2\,y^2 + y^4 + x^4}{(y^2+x^2+z^2)^{7/2}}$$

```
> brand_invariant:=simplify(innerprod(BRAND,BRAND));
```

$$brand\_invariant := \frac{2\,y^2\,z^2 + 2\,x^2\,z^2 + 2\,x^2\,y^2 + y^4 + x^4}{(y^2+x^2+z^2)^4}$$

```
> factor(stretch/enstrophy);factor(brand_invariant/enstrophy);
```

$$\frac{(y^2+x^2)^2}{\sqrt{y^2+x^2+z^2}\,(6\,y^2\,z^2 + 6\,x^2\,z^2 + x^4 + 2\,x^2\,y^2 + y^4 + 4\,z^4)}$$

$$\frac{(y^2+x^2)\,(y^2+x^2+2\,z^2)}{(y^2+x^2+z^2)\,(6\,y^2\,z^2 + 6\,x^2\,z^2 + x^4 + 2\,x^2\,y^2 + y^4 + 4\,z^4)}$$

Again note that the ratios are a reciprocal length and a reciprocal length squared - similar to some form of curvature measures. The denominators are

```
> restart:
```

Fourth EXAMPLE

The Helicoid

```
>  with (linalg): with(plots):setup[x,y,z]:
> DR:=([dx,dy,dz]);
Warning, new definition for norm
Warning, new definition for trace
```

$$DR := [dx, dy, dz]$$

```
> defform(x=0,y=0,qz=0,v=0,u=0,vz=0,px=0,py=0,pz=0,t=0,s=0,a=const,b=const,c=const
,k=const,mu=const,m=const);
```

$$\mathrm{defform}(x = 0, y = 0, qz = 0, v = 0, u = 0, vz = 0, px = 0, py = 0, pz = 0, t = 0, s = 0, a = const, b = const, c = const, k = const, \mu = const, m = const)$$

Define an implicit surface function:

```
> Phi:=(cos(z)*y-x*sin(z));
```

$$\Phi := \cos(z)\, y - x \sin(z)$$

```
>
> NN:=eval(([diff(Phi,x),diff(Phi,y),diff(Phi,z)]));
> magn:=(a*NN[1]^p+b*NN[2]^p+c*NN[3]^p)^(n/p);
```

$$NN := [-\sin(z), \cos(z), -\sin(z)\, y - x \cos(z)]$$

$$magn := (a\,(-\sin(z))^p + b \cos(z)^p + c\,(-\sin(z)\, y - x \cos(z))^p)^{\left(\frac{n}{p}\right)}$$

```
> p:=2;n:=1;
> a:=1;b:=1;c:=1;
```

$$p := 2$$
$$n := 1$$
$$a := 1$$
$$b := 1$$
$$c := 1$$

```
> unitnormal:=evalm(NN/magn);
```

$$unitnormal := \left[-\frac{\sin(z)}{\sqrt{\sin(z)^2 + \cos(z)^2 + \%1^2}}, \frac{\cos(z)}{\sqrt{\sin(z)^2 + \cos(z)^2 + \%1^2}}, \frac{\%1}{\sqrt{\sin(z)^2 + \cos(z)^2 + \%1^2}}\right]$$
$$\%1 := -\sin(z)\, y - x \cos(z)$$

Compute the Jacobian matrix of the unit normal field:

```
> JACNN:=jacobian(unitnormal,[x,y,z]);
```

