INTRINSIC HYDRODYNAMICS WITH APPLICATIONS TO SPACE-TIME FLUIDS

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Abstract

Cartan's methods of exterior differential forms are utilized to derive conformal and absolute conservation equations for various field quantities in relativistic hydromechanics In particular, Fridman's results for the conservation of vector lines are extended to space time situations. The invariance concepts of Hclmholtz and Bernoulli are formulated for arbitrary frames of reference, and the separate conscrvation concepts of total mass and total number in space time are distinguished and associated with the concept of turbulence.

1 INTRODUCTION

Helmholtz, in his study of vortices, established two fundamental theorems on the conservation of vortex "lines" and the conservation of vortex tubes, These notions have been applied to arbitrary vector lines in Euclidean three-space by Kochin et. al., following the development of Fridman and others [1]. The derivations of the theorems, as given by Kochin, are somewhat awkward, and do not directly extend to vector fields in space-time or on arbitrary manifolds.

An initial motivation of this note was to present the concepts of invariance of vector lines and intensities of vector tubes in a coordinate free manner, valid on arbitrary manifolds of any dimension, especially space-time. This goal is accomplished, and Fridman's results are reproduced as a special case, with a domain of validity restricted by the constraint that either the flow is steady, or the vector field of interest is entirely space-like. These restrictions were not presented by Kochin. However, once the mathematical tools of Cartan are applied to the Fridman problem, the in variance concepts, intrinsically formulated, are easily extended to other types of tensor fields of use in hydrodynamics. In particular, by means of the Cart an formalism, the Helmholtz and Bernoulli invariance concepts are directly extendable to the space time situation, but the invariance of mass concept requires further examination. In the first section, a summary of the various types of invariance notions which appear in the literature is made, but herein the ideas are redefined and constructed in the Cartan language of forms; in the second section, a number of applications are made to hydrodynamic systems of the relativistic and non-relativistic varieties.

2 INVARIANCE OF FORMS

2.1 The Lie derivative

The mathematical notions to be used herein are developed in the language of differential forms, and intrinsic operators on forms, such as the Lie derivative. These concepts, developed by Cartan, are well exposed in a style palatable to engineers and physicists in the textbooks by Loomis and Sternberg, Flanders or Bishop and Goldberg [2].

The technique is straight forward for hydrodynamic flows represented by the contravariant vector field V in N-dimensions. The interesting physical fields or functions are cast into coordinate free differential form language, and then as mathematical objects, ω, they are propagated down the flow vector lines given by $\mathbf{V}$. The question is asked: How does the object, ω, change as it propagates with the flow $\mathbf{V}$?

For a differential form, ω, the Lie derivative may be viewed, essentially, as the propagator of the form, ω, down the trajectories of the vector field, $\mathbf{V}$. If the Lie derivative of a form, ω, with respect to a vector field, $\mathbf{V}$, vanishes, then the form ω is said to be invariant with respect to the flow, $\mathbf{V}$. It is important to note that the Lie derivative is a somewhat broader idea than the covariant derivative, for the Lie derivative requires neither a metric nor a connection.

In terms of operators on forms, a particularly useful formulation of the Lie derivative is available[1]; namely, in terms of the interior product operator, $i(\mathbf{V})$, w.r.t. a vector field $\mathbf{V}$, and the exterior differential operator, d, the Lie derivative on a form, ω, can be written as:

$$L_{(\mathbf{V})}\omega = \{i(\mathbf{V})d + di(\mathbf{V})\}\omega. \tag{1}$$

This concept is coordinate free, it is metric free, it is connection free, it is measure free; it is useful in any reference system whether it be inertial, non-inertial, Lorentzian, non-Lorentzian, deformed, non-deformed, 3-dimensional or 1024 dimensional. More importantly, the Lie derivative is well behaved with respect to a submersion from inertial to non-inertial frames of reference.

