MACH'S PRINCIPLE IN A MIXED NEWTON-EINSTEIN CONTEXT

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Abstract: A closed physical space, in conjunction with scalar versus pseudo scalar distinctions, and an accordingly adapted Gauss theorem, reveal unexpected perspectives on Mach's principle, the mass-energy theorem, and a bonus insight into the nature of the solutions of the Einstein field equations of gravity.

Preamble

The following discussion of Mach's principle in the context of the general theory of relativity follows largely a contribution to the September '96 meeting in London of the British Society for the Philosophy of Science.[1] The items selected here for discussion are two extensions of Gauss' law, which have remained somewhat unexplored for the purposes of physics. First the embedding manifold is taken to be closed instead of Euclidian and secondly the ramifications of extending the results from scalar-valued to pseudo scalar-valued integrals are assessed.

The last part of this communication delineates an overlap with an earlier assessment of Mach's principle that is due to Schroedinger.

Two-dimensional Residue Integrals in a Three-dimensional Physical Space

Gauss' law of electrostatics says: a closed surface integral of the dielectric displacement **D** equals the algebraic sum of electric charges ±e enclosed by its integration cycle c_2:

$$\S_{c_2}\mathbf{D}\cdot d\mathbf{S} = \Sigma\pm e. \qquad 1$$

Mathematically, Gauss' law summarizes and ***extends*** implications of Coulomb's inverse square law of attraction between charges of opposite polarity and of repulsion for charges of equal polarity.

The field force **E** per unit charge relates to **D as**

$$\mathbf{D}=\varepsilon_0\mathbf{E}\ , \qquad 2$$

in which ε_0 is taken to be constant.

The Neumann-Brewster symmetry principle of crystal physics dictates that **D**, **E** and e *change sign under spatial inversions.*

Coulomb's law has the "inverse square" law behavior in common with Newton's law of gravity. Hence a similar statement as that of Eq.1 can be expected for the interaction of point-masses m_k:

$$\S_{c_2}\mathbf{m}\cdot d\mathbf{S} = \Sigma\ m_k. \qquad 3$$

In Eq.3, the vector field **m** is analogous to **D** in Eq.2 and can be referred to as vector of mass-displacement $[\mathbf{m}]=[ml^{-2}]$. It similarly relates to a vector of force **g** per unit mass, known as gravity acceleration

g =

Error!

$$[\kappa]=[m^{-1}l^3\ t^{-2}]\ .$$

The standard geometric backdrop chosen in physics for the just mentioned laws is an infinitely extended three- dimensional Euclidian space. In mathematics this space is referred to as neither *closed* nor *compact*.

In a Euclidian context the notion of enclosing by a clsoed surface is unambiguous; inside the two dimensional enclosure is a finite domain, whereas outside is the infinity of Euclidian space. Hence in a Euclidian context there is no question whatsoever as to what is inside and what is outside.

This distinguishability between inside and outside no longer has that absolute status, if the space under consideration is taken to be closed. As a visual example consider **a closed loop on the surface of a sphere . The loop divides that spherical surface into two separate finite domains. Whatever part is called inside or outside is now purely a matter of choice.** There can at best be a bias for referring to the smallest part as the inside.

The theory of complex functions envisions exactly such topological situations. Applications of Cauchy's residue theorem require consideration of *residues on either side of the integration loop. The residues* are counted with different signs according to whether they are encircled in clock- or counter clockwise fashion.

After comparison with the just cited purely mathematical procedure that has helped in the correct evaluation of numerous integrals, it is now instructive to go up actually one step in dimension from the complex plane to real physical space.

For the purpose of finding what conceivably could happens at infinity the Euclidian 3-dimensional space is now replaced by **a closed** three-dimensional space; **a three-dimensional sphere** M_3 if you will. Locally these two options are indistinguishable, yet their global structures are very different. Each has its own problem of visualization. The following is an attempt at establishing which of the two options is closest to what is considered to be good epistemic reality.

The 3-shere is separated into two domains by a closed 2-dimensional surface c_2, which shall be considered as an integration cycle. The integration cycle now not only encloses residues perceived as on one side of c_2, it also encloses (with opposite sign to be sure) residues on the other side.

