Topological Torsion, Negative Pressure and the Internal Energy of a Non-equilibrium Fluid in 4D

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Abstract The thermodynamic Internal Energy of a closed, non-equilibrium, rotating fluid or gas can have a classical component related to potential energy and a non-classical component related to the spiral helicity density, $\mathbf{v} \circ curl\ \mathbf{v}$, of the gas dynamics. The existence of non-zero helicity density is a topological property (of topological, not affine, torsion) not found in isolated equilibrium thermodynamic systems. For a non-equilibrium gas, whose molecules are stars, the spiral helicity density could play the role of negative pressure, thereby giving a palatable explanation for the observed rotational velocities of spiral galaxies.

Introduction

Consider a non-equilibrium, closed, thermodynamic system (like a fluid or a gas) that can be encoded in terms of a 1-form of Action, A, on a 4-dimensional domain of space-time [2]. Such a system must be of Pfaff topological dimension 3, which implies that the rank of the 4x4 antisymmetric matrix, $[\mathbb{F}]$, as constructed from the derived 2-form, $F = dA$, is not maximal. Therefore, the degenerate antisymmetric matrix, $[\mathbb{F}]$, has two 4-component real vector eigendirection fields, $\{|\mathbf{v}1\rangle, |\mathbf{v}2\rangle\}$, both with eigenvalue zero. In addition, the matrix, $[\mathbb{F}]$, has two Spinor eigendirection fields, $\{|\boldsymbol{\sigma}^+\rangle, |\boldsymbol{\sigma}^-\rangle\}$, with pure imaginary, conjugate, eigenvalues, $\pm\sqrt{-Y_G}$. The term, Y_G, is related to the Gaussian (quadratic) curvature of the implicit hypersurface generated by the Characteristic Polynomial of $[\mathbb{F}]$. Note that the Mean curvature of the Hypersurface is zero.

Any thermodynamic process, $|V\rangle$, acting on the the thermodynamic system, A, can be expanded in terms of the eigendirection fields of $[\mathbb{F}]$,

$$|V\rangle = \alpha\,|\mathbf{v}1\rangle + \beta\,|\mathbf{v}2\rangle + \lambda_1^+\,|\boldsymbol{\sigma}_1^+\rangle + \lambda_1^-\,|\boldsymbol{\sigma}_1^-\rangle \tag{1}$$

From a thermodynamic point of view, based on Cartan's Magic Formula [1], the definitions of the Work, W, and Internal Energy, U, are given by the expressions,

$$\begin{aligned} W &= i(V)dA, \quad (2)\\ U &= i(V)A, \quad (3)\end{aligned}$$

such that the Lie differential expresses the cohomological properties of the first law of thermodynamics:

$$L(V)A = i(V)dA + d\{i(V)A)\} = W \;\; + dU = Q. \tag{4}$$

In this formula, Q is the heat 1-form.

Only the Spinor components of the process field, $|V\rangle$, give rise to non-zero contributions to the Work 1-form, W. However, both the Vector and Spinor components give contributions to the internal energy, U.

$$\begin{aligned} U &= i(V)A = i(V)\{A_x dx + A_y dy + A_z dz + A_t dt\} = i(V)\{\mathbf{A}_k dx^k\}, \quad (5)\\ &= +\alpha\,\langle A_k|\circ|\mathbf{v}1\rangle + \beta\,\langle A_k|\circ|\mathbf{v}2\rangle + \lambda_1^+\,\langle A_k|\circ|\boldsymbol{\sigma}_1^+\rangle + \lambda_1^-\,\langle A_k|\circ|\boldsymbol{\sigma}_1^-\rangle. \quad (6)\end{aligned}$$

Consider the case where

$$[\mathbb{F}] = \begin{bmatrix} 0 & b_3 & -b_2 & 0 \\ -b3 & 0 & b_1 & 0 \\ b_2 & -b_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{7}$$

This simplification implies that

$$\begin{aligned} \mathbf{b} &= curl\ \mathbf{A}, \text{ the fluid vorticity,} \quad (8)\\ \mathbf{e} &= -\partial\mathbf{A}/\partial t - grad\ \phi \Rightarrow 0, \quad (9)\\ \phi &= -A_t. \quad (10)\end{aligned}$$

and

$$|\mathbf{v}1\rangle = \left|\begin{matrix} 0 \\ 0 \\ 0 \\ 1 \end{matrix}\right\rangle, |\mathbf{v}2\rangle = \left|\begin{matrix} b_1 \\ b_2 \\ b_3 \\ 0 \end{matrix}\right\rangle = |\mathbf{b}\rangle. \tag{11}$$

The simplification is equivalent to the state of a plasma for which the Lorentz force law is zero, and to the state of a fluid where the velocity field is in the direction of the vorticity (typical of a Contact structure).

The internal energy (ignoring the Spinor terms) becomes,

$$U = \alpha \langle A_t | \circ | \mathbf{v}1 \rangle + \beta \langle \mathbf{A} | \circ | \mathbf{b} \rangle ..., \tag{12}$$

$$= -\alpha\phi + \beta(\mathbf{A} \circ curl\ \mathbf{A}) + ... \tag{13}$$

It is apparent that the Internal Energy, U, of the non-equilibrium fluid has a classical component related to the potential energy, ϕ, and a non-classical dynamical component related to the spiral helicity density (topological torsion), $(\mathbf{A} \circ curl\ \mathbf{A})$. If the fluid is in an isolated equilibrium state, then the Pfaff topological dimension of the Action 1-form is 2 or less, and it follows that the topological torsion 3-form, $A\hat{}dA$, vanishes. The 3-form component (helicity density) yields a non-zero contribution to the internal energy only in non-equilibrium systems.

Cosmological Applications

Now consider a closed non-equilibrium thermodynamic gas, whose molecules are stars, executing spiral rotational motion. The remarkable conclusion is that the thermodynamic internal energy of such a non-equilibrium, closed, system apparently has a component related to its spiral helicity density, as well as to its potential energy. The conjecture is that this non-classical helicity component of internal energy, a consequence of topological torsion (not affine torsion) and non-equilibrium thermodynamics, could be related to a palatable explanation of negative pressure effects and the non-equilibrium dynamics of the spiral galaxies. These are topological ideas independent from a choice of metric and/or connections.

References

[1] Marsden, J.E. and Riatu, T. S. (1994) "Introduction to Mechanics and Symmetry", Springer-Verlag, 122.

[2] Kiehn, R. M. (2004) Non-Equilibrium Thermodynamics, "Non-Equilibrium Systems and Irreversible Processes Vol 1", see (http://www.lulu.com/kiehn).