[1] The Lie derivative was introduced into the Schouten school by Slebodzinski, and championed there by Van Dantzig in the early thirties, and later, by Post. Cartan had the concept essentially defined (for forms) in his 1922 text (see Ref. 4 below). For an equivalent formulation in the somewhat ponderous kernel-index method, see Ref. [3].

2.2 Invariance

For some mathematical object, ω, propagated down the trajectories of $\mathbf{V}$ by means of the Lie derivative, there exist several distinct notions of the invariance concept: it is possible to distinguish between invariance, absolute invariance, conformal invariance, integral invariance, and relative integral invariance with respect to the field $\mathbf{V}$.

The first concept, of invariance, is interpreted as the idea that the object, ω, does not change as it propagates down the trajectories of V. The coefficients of the invariant form, ω, transform as tensor invariants (preservation of functional form) w.r.t. the vector field V. The requirement that the form, ω, be invariant w.r.t. the flow, $\mathbf{V}$, is realized by the statement that the Lie derivative of the form vanishes:

$$L_{(\mathbf{V})}\omega = 0. \tag{2}$$

2.3 Absolute invariance

The second concept, of absolute invariance[2], is the idea that ω does not change as it propagates down the solution curves of $\mathbf{V}$, independent of how rapidly the transport is accomplished; i.e. the notion of absolute invariance depends only on the solution curves to the vector field, $\mathbf{V}$, and not on the parameterization of the solution curves. The requirement that a form, ω, be an absolute invariant w.r.t. the vector field, $\mathbf{V}$, is realised by the statement that the Lie derivative of the form with respect to $\gamma\mathbf{V}$ vanish for all γ:

$$L_{(\gamma\mathbf{V})}\omega = 0 \quad all \;\; \gamma. \tag{3}$$

2.4 Conformal invariance

The third concept, of conformal invariance, implies that the change of ω when propagated down the trajectories of $\mathbf{V}$ is proportional to itself via a conformality function, Γ. The requirement that a form, ω, be a conformal invariant w.r.t. the vector field, $\mathbf{V}$, is realized by the statement:

$$L_{(\mathbf{V})}\omega = \Gamma\omega. \tag{4}$$

In the conformal case[3], the form, ω, is not explicitly an invariant w.r.t. $\mathbf{V}$, but it is possible to distort the form ω by multiplying by a function, β, and the function β may be chosen such that the distorted form, $\beta\omega$, is an invariant w.r.t. $\mathbf{V}$. Hence, using (4),

[2] Herein, absolute invariance is distinguished from invariance in order to fit Cartan's concepts (Ref. 4); compare to the definitions of Schouten (Ref. 3) where the words, absolute invariance, are used in the sense of equation (2) and, absolute invariance w.r .t. streamlines, are used in the sense of equation (3). Note that (3) is equivalent to $i(\mathbf{V})d\omega = 0$ and $i(\mathbf{V})\omega = 0$.

[3] Note added 03/27/2003: Conformal invariance also implies that the form ω is homogeneous of degree Γ, and Γ need not be an integer.

$$L_{(\mathbf{V})}\beta\omega = \{L_{(\mathbf{V})}\beta \ + \beta\}\omega. \tag{5}$$

and if β is chosen such that the bracket factor in (5) vanishes, then $\beta\omega$ is invariant w.r.t. $\mathbf{V}$. Note that In β is not unique as its non-principle parts may be arbitrary first integrals of $\mathbf{V}$.

The conformal case, exhibited by (4), is the basis of theorems on conservation of vector lines, and the notions of "frozen in" field lines. In a physical, hydrodynamic interpretation, one states that if the points along a solution curve of an arbitrary vector field, $\mathbf{Z}$, are composed of fluid particles whose trajectories are governed by the solution curves to the flow field, $\mathbf{V}$, then the same fluid particles which make up one solution curve of $\mathbf{Z}$ at $t = t_1$ will make up another solution curve of $\mathbf{Z}$ at $t = t_2$. However, solution curves to a vector field, $\mathbf{Z}$, and a vector field, $\beta\mathbf{Z}$, are the same. Hence, invariance of vector lines of $\mathbf{Z}$ w.r.t. $\mathbf{V}$ does not require invariance of $\mathbf{Z}$, but does require conformal invariance of $\mathbf{Z}$ w.r.t. $\mathbf{V}$.