It now follows that Gauss' law, applied to the vector fields **m** and **g** , as defined on M_3, assumes the generalized form given in Eq.5; the difference in sign between 'inner' and 'outer' residues is, similarly as for the Cauchy theorem, determined by a matching of surface to volume orientation conventions:

$$\oint_{c_2} \text{2-form} = \Sigma \text{ inner residues} - \Sigma \text{ outer residues}, \quad 5$$

Gauss' law in closed compact Manifold M_3

A comparison with the traditional renditions Eqs1 and 3, somehow shows how, during all those years, **the convenient choice of a Euclidian backdrop has provided for a tacit rationale to simply disregard the Euclidian outer "world" at infinity. In retrospect it is now not surprising why the traditional Euclidian approach fails to get a quantitative handle on Mach's principle.** The latter is exactly a proposition about finite influences of that outer world. If outer influences are suspected, dealing with them means a choice of manifold structure that at least permits us to do something. The Eulidian proposition has presented insurmountable hurdles in this respect.

In pursuing the implications of the mentioned manifold specifications of closure and compactness,* we do well by making first a routine examination whether Eqs.1 and 3 meet the mathematical requirements for residue integration. Apart from the familiar Diffeo-invariance** and scalar- or pseudo-scalar valuedness of the integrals, the conditions for residue integration are:

1) The **differential forms** defined by the integrand of the integrals are **closed;** in the present context, this means their **exterior derivative** vanishes in subdomains of space that are charge-free and/or mass-free.

2) The integration cycles c_2 reside where the exterior derivative of these differential form vanishes. This property gives the residues invariance under c_2 deformations in the subdomain where the exterior derivative vanishes.

3) Residues are topological, scalar or pseudo scalar *domain* invariants. They remain additive under all reference changes.

Since the divergence operations div**D** and div**m** translate into **exterior derivatives**, Eq.1, without exception, meets all three requirements. This makes Gauss' law of electrostatics an historical prototype of a residue integral for mathematics and physics both.

One may argue that Newton was indeed close to indicating a near-valid precursor of Gauss' law, and indeed he was. The Diffeo-invariant nature of the generalized Gauss-Stokes integral theorems began to surface earlier this century. The residue integral concept first appears explicitly in Gauss' theorem of electrostatics. One may assume that Gauss was well aware of its Diffeo invariance. **Ironically, Physics' first residue integral was pseudo scalar-valued.**

An inspection of Eqs. 3 and 4 in conjunction with Eqs.1 and 2 reveals that also the gravity case is very close to meeting all three residue integral requirements. Closer scrutiny, though, shows that gravity does not quite meet the condition of additivity for the mass residues, because according to relativity, additivity of masses does not hold. Gravity interaction between masses, as presently understood, invokes negative energies producing small defects. such as are evident in the periodic table of atomic weights for the much stronger atomic interactions. Since gravity is the weakest of interactions, the following proposition is taken to hold with a fair degree of approximation:

This approximate status mass additivity leads us to admit here Eq.3 as a near-valid residue integral manifestation.

It is now necessary to emphasize basic physical and mathematical difference between the two cases: *e.g.*,

a) The residues of Eq.1 have polarity, the residues of Eq.3 don't!

b) The polarity of Eq.1 makes the differential form defined by **D** an **impair** form, whereas the differential form defined by **m** is a **pair** form. **Pair forms are invariant under inversion, they define scalars. Impair forms change sign under inversion and they define pseudo scalars.**

Explicit definitions of pair and impair differential forms have been introduced by de Rham[2] for the purpose of dealing appropriately with orientation sensitive matters. When reading de Rham's text one finds that an explicit use of impair forms remains sort of dormant. From earlier de Rham work it appears that topological implications of Maxwellian theory may have induced de Rham to maintain the pair-impair distinction. In mathematical follow-ups (now known as de Rham cohomology) impair forms have disappeared, in part due to leads given in ref.2.

A need for making pair-impair distinctions in physics becomes absolutely mandatory in crystal physics. Since tensors are the standard mathematical tools for crystal physicists, it now becomes essential that a unique correspondence is established between tensor species and the pair-impair forms of de Rham.