$$JACNN := \begin{bmatrix} -\frac{\sin(z)\,\%1 \cos(z)}{\%2^{3/2}} & -\frac{\sin(z)^2\,\%1}{\%2^{3/2}} & \frac{\sin(z)\,\%1\,\%3}{\%2^{3/2}} - \frac{\cos(z)}{\sqrt{\%2}} \\ \frac{\cos(z)^2\,\%1}{\%2^{3/2}} & \frac{\sin(z)\,\%1 \cos(z)}{\%2^{3/2}} & -\frac{\cos(z)\,\%1\,\%3}{\%2^{3/2}} - \frac{\sin(z)}{\sqrt{\%2}} \\ \frac{\%1^2 \cos(z)}{\%2^{3/2}} - \frac{\cos(z)}{\sqrt{\%2}} & \frac{\%1^2 \sin(z)}{\%2^{3/2}} - \frac{\sin(z)}{\sqrt{\%2}} & -\frac{\%1^2\,\%3}{\%2^{3/2}} + \frac{\%3}{\sqrt{\%2}} \end{bmatrix}$$
$$\%1 := -\sin(z)\, y - x \cos(z)$$
$$\%2 := \sin(z)^2 + \cos(z)^2 + \%1^2$$
$$\%3 := -\cos(z)\, y + x \sin(z)$$

```
> DETJAC:=det(JACNN);
```

$$DETJAC := 0$$

```
> MEAN_CURVATURE:=factor(simplify(trace(JACNN)/2,trig));
```

$$MEAN\_CURVATURE := \frac{1}{2}\frac{-\cos(z)\,y + x\sin(z)}{(1 + y^2 - y^2\cos(z)^2 + 2\sin(z)\,y\,x\cos(z) + x^2\cos(z)^2)^{3/2}}$$

```
>
```

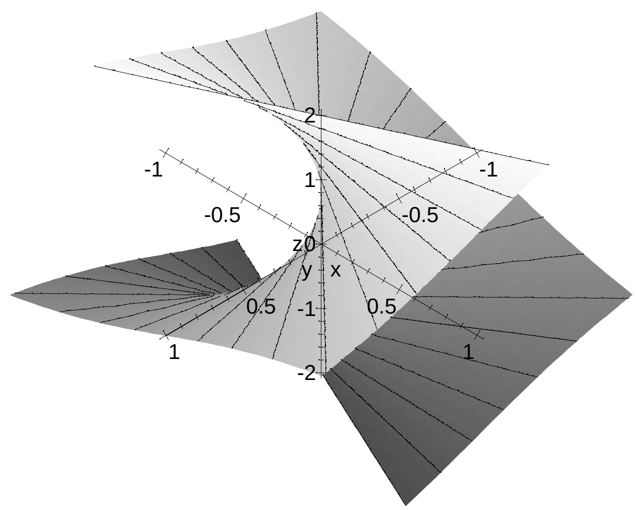
Note that on the zero set of Phi (the surface) the Mean Curvature vanishes!!
Hence the catenoid is a Minimal surface.

```
> ADJOINTJACNN:=(adj(JACNN)):
> GAUSS_CURVATURE:=factor((trace(ADJOINTJACNN)));
```

$$GAUSS\_CURVATURE := -\frac{(\sin(z)^2 + \cos(z)^2)^2}{(\sin(z)^2 + \cos(z)^2 + \sin(z)^2 y^2 + 2\sin(z)\,y\,x\cos(z) + x^2\cos(z)^2)^2}$$

```
> implicitplot3d(Phi,x=-1..1,y=-1..+1,z=-2..2,numpoints=12000,style=PATCHCONTOUR,a
xes=NORMAL,shading=ZGREYSCALE,title=`Ruled Helicoid`);
```

Ruled Helicoid




```
>
> VORTICITY:=evalm(curl(unitnormal,[x,y,z])):CURLx:=simplify(VORTICITY[1],trig);CU
RLy:=simplify(VORTICITY[2],trig);CURLz:=simplify(VORTICITY[3],trig);
```

$$CURLx := \frac{y\,(\sin(z)\,y + x\cos(z))}{(1 + y^2 - y^2\cos(z)^2 + 2\sin(z)\,y\,x\cos(z) + x^2\cos(z)^2)^{3/2}}$$