Note the difference between the concepts of absolute invariance and conformal invariance. For the absolute invariance (homological) case the flow vector field (chain) is deformed by γ and the form is left unchanged. In the conformal (co-homological) case, the form is deformed by β, but the vector field (chain) is left undisturbed. The two notions are not equivalent. Suppose ω is a conformal invariant w.r.t. $\mathbf{V}$. Then, while it is always possible to find a β which will leave $\beta\omega$ invariant w.r.t. $\mathbf{V}$, it is not always possible to find a γ which will leave ω invariant w.r.t. $\gamma\mathbf{V}$.

The word conformal applied to the invariance concept given by (4) is an extension of the word meaning historically associated with those cases where the object, ω, is restricted to be the line element, $\omega = g_{\mu\nu}dx^{\mu}dx^{\nu}$

2.5 Integral invariance

The fourth concept, of integral invariance, considers the invariance properties of the integral of the form, ω, over an open integration chain, c, with respect to a vector field, $\mathbf{V}$. The concept is realized by the statement that

$$L_{(\mathbf{V})}\int_c \omega = \int_c L_{(\mathbf{V})}\omega = 0. \tag{6}$$

which implies that integral invariance for open chains, c, requires differential invariance, $L_{(\mathbf{V})}\int_c \omega = 0$. If the integral is invariant for all parameterizations of $\mathbf{V}$, then $L_{(\gamma\mathbf{V})}\int_c \omega = 0$ and $\int_c \omega$ is said to be an absolute integral invariant w.r.t. $\mathbf{V}$; an absolute integral invariant retains its value for arbitrary deformations of the chain c so long as the points which make up the chain are confined to the same system of trajectories of $\mathbf{V}$. For an absolute integral invariant, note that the integration chain need not be a collection of equal time points; in this sense, Cartan's concept of absolute integral invariance is somewhat broader than Poincare's concept, which was confined to equal time point sets. For an absolute integral invariant, the integration chain may be grossly deformed, as long as the points which make up the integration chain are confined to the same system of trajectories of $\mathbf{V}$; that is, a point p of the integration chain on a given solution trajectory may be displaced along the given solution trajectory of $\mathbf{V}$ to another point, p'', and if the new integration chain c' consists of all of these points, p'', so displaced by a reparameterization of $\mathbf{V}$, then the integral of an absolute integral invariant, ω, over c' is the same as the integral of ω over c. Note that the end points of the integration chain (boundaries) are not necessarily fixed. Vector fields that leave the open integral of the object ω an absolute integral invariant are vector fields which are said to be extremal and associated to ω. If the integration end points are fixed, but the rest of the integration chain is deformed, then the vector field that leaves the integral invariant is said to be an extremal vector field, and a search for such fields is equivalent to the classic problem in variational calculus.

2.6 Relative integral invariance

The fifth concept, of relative integral invariance, considers the invariance of closed integrals over forms with respect to vector fields, $\mathbf{V}$. The integration chain is a cycle, z, and the concept is realized by the statement:

$$L_{(\mathbf{V})} \int_z \omega = \oint L_{(\mathbf{V})} \omega = 0. \tag{7}$$

Use of (1) implies that a relative integral invariant requires the constraint $L_{(\mathbf{V})} \oint d\omega = 0$, or equivalently, the idea that $i(\mathbf{V})d\omega$ be exact. If the closed integral is an absolute relative integral invariant with respect to $\gamma\mathbf{V}$ for all γ, then it must be true that $i(\mathbf{V})d\omega = 0$; $\mathbf{V}$ is then called an extremal vector field for ω.