Yet, most tensor books, written for the purposes of physics, have ignored the needs of crystal physics. Therefore, tensor species corresponding with de Rham's impair forms are missing **Hence. physics and mathematics both are guilty of having completely abandoned the impair differential forms.***

All of this shows how most textbook writers tend to be talking too much to themselves, with inadequate awareness as to what their readers can do with their creations. **These things had to be mentioned, because the pair-impair distinction is far too fundamental to continue the presently customary ad hoc treatments of those aspects.**

At least one general tensor text [3]exists in which the inversion features of tensors are well acknowledged so that the exigencies of crystal physics can be met. Since differential

* Compactness means a **finite** atlas maps M_3 on Euclidian neighborhoods; it makes proofs easier!

** Diffeo is short for genral transformations that are invertable and differentiable.

* This measure would in effect throw out the first residue integral ever: *i.e.*, Gauss' law of electrostatics.

forms and tensors are rarely treated concurrently, a dictionary of how the a one to one correspondence between differential forms and tensor species works out would, of course, be helpful.

However, in the absence of such dictionary, the following discussions attempt to bridge the gap as well as possible. Perhaps, overseers of our textbook literature may consider in the future a joint textbook for tensors and forms covering orientability in non ad hoc fashion. These are the conditions to establish a dictionary with extensive physical identifications.

Crystal physics makes it necessary to identify **D** as an **impair differential** 2-form, thus making electric **charge a pseudo-scalar** changing sign under spatial inversion.

Pre-relativity mass, by contrast, is an absolute scalar, not changing sign under inversion. Mass is physically perceived as a quantity only assuming one sign; say positive values. The so-called mass defect, which is perceived as negative, only modifies the inherently single sign positive nature of mass. This identifies **m** as defining a **pair differential** 2-form with all positive scalar residues.

A global exploration of these presumed period integrals is now in order. Consider the possibility that three dimensional physical space M_3 is closed and compact so that the cycle c^2 has the Jordan-Brouwer property of separating physical space into two domains, which now can only *subjectively* be referred to as inner domain and outer domain.

Once 'closed' and 'compactness' govern M_3, the notions of inner and outer domain are interchangeable except for a change of sign due to the matching of surface and spatial orientations. Hence if c_2 is a cycle in M_3, Gauss' law now reads according to Eq.5:

$$\S_{c_2} \text{ 2-form} = \Sigma \text{ inner residues} - \Sigma \text{ outer residues} \qquad 5$$

If c_2 were to be contracted to a point, it could say

$$\Sigma \text{ of all residues in } M_3 ‘ 0. \qquad 6$$

The latter condition is indeed easily met for the Gauss integral of electrostatics Eq.1, if the proviso is met that electric charges only occur as pairs. *There are no isolated unpaired charges of either polarity.* So, the number N_+ of positive elementary charges in a closed compact universe equals the number N_- of negative charges:

$$N_+ = N_- \quad . \qquad 6$$

Applications of Gauss' law of electrostatics in a Euclidian context does not invite us to enter unduly into far-reaching specifications about the nature of the physical universe. ***The conditions expressed by Eqs.5 and 6 clearly hinge on the existence of a universal unit of elementary charge ±e and its polarity.***

While global explorations based on closed and compactness are in ideal conformance with Eq. 1, no such easy conformity is within reach for the gravity counterpart Eq.3. There is no unique standard of mass, which appears as beautifully additive as electric charge. Moreover, notwithstanding the notion of anti-matter, so far, present knowledge does not reveal the existence of a mass polarity. Mass is taken to be inherently positive, hence Eq.6 has no chance of being met for the mass distribution in M_3.

Although relativity calls for change in Newtonian gravity, the latter's asymptotic closeness to relativity peaks the curiosity about exactly when the ensuing discrepancies become intolerable. How, and in what way, do the distant masses of the universe affect our local conditions? The near masses give us gravity, approximately according to Newton's description. The distant masses of the universe, according to Mach,[4,5] assume a role in mass inertia. Gravity- and inertia forces display an opposing counteracting function in physical descriptions; a feature qualitatively in accord with the opposite signs attributed to the near inner domain of gravity influence and the presumed outer domain of far away inertia influences.

For gravity the condition of Eq.6 would have to be abandoned. For mass residues an alternative of a finite sum of residues needs to be considered;

$$\Sigma_{M_3} \textbf{ mass residues} = \text{finite.} \qquad 5a$$

The proposition expressed by Eq. 5a is mathematically permissible, yet has no obvious support from a traditional physical angle, because Eq.3 registers no influence of distant masses.