$$CURLy := -\frac{x\,(\sin(z)\,y + x\cos(z))}{(1 + y^2 - y^2\cos(z)^2 + 2\sin(z)\,y\,x\cos(z) + x^2\cos(z)^2)^{3/2}}$$

$$CURLz := -\frac{\sin(z)\,y + x\cos(z)}{(1 + y^2 - y^2\cos(z)^2 + 2\sin(z)\,y\,x\cos(z) + x^2\cos(z)^2)^{3/2}}$$

```
> BRAND:=innerprod(JACNN,VORTICITY);
```

$$BRAND := \left[\frac{(\sin(z)\,y + x\cos(z))\cos(z)\,(\sin(z)^2+\cos(z)^2)}{(\sin(z)^2+\cos(z)^2+\sin(z)^2 y^2 + 2\sin(z)\,y\,x\cos(z) + x^2\cos(z)^2)^2}, \frac{\sin(z)\,(\sin(z)\,y + x\cos(z))\,(\sin(z)^2+\cos(z)^2)}{(\sin(z)^2+\cos(z)^2+\sin(z)^2 y^2 + 2\sin(z)\,y\,x\cos(z) + x^2\cos(z)^2)^2}, 0 \right]$$

```
> HELICITY:=simplify(innerprod(unitnormal,VORTICITY));
```

$$HELICITY := 0$$

It is remarkable that the "Vorticity" of the Helicoid is not zero, but the Helicity is zero. The system has apparent Helical structure but it does not have Helicity!! The Surface is an implicit surface, and as such it is integrable. Then unit norm has zero Helicity. Helical appearance is necessary but not sufficient for Helicity (topological torsion) to be non-zero. One would say

```
> MAGNUS_FORCE:=(crossprod(unitnormal,VORTICITY));
```

$$MAGNUS\_FORCE := \left[\frac{\cos(z)\left(\frac{\cos(z)^2\,\%2}{\%3^{3/2}} + \frac{\sin(z)^2\,\%2}{\%3^{3/2}}\right)}{\sqrt{\%3}} - \frac{\%2\left(\frac{\sin(z)\,\%2\,\%1}{\%3^{3/2}} - \frac{\%2^2\cos(z)}{\%3^{3/2}}\right)}{\sqrt{\%3}}, \frac{\%2\left(\frac{\%2^2\sin(z)}{\%3^{3/2}} + \frac{\cos(z)\,\%2\,\%1}{\%3^{3/2}}\right)}{\sqrt{\%3}} + \frac{\sin(z)\left(\frac{\cos(z)^2\,\%2}{\%3^{3/2}} + \frac{\sin(z)^2\,\%2}{\%3^{3/2}}\right)}{\sqrt{\%3}}, -\frac{\sin(z)\left(\frac{\sin(z)\,\%2\,\%1}{\%3^{3/2}} - \frac{\%2^2\cos(z)}{\%3^{3/2}}\right)}{\sqrt{\%3}} - \frac{\cos(z)\left(\frac{\%2^2\sin(z)}{\%3^{3/2}} + \frac{\cos(z)\,\%2\,\%1}{\%3^{3/2}}\right)}{\sqrt{\%3}} \right]$$

$$\%1 := -\cos(z)\,y + x\sin(z)$$

$$\%2 := -\sin(z)\,y - x\cos(z)$$

$$\%3 := \sin(z)^2 + \cos(z)^2 + \%2^2$$

```
> # Brandt invariants
> enstrophy:=subs((sin(z))^2+(cos(z))^2=1,factor(innerprod(VORTICITY,VORTICITY)));
```

$$enstrophy := \frac{(\sin(z)\,y + x\cos(z))^2\,(y^2+x^2+1)}{(\sin(z)^2+\cos(z)^2+\sin(z)^2 y^2 + 2\sin(z)\,y\,x\cos(z) + x^2\cos(z)^2)^3}$$

```
>
> stretch:=subs((sin(z))^2+(cos(z))^2=1,factor(innerprod(BRAND,VORTICITY)));
```