2.7 Extensions and interpretations

It should be noted that interpretation of the problem may be reversed; i.e. rather than being given the vector field and searching for those invariant objects, ω, it is possible to solve for those vector fields that leave an object, ω, invariant. This procedure is the basis of Cartan's analysis of Hamiltonian dynamics [4][4]; he assumes that the closed integral of action is an absolute invariant of some vector field, $\mathbf{V}$, in state space, and solves for this vector field. The vector field turns out to be unique, and its components are given by Hamilton's equations.

The equations discussed above cover the basic conservation theorems usually exploited in hydrodynamics, and yield constraints on vector fields $\mathbf{Z}$ such that (a) the vector lines, or solution curves, of $\mathbf{Z}$ are invariants w.r.t. $\mathbf{V}$, and (b) the currents (or flux integrals) of vector lines of $\mathbf{Z}$ are invariants w.r.t. V. For case (a) equation (4) must be true; for case (b) equation (2) must be true. The ideas expressed in these constraint relations of invariance are not limited

[4] Intrinsic methods have been utilized in electromagnetic theories by a number of authors (Bateman, Weyl, Van Dantzig, Post), but very minimal efforts have been devoted to hydrodynamics.

to three dimensions, nor to Euclidean manifolds, nor to N-I forms built from vector fields (contra-tensor densities of rank I).

It should be noted that once some invariants are known, other invariants may be computed through either exterior differentiation, or by forming interior products. The alternating applications of these operators will lead to ladders of invariants, unless the application of the operator annihilates the object upon which it is operating. These statements are justified by the following argument: As the Lie derivative commutes with $i(\mathbf{V})$ and d, and if ω is an invariant w.r.t. $\mathbf{V}$ such that $L_{(\mathbf{V})}\omega = 0$, then it is also true that $d\omega$ is an invariant and $i(\mathbf{V})\omega$ is an invariant, for $L_{(\mathbf{V})}d\omega = d(L_{(\mathbf{V})}\omega) = 0$ and $L_{(\mathbf{V})}i(\mathbf{V})\omega = i(\mathbf{V})(L_{(\mathbf{V})}\omega) = 0$

Some applications are presented below.

3 APPLICATIONS

3.1 Contravariant vector fields and currents

The first application will be to the Fridman problem which is concerned with the preservation of vector lines in a flow $\mathbf{V}$. Consider the vector field (of flow) $\mathbf{V} = \{V^\mu(x^\nu)\}$ of contravariant components on a variety with some local coordinate chart, x^ν. In a relativistic hydrodynamic sense, $\mathbf{V}(x, y, z, t)$ is taken to be the flow field of a fluid in space-time.

Consider another vector field $\mathbf{Z} = \{Z^\mu(x^\nu)\}$, and a volume element with measure function $J(x^\nu)$ represented as an (odd) n-form, Ω;

$$\Omega = J(x^\nu)dx^1\hat{}...\hat{}dx^N. \tag{8}$$

Construct from the vector field, $\mathbf{Z}$, and the volume element, Ω, the (odd) N-I form, z, by contraction:

$$\begin{aligned} z &= i(\mathbf{Z})\Omega = JZ^\mu\{dx^1\hat{}...\hat{}\widehat{dx^\mu}\hat{}...\hat{}dxN \\ &= J\{Z^1dx^2\hat{}...\hat{}dx^N - Z^2dx^1\hat{}dx^3...\hat{}dx^N + ...\}. \end{aligned} \tag{9}$$

Now ask the question: Are the vector lines of $\mathbf{Z}$ invariants of the flow field $\mathbf{V}$? The answer is yes by the argument of the preceding section, if z is a conformal invariant of the flow. Hence, for invariance of vector lines of $\mathbf{Z}$ w.r.t. $\mathbf{V}$,

$$L_{(\mathbf{V})}z = \Gamma z. \tag{10}$$

Using (1) the constraint relation for invariance leads to the relation $\{i(\mathbf{V})dz + di(\mathbf{V})z\} = \Gamma z$, which has a coordinate expression in terms of the equations:

$$V^\mu\ \partial Z^\nu/\partial x^\mu - Z^\mu\ \ \partial V^\upsilon/\partial x^\mu + Z^\nu(\ \partial(JV^\upsilon)/J\partial x^\mu) - \Gamma Z^\nu = 0. \tag{11}$$