A measure for the gravity-inertia interaction due to the outer masses of the universe can be extracted from Newtonian potential theory, provided an **artifact is used that, in a permissible way, pulls the distant outer world within a Newtonian realm.**

Let the potential f of the acceleration of gravity **g** be defined through the gradient relation

$$\mathbf{g} = -\# f .$$

From Gauss' integral theorem it follows

$$\operatorname{div} \mathbf{m} = ®,$$

in which ® is the mass density (mass per unit volume).

Using Eq.4 gives the Poisson equation for gravity

$$\#^2 f = -4¹ \kappa ®,$$

which has a Euclidian-based solution

$$f = \kappa \int_{M_3} \frac{®}{r}\, dV. \qquad 7$$

For a closed M_3, Eq.7 is hopelessly extended beyond the Newtonian realm of validity. To make up for an impermissible act of using Euclidian results in a non-Euclidian context, it is now necessary to take recourse to a bold **artifact.**

The distant universe be replaced by a spherical shell of effective mass M and effective radius R. Our local Euclidian world of interest be inside of this shell. This physical substitution is, for all practical purposes, analogous to replacing spherical mass by a mass-point. The artifact of the giant massive sphere is meant to extend the realm of Newton's potential.

Similar as in the electrical case, the shell now acts as ***a gravitational Faraday cage, inside of which a huge, yet constant, gravitational potential exists.*** It can be written in the form:

$$f = \kappa \frac{M}{R} \quad . \qquad 8$$

In Eq.8, M and R are to be regarded as equivalent measures of the mass and radius simulating the action of a distant Universe.

Since the gravitational Faraday cage effect makes the potential f, as given by Eq.8, a constant, **grad**f = 0. Hence **no net gravity forces are exerted on massive objects inside this shell, yet there is a tremendous gravitational potential.**

The intense potential field of Eq.8, in which massive objects inside the shell are immersed, has been perceived as instrumental in the manifestation of inertia of massive objects. The balanced gravitational pull of the universe creates an opposing "inertia" of massive objects in responding to local disturbances.

All inferences, so far, are obtained, if you will, with the help of a speculative global extension of the Newtonian picture. Let us compare these results with a local premise of relativity known as the geodesic line axiom.

Light rays and point masses exposed to gravity and inertia travel along a geodesic spacetime path:

$$\ddot{x}^{\neg} + \Gamma^{\lambda}_{\nu\kappa}\,\dot{x}^{\nu}\ \ddot{x}^{\kappa} = 0, \qquad 9$$

in which $\Gamma^{\lambda}_{\nu\kappa}$ jointly accounts for gravity as well as inertia forces. This object is known as a Christoffel symbol, it is expressed as a function of the spacetime metric. The latter, in turn, expressed in its Cartesian, inertial frame appearance is

$$g_{\lambda\nu} = \{c^2, -1, -1, -1\}.$$

For these inertial frame conditions, Eq.9 simplifies to a nearly Newtonian form, which is a simple balance between inertia of acceleration $\ddot{x}^{\neg}$ and gravity forces **grad** $(c^2/2)$, both taken per unit mass:

$$\ddot{\mathbf{x}} + \mathbf{grad}\,\frac{c^2}{2} = 0. \qquad 10$$

Eq.10 reveals that one of the most accurately determined physical "constants" (known as the velocity of light c) is not a constant after all. In fact, $(c^2/2)$ assumes the surprising role of a near-constant gravity potential due to the rest of the universe.

Establishment physics has remained suspiciously uncommitted about this silent contradiction between an experimental result that accepts c as a constant and a body of fairly well accepted theory (*i.e.*, relativity) that has c as not constant.

A comparison between Eqs.8 and 10 invites an identification of the light velocity squared c^2 as a gravitational potential. The latter being determined by the artifact of an **effective** mass M and radius R of the Universe:*

$$c^2 = 2\kappa\, M/R\,. \qquad 11$$

Multiplication of Eq.11 with an object mass m inside the gravitational Faraday cage reveals an interesting genesis of the expression mc^2 as a measure of potential energy of m in the **potential** field of the universe:

$$mc^2 = 2\kappa\, m\, M/R\,. \qquad 12$$

Note the factor 2 in Eq.12, as compared to an identification of mc^2 with a standard Newtonian potential energy: $\kappa mM/R$.