$$stretch := -\frac{(\sin(z)\,y + x\cos(z))^2\,(-\cos(z)\,y + x\sin(z))}{(\sin(z)^2+\cos(z)^2+\sin(z)^2 y^2 + 2\sin(z)\,y\,x\cos(z) + x^2\cos(z)^2)^{7/2}}$$

```
> brand_invariant:=subs((sin(z))^2+(cos(z))^2=1,factor(innerprod(BRAND,BRAND)));
```

$$brand\_invariant := \frac{(\sin(z)\,y + x\cos(z))^2}{(\sin(z)^2+\cos(z)^2+\sin(z)^2 y^2 + 2\sin(z)\,y\,x\cos(z) + x^2\cos(z)^2)^4}$$

```
> stretch/enstrophy;brand_invariant/enstrophy;
```

$$-\frac{-\cos(z)\,y + x\sin(z)}{\sqrt{\sin(z)^2+\cos(z)^2+\sin(z)^2 y^2 + 2\sin(z)\,y\,x\cos(z) + x^2\cos(z)^2}\,(y^2+x^2+1)}$$

$$\frac{1}{(\sin(z)^2+\cos(z)^2+\sin(z)^2 y^2 + 2\sin(z)\,y\,x\cos(z) + x^2\cos(z)^2)\,(y^2+x^2+1)}$$

Notice that when the surface is a minimal surface the "vortex stretching" term vanishes!!
The first and third Brand invarinats are proportional.
The surprise of this exercise is that the apparent helical structure does not have helicity.!!!
The surface does NOT have topological Torsion A^dA=0 !!!

```
> restart;
```

Example 5

The Quartic Norm

```
>
> with (linalg):
Warning, new definition for norm
Warning, new definition for trace
```

The sphere as an implicit surface phi(x,y,z) mapped to zero. The normal field will be constructed as the gradient and then scaled by the Holder Quartic norm. Riemann in his lectures on geometry mentioned the possibility of a quartic definition of distance, but little application has been made. The key idea is that the scaling or norm function should be homogeneous of degree 1. Chern in his Houston lectures often mentioned that the quartic norm ought to be studied.

As will be developed below, the quadratic norm eliminates torsion of the second kind (Big Omega), but the torsion coefficients of the second kind do not dissappear for the quartic norm. The Repere Mobile method must be used to determine these effects.

What is the physical significance of the quartic norm? That is an open question, but recall that in electromagnetic theory, the Fresnel ellipsoids representing the wave and ray surfaces of electromagnetic discontinuity propagation involve fourth order Kummer surfaces. (Quartic forms).

```
> a:=1;b:=1;c:=1;Phi:=x^(2)+y^(2)+z^(2)-a;
```

$$a := 1$$

$$b := 1$$

$$c := 1$$

$$\Phi := x^2 + y^2 + z^2 - 1$$

Construct the gradient of the surface function to give a 4 dimensional "surface gradient" vector in the space of variables {x,y,z,u}.

```
> Dr:=([d(x),d(y),d(z),d(u)]):ZZ:=grad(Phi,[x,y,z]);
```

$$ZZ := [2\,x, 2\,y, 2\,z]$$

```
> NN:=grad(Phi,[x,y,z,u]);magn:=(NN[1]^4+NN[2]^4+NN[3]^4)^(1/4);ACTION:=wcollect(i
  nnerprod(NN,Dr))-magn*d(u);FIELD:=d(ACTION);unitnorm:=evalm([NN[1]/magn,NN[2]/ma
  gn,NN[3]/magn]);
```

$$NN := [2\,x, 2\,y, 2\,z, 0]$$

$$magn := (16\,x^4 + 16\,y^4 + 16\,z^4)^{1/4}$$

$$ACTION := \mathrm{wcollect}(2\,x\,\mathrm{d}(x) + 2\,y\,\mathrm{d}(y) + 2\,z\,\mathrm{d}(z)) - (16\,x^4 + 16\,y^4 + 16\,z^4)^{1/4}\,\mathrm{d}(u)$$