For four dimensional space time with $\mathbf{V} = \{\mathbf{v}, V^4\}$, $\mathbf{Z} = \{\mathbf{z}, Z^4\}$, $J = 1.0$, the four constraint relations given by (11) become

$$0 = V^4 \partial \mathbf{z}/\partial t + \mathbf{v} \cdot grad \mathbf{z} - \mathbf{z} \cdot grad \mathbf{v} - Z^4 \partial \mathbf{v}/\partial t + (\{div \mathbf{v} + \partial V^4/\partial t\} - f)\mathbf{z} = 0, \quad (12)$$

and

$$0 = V^4 \cdot \partial Z^4/\partial t + \mathbf{v} \cdot grad Z^4 - \mathbf{z} \cdot grad V^4 - Z^4 \partial V^4/\partial t + (div \mathbf{v} + \partial V^4/\partial t - f) Z^4. \quad (13)$$

If one assumes $V^4 = 1.0$ and either $Z^4 = 0$, or $\partial \mathbf{v}/\partial t = 0$, then (12) reduces to Fridman's requirement for invariance of vector lines of **Z** w.r.t. **V** (define $d\mathbf{z}/dt = \partial \mathbf{z}/\partial t + \mathbf{v} \cdot grad \mathbf{z}$, and compare to equation (5.5.1) of Ref. [1]):

$$d\mathbf{z}/dt - (\mathbf{z} \cdot grad)\mathbf{v} + \mathbf{z}(div \mathbf{v} - f) = 0. \quad (14)$$

The formulas (12) and (13) extend Fridman's result to relativistic fluids without the constraints of constant time like components, or steady flow.

For invariance of currents of vector lines (integral invariance), equation (2) requires that the function f must vanish in the preceding equations. Again, this result agrees with Fridman's analysis in the special case that the flow fields have constant time like components.

Note that integrals which yield currents of vector lines have as integration chains N -1 dimensional point sets. Only for three dimensional space is the N -1 chain a two dimensional surface, such that notions of flux (intensity) and notions of currents (quantity) may be compared. The notion of flux is associated with a covariant tensor of rank two integrated over a two dimensional point set. Historically, intensities were associated with the covariant fields of vorticity, which can be put into correspondence with 2-forms. In three dimensions, the natural correspondence between 2-forms and N -1 forms leads to a degeneracy of the distinct concepts of flux (objects of intensity) and currents (objects of quantity). It is proper to speak about intensities of vortex tubes and the currents of contravariant vector densities in any dimension, but the notion of "vortex lines" or intensities of vector densities only makes sense in three dimensions.

For example, from the covariant velocity field, V_μ a 1-form, $A = V_\mu dx^\mu$ may be constructed. The exterior derivative of this one form leads to a 2-form whose components are the components of vorticity, W:

$$W = dA = 1/2(\partial V_\mu/\partial x^\nu - \partial V_\nu/\partial x^\mu)\ dx^\nu \hat{}\, dx^\mu. \quad (15)$$

In three dimensions, this object is dual to a N -1 = 2 form, a contravariant vector density of three components, and this correspondence-a fluke of 3-dimensions-admits of a "vector" interpretation. In space time, however, the dual to W has 6 components; it is not at all equivalent to a N -1 form built from a vector density with four components. The equations for invariance of intensities and integrals of intensities (fluxes) are not the same as the relationships (12) and (13) developed for vector lines and currents.

3.2 Covariant tensor fields; vorticity

The concept of vorticity, W, is built on a covariant two form, which is the form derived by exterior differentiation of a one form, A, called the action per unit mass (see (15)). It is possible to examine the invariance questions for vorticity by use of equations (2-7). The intensity of vorticity will be an absolute invariant w.r.t. the flow $\mathbf{V}$ if

$$L_{(\gamma\mathbf{V})}W = 0 \;\; all \;\; \gamma \tag{16}$$

In coordinate language (16) becomes a system of constraint relations on the covariant components of the velocity of the form,

$$i(V)dA = 0, \tag{17}$$

which in coordinate language becomes

$$V^{\nu}(\partial V_{\mu}/\partial x^{\nu} - \partial V_{\nu}/\partial x^{\mu}) + \partial(V^{\nu}V_{v})l\partial x^{\mu} = o. \tag{18}$$

These equations of constraint are not at all equivalent to (11).

3.3 Helmholtz and Bernoulli invariants

The physical constraints for invariance of vortex intensities in space-time may be retrieved from a coordinate free representation of the Navier-Stokes equations: Consider a 1-form of action per unit mass, $A = \mathbf{v}_k dx^k - (\Phi + \mathbf{v}^k\mathbf{v}_k/2)dt$, built on the covariant components of the velocity field and the scalar potential Φ. The Hamiltonian per unit mass is defined as the sum of the potential and kinetic energies per unit mass, $H = (\Phi + \mathbf{v}^k\mathbf{v}_k/2)$. The Navier-Stokes equations are an image of the coordinate free form statement (ν is the shear viscosity, λ is the bulk dissipation),

$$L_{(\mathbf{V})}A = -1/\rho \; dP + v\Delta A - (\lambda + v)d\delta A + d\mathcal{L}, \tag{19}$$

where Δ is the Laplacian $(\delta d + d\delta)$, δ is the co-derivative operator, and $\mathcal{L}$ is the Lagrange function $(-\Phi + \mathbf{v}^k\mathbf{v}_k/2)$. For a Euclidean metric, the first three components of (19) yield,

$$\partial\mathbf{v}/\partial t + grad(\mathbf{v}\cdot\mathbf{v}/2) - \mathbf{v}\times \; curl\,\mathbf{v} = -grad\Phi - (1/\rho)\; gradP + v\Delta A - (\lambda + v)\; grad\, div\mathbf{v}, \tag{20}$$

which is the classical format of the Navier-Stokes equation. The exterior derivative of (19) yields a frame independent expression for the Lie derivative of the vorticity; i.e.

$$dL_{(\mathbf{V})}A = L_{(\mathbf{V})}dA = L_{(\mathbf{V})}W = (1/\rho^2)d\rho\,\hat{}\; dP + vd\delta W. \tag{21}$$

If the r.h.s. of (21) vanishes, then the vortex intensity and the flux of vortex tubes, $\iint W$, over any two-dimensional chain, are invariants of the flow. For the ideal non-viscous fluid ($v = 0$), with a barotropic equation of state ($dp\hat{}dP = 0$), the r.h.s. of (21) does indeed vanish, and the Helmholtz theorem is retrieved, in space time. Equation (21) indicates that vorticity can be created and destroyed, even in non-viscous fluids, if the fluid is not barotropic. The usual example is the atmosphere.

Note that if the Helmholtz theorem is true in the integral sense, then the Kelvin circulation theorem is also true; i.e., in space time, by Stokes theorem which requires that the closed integration chain, z, be a boundary, b;

$$L_{(\mathbf{V})} \iint dA = 0 = L_{(\mathbf{V})} \oint_b A. \tag{22}$$

Invariance of the intensity of vortex tubes implies invariance of the circulation.

It is of some interest to note that harmonic components of vorticity (vortex wave solutions of the equation $\Delta W = (\delta d + d\delta)W = 0$), do not contribute to the change in vortex intensities, even in viscous fluids.