$$FIELD := \mathrm{d}(\mathrm{wcollect}(2\,x\,\mathrm{d}(x) + 2\,y\,\mathrm{d}(y) + 2\,z\,\mathrm{d}(z)) - (16\,x^4 + 16\,y^4 + 16\,z^4)^{1/4}\,\mathrm{d}(u))$$

$$unitnorm := \left[2\frac{x}{(16\,x^4 + 16\,y^4 + 16\,z^4)^{1/4}}, 2\frac{y}{(16\,x^4 + 16\,y^4 + 16\,z^4)^{1/4}}, 2\frac{z}{(16\,x^4 + 16\,y^4 + 16\,z^4)^{1/4}} \right]$$

Use the 3D euclidean norm to form the unit "normal" field. Consider only the first three components of the "normal" field and construct its Jacobian. (Note that this is the same as the Gauss map where the normal field is divided by the 3D euclidean norm). Form the Jacobian with respect to [x,y,z] of the first three components creating a 3 x 3 matrix. This matrix (JACNN) is equivalent to the Shape matrix which will be computed in another way below: The matrix has three similarity invariants. Its determinant is always zero (for the norm chosen). Its trace divided by 2 is the Mean curvature of the implicit surface. The determinant of the adjoint matrix is the Gauss curvature of the implicit surface.

```
> JACNN:=evalm(jacobian(ZZ/magn,[x,y,z]));DETJACNN:=simplify(det(JACNN),trig);JMEAN:=factor(simplify(trace(JACNN)/2,trig));JGAUSS:=factor(trace(adjoint(JACNN)));
```

$$JACNN := \begin{bmatrix} 2\frac{1}{\%1^{1/4}} - 32\frac{x^4}{\%1^{5/4}} & -32\frac{x\,y^3}{\%1^{5/4}} & -32\frac{x\,z^3}{\%1^{5/4}} \\ -32\frac{y\,x^3}{\%1^{5/4}} & \frac{2}{\%1^{1/4}} - 32\frac{y^4}{\%1^{5/4}} & -32\frac{y\,z^3}{\%1^{5/4}} \\ -32\frac{z\,x^3}{\%1^{5/4}} & -32\frac{z\,y^3}{\%1^{5/4}} & \frac{2}{\%1^{1/4}} - 32\frac{z^4}{\%1^{5/4}} \end{bmatrix}$$

$$\%1 := 16\,x^4 + 16\,y^4 + 16\,z^4$$

$$DETJACNN := 0$$

$$JMEAN := 32\,\frac{x^4 + y^4 + z^4}{(16\,x^4 + 16\,y^4 + 16\,z^4)^{5/4}}$$

$$JGAUSS := \frac{1}{\sqrt{x^4 + y^4 + z^4}}$$

The mean and Gauss curvatures are of the proper dimension even though the norm is a quartic rather that a quadratic.

```
> VORTICITY:=evalm(curl(unitnorm,[x,y,z])):CURLx:=factor(VORTICITY[1]);CURLy:=factor(VORTICITY[2]);CURLz:=factor(VORTICITY[3]);
```

$$CURLx := 32\,\frac{z\,y\,(z-y)\,(z+y)}{(16\,x^4 + 16\,y^4 + 16\,z^4)^{5/4}}$$

$$CURLy := -32\,\frac{z\,x\,(z-x)\,(z+x)}{(16\,x^4 + 16\,y^4 + 16\,z^4)^{5/4}}$$

$$CURLz := 32\,\frac{y\,x\,(y-x)\,(y+x)}{(16\,x^4 + 16\,y^4 + 16\,z^4)^{5/4}}$$

```
> BRAND:=innerprod(JACNN,VORTICITY);
```