The Bernoulli invariant may be obtained by contracting (19) with the vector field V :

$$i(\mathbf{V})(L_{(\mathbf{V})}A) = L_{(\mathbf{V})}i(\mathbf{V})A = L_{(\mathbf{V})}\mathcal{L} = i(\mathbf{V})f \tag{23}$$

where f is defined to be the r.h.s. of (19). If the fluid is isentropic ($1/\rho dP = d\Psi$), and if the shear viscosity is negligible ($v = 0$), then (23) reduces to the equation

$$\begin{aligned} L_{(\mathbf{V})}\mathcal{I} &= i(\mathbf{V})d\{H + \Psi + \lambda\delta A\} \\ &= i(\mathbf{V})d\{(\Phi + \mathbf{v}^k\mathbf{v}_k/2) + \Psi + \lambda\delta A\} = 0 \end{aligned} \tag{24}$$

The variable, $\mathcal{I} = \{\Phi + \mathbf{v}^k\mathbf{v}_k/2 + \Psi + \lambda\delta A\}$, defines the specific enthalpy and according to (24) is an invariant (the Bernoulli invariant) of the flow. Again, this result is coordinate free and is valid in space-time.

The formulations of the Helmholtz and Bernoulli theorems given by (19)-(22) give the impression that these ideas are intrinsic concepts, independent from the notions of coordinates and dimensions of the manifold over which they are constructed. Indeed, it is possible to abstract the system in terms of the idea:

For a parameterization ρ of a vector field $\mathbf{V}$, what equations must be necessarily satisfied if the closed integral of action per unit mass is to be a conformal invariant of the flow?

The necessary conditions that must be satisfied are given essentially by equation (19) with the conformality factor, Γ, related to the viscosity coefficient, ν. These necessary conditions become the fundamental equation of motion, valid in any reference system, inertial or not; the fundamental equation is of the form

$$(L_{(\mathbf{V})}A) = f. \tag{25}$$

The Helmholtz theorems and Bernoulli theorems are results of applying the ladder operations to (23).

3.4 Invariance of mass

Consider the three form, m, of differential mass in space time:

$$m = \mu(x, y, z, t) dx\hat{}dy\hat{}dz. \tag{26}$$

Is m an invariant of the flow $\mathbf{V}$? The answer is given again by (2); i.e. if

$$(L_{(\mathbf{V})}m) = \{V^4 \partial\mu/\partial t + \partial(\mu \mathbf{v}^k)/\partial x^k\} dx\hat{}dy\hat{}dz - \mu\partial(\mathbf{v}^x)/\partial t\; dt\hat{}dy\hat{}dz + \mu\partial(\mathbf{v}^y)/\partial t\; dt\hat{}dx\hat{}dz - \mu\partial(\mathbf{v}^z)/\partial t\; dt\hat{}d \tag{27}$$

then m is an invariant of the flow. If it is assumed that $V^4 = 1.0$, then the vanishing of the first bracket yields the usual equation of continuity. However, for invariance of m in space time, it must also be true that the flow is steady, $\partial(\mathbf{v})/\partial t = 0$.

In space-time, two requirements are necessary if m is to be an invariant of the flow:

(I) The equation of continuity must be satisfied, and
(2) The flow must be steady.

If the total mass is constructed by integrating m over an open 3-chain, the same two requirements are necessary for $M = \iiint_c m$ to be a flow invariant — unless the integration chain is not arbitrary and is restricted to an equal time point set ($dt = 0$). Of course, this restriction is the usual assumption of Galilean mechanics, but in space-time, it is not always true that such equal time integration chains are applicable. For an optical observer using energy rate detectors, the integration chain is given by a point set constructed from photons which are simultaneously received. These points are not necessarily a set of equal time emission points, hence, $dt \neq 0$, and total mass invariance, indeed, requires steady flow.

If the integration 3-chain is closed (in 4-dimensions) a slightly less restrictive invariance concept can be established. The criteria for relative integral invariance, (7), requires that $d(i(\mathbf{V})dm) = 0$, or

$$\partial\{(\partial\mu/\partial t)V^\sigma\}/\partial x^\sigma = 0. \tag{28}$$

Hence, $\iiint_z m$ is an invariant of the flow if $(\partial\mu/\partial t)$ is an integrating factor for $\mathbf{V}$. If $(\partial\mu/\partial t)$ vanishes, then all closed integrals of m are invariant, but open integrals are not necessarily invariant. Equation (26) is an equation of continuity in terms of the function $(\partial\mu/\partial t)$, and presents a requirement for invariance which differs from (25) and is not equivalent to the results obtained from the classical Reynold's transport theorem [5]. The classical development presumes that the integration chain is over a set of equal time points, and destroys the generality of the theorem in relativistic space-time.

If the integrals of m over closed chains are defined to be total "particles," or total number, and if the open integrals of m are defined to be total mass, then the above notions indicate that a flow may preserve total particles without preserving total mass, but if total mass is preserved, it follows that total number is conserved as well.

3.5 Comments on fluids in space time

The conservation of total mass in space-time requires two constraints on the flow field, $\mathbf{V}$, and mass density, μ. It is possible to inquire if a reparameterization of the flow to the form $\gamma\mathbf{V}$ might satisfy these requirements, even though the original flow $\mathbf{V}$ did not. Effectively, the problem becomes a search for a factor γ such that the second constraint is satisfied in the form,

$$\partial(\gamma\mathbf{V})/\partial t = 0. \tag{29}$$

This requirement says that the flow is reducible to a steady flow by means of the reparameterization factor γ, and implies that $\mathbf{V}$ is factorable into functions of space multiplied by functions of time alone. Such a factorization is, by Lie's theorem, the necessary and sufficient condition that the flow field be the generator of a semi-group with time as the group parameter. It follows that the conservation of total mass is associated with the group structure of the flow, and alternatively, if the total mass is not conserved for a continuous flow, the flow does not admit group structure. The observation that turbulent flows are not autonomous, nor reducible to autonomous flows (by a reparameterization, γ), implies that turbulent flows do not conserve total mass in a relativistic sense and do not admit a group structure in space-time. Note that a turbulent flow may preserve total number, but not total mass.

Another point of the space-time analysis that requires comment is the assumption that arises both in the Fridman equations (14) and in the equation of continuity. The assumption in question is the statement that $V^4 = 1.0$ everywhere over the manifold. Such an assumption implies that 4-dimensional space time is reducible to a 3 +1I submanifold structure. In general relativity such a structure is called a time orthogonal reference system. It is not obvious that all physical systems admit a reduction to time orthogonal frames of reference (by means of a suitable submersion). Galilean mechanics not only assumes such a reduction capability, but also it assumes that t is a constant on such submanifolds. Special relativity assumes a time orthogonal reduction capability, but the submanifolds need not be t = constant hypersurfaces. A general relativistic, accelerating, radiating system may not admit a 3 + 1submanifold representation; there may not exist a unique parameterization of the flow $\mathbf{V}$ such that $V^4 = 1.0$ everywhere in the manifold. A mathematical example of such a system is a manifold endowed with a Godel metric [6].

So little is known about such systems that physical examples are hard to construct, but the notion that V^4 is intimately connected with the continuity constraint equation, and not with the autonomicity portion of the two constraints on total mass invariance, leads to the conjecture that a cavitating fluid may be a physical counter example to the assumption that $V^4 = 1.0$ everywhere. The conjecture is further strengthened by the observation that the constraint

(26), which is the requirement for total number invariance, may not be satisfied by the hypothesis that $V^4 = 1.0$ for arbitrary μ. In particular, at a spatial stagnation point, $\mathbf{v} = 0$; the conditions for conservation of total number will not be satisfied for $V^4 = 1.0$, if the density function, μ is non-linear in time. On the other hand, at a stagnation point it is always possible to choose a $V^4 \neq 1.0$ such that the constraint is satisfied; furthermore, if the partial time derivative of the mass density may be factored into a product of spatial functions and a function of time, $V^4(t)$ may be chosen such that the conservation of number constraint is satisfied by merely regraduating clocks. However, the relationship between cavitation and conservation of total number is still somewhat indistinct, so that this interesting idea must at this time remain as a physical conjecture.

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