$$BRAND := \left[\frac{z\,y\,(-y^2+z^2)}{(x^4+y^4+z^4)^{3/2}},\ -\frac{z\,x\,(z^2-x^2)}{(x^4+y^4+z^4)^{3/2}},\ \frac{y\,x\,(-x^2+y^2)}{(x^4+y^4+z^4)^{3/2}}\right]$$

```
> HELICITY:=simplify(innerprod(unitnorm,VORTICITY));
```

$$HELICITY := 0$$

```
> MAGNUS_FORCE:=(crossprod(unitnorm,VORTICITY));
```

$$MAGNUS\_FORCE := \left[2\,\frac{y\left(-32\frac{y\,x^3}{\%1^{5/4}} + 32\frac{x\,y^3}{\%1^{5/4}}\right)}{\%1^{1/4}} - 2\,\frac{z\left(-32\frac{x\,z^3}{\%1^{5/4}} + 32\frac{z\,x^3}{\%1^{5/4}}\right)}{\%1^{1/4}},\right.$$

$$2\,\frac{z\left(-32\frac{z\,y^3}{\%1^{5/4}} + 32\frac{y\,z^3}{\%1^{5/4}}\right)}{\%1^{1/4}} - 2\,\frac{x\left(-32\frac{y\,x^3}{\%1^{5/4}} + 32\frac{x\,y^3}{\%1^{5/4}}\right)}{\%1^{1/4}},$$

$$\left.2\,\frac{x\left(-32\frac{x\,z^3}{\%1^{5/4}} + 32\frac{z\,x^3}{\%1^{5/4}}\right)}{\%1^{1/4}} - 2\,\frac{y\left(-32\frac{z\,y^3}{\%1^{5/4}} + 32\frac{y\,z^3}{\%1^{5/4}}\right)}{\%1^{1/4}}\right]$$

$$\%1 := 16\,x^4 + 16\,y^4 + 16\,z^4$$

```
> # Brandt invariants
```

```
> enstrophy:=subs((sin(z))^2+(cos(z))^2=1,factor(innerprod(VORTICITY,VORTICITY)));
```

$$enstrophy := \frac{y^6 z^2 - 2 y^4 z^4 + y^2 z^6 + x^2 z^6 - 2 z^4 x^4 + x^6 z^2 + x^6 y^2 - 2 x^4 y^4 + x^2 y^6}{(x^4 + y^4 + z^4)^{5/2}}$$

```
>
> stretch:=subs((sin(z))^2+(cos(z))^2=1,factor(innerprod(BRAND,VORTICITY)));
```

$$stretch := 32 \frac{y^6 z^2 - 2 y^4 z^4 + y^2 z^6 + x^2 z^6 - 2 z^4 x^4 + x^6 z^2 + x^6 y^2 - 2 x^4 y^4 + x^2 y^6}{(x^4 + y^4 + z^4)^{3/2} (16 x^4 + 16 y^4 + 16 z^4)^{5/4}}$$

```
> brand_invariant:=subs((sin(z))^2+(cos(z))^2=1,factor(innerprod(BRAND,BRAND)));
```

$$brand\_invariant := \frac{y^6 z^2 - 2 y^4 z^4 + y^2 z^6 + x^2 z^6 - 2 z^4 x^4 + x^6 z^2 + x^6 y^2 - 2 x^4 y^4 + x^2 y^6}{(x^4 + y^4 + z^4)^3}$$

```
> stretch/enstrophy;brand_invariant/enstrophy;
```

$$32 \frac{x^4 + y^4 + z^4}{(16 x^4 + 16 y^4 + 16 z^4)^{5/4}}$$

$$\frac{1}{\sqrt{x^4 + y^4 + z^4}}$$

```
>
```

The second and third Brand invariants have a common factor equal to the enstrophy times the mean and the Gauss curvatures!

This result is remarkable in that when the quadratic scaling is used all of the Brand invariants vanish.