Since c seems generated by the distant mass of the universe it stands to reason that c could be changing in the neighborhood of a local gravitating body, placed inside the equivalent "shell" of the universe. Let a mass m be placed inside this gravitational Faraday cage. Now using the potential additivity as prevailing in the Newtonian realm, the new potential at a distance r from the gravitational center of m must equal the **difference** between "universe potential" and local potential; *i.e.*, M being in the outer and m in the inner realm of the cycle c_2, **Gauss' law in the form of Eq.5 now requires:**

$$f = \kappa \frac{M}{R} - \kappa \frac{m}{r}\ . \qquad 13$$

Let the primed c' be the gravity-modified light velocity and unprimed c the velocity for m = 0, one then obtains according to Eqs.11 and 13:

$$(c')^2 = c^2 \left[1 - 2\kappa \frac{m}{rc^2} \right], \qquad 14$$

which is compatible with the value for g_{44} of the Schwartzschild solution of the Einstein gravitational field equations.

In view of the local r dependence of c', light will be diffracted near the gravitating body m, leading to standard predictions of the general theory of relativity, without a need for calling on the field equations.

It thus appears that parts of relativity are almost within the realm of Newtonian theory. The mere act of specifying things, where Euclidian space leaves matters unspecified by necessity, extends Newton's realm. The verification of results of the general theory of relativity lends a measure of support to the global process as a complementary procedure. While insight into Mach's principle is not well possible via the local procedures of the general theory, the global complement is found to compensate for those shortcomings. Notwithstanding the slightly stretched application of the gravity residue integrals, the ensuing asymptotic perspectives have some undeniable conceptual merits. **It reveals an emerging global angle on the mass-energy theorem of relativity and Mach's principle, contingent on an enhanced relevance of a pair-impair distinction.**

Thomas E. Phipps jr alerted me to a 1925 paper by Schroedinger[6] in which the simulation of the universe by a massive hollow sphere is also used to evaluate a gravity potential due to distant masses. Instead of using an extended Gauss theorem and the asymptotic comparison with the geodesic equation, Schroedinger explicitly performs the integration and establishes a relation to the light velocity which differs slightly from the result obtained here.

* According to Hermann Poeverlein formerly at the US Airforce Lab. at Hanscom Field in MA emiritus Darmstadt Tech. Un.) hunches comparable to Eq.11 have been around for a long time.

Yet, in order to see this comparison in a propre perspective, it needs to be pointed out that Schroedinger tackles a more ambitious problem. Here is a sketch of the rationale used by Schroedinger.

A static Coulomb interaction undergoes a slight modification if the charges or the mass- points are in motion with respect to one another. This effect is known as Weber's *velocity* correction. Phipps[7] has cast this Weber change in the form of a familiar factor $[1-\sqrt{1-v^2/c^2}\,]$. Multiplied by mc^2 it gives the kinetic energy $(1/2)mv^2$ as a first order result. Hence correcting the Coulomb energy with this factor has a very small effect, because mc^2 is much bigger than the Coulomb energies here considered. **Either for that reason, or possibly for no reason at all, establishment physics has been neglecting this term.**

It is, therefore, interesting to note that, in a relatively unknown paper, Schroedinger went out of his way to salvage this Weber correction for a gravity application. He shows how this dynamic Weber term identifies the kinetic energy as a manifestation of interaction with the outer masses of the universe; thus substantiating Mach's assertion.

By contrast, the **static** global approach here considered identifies mc^2 instead of $(1/2)mv^2$ as a manifestation of interaction with the outer universe. Despite the cited marginal additivity of mass residues, a wider-ranging overlap between local and global methododologies seems to be evolving.

It seems an interesting irony of fate that the **global** assessment of gravity exhibits a pronounced **Galilean** character, whereas the **local** assessment of gravity with the help of Einstein's field equations is inextricably interwoven with **spacetime** description. Those compelled to see things either black or white, and nothing in between, are invited to make their choices. Keep in mind though that extremism in either direction can blind us for the more subtle things in life.

1 E J Post and M Berg, Proceedings of the 1996 September meeting on **Interpretations of Relativity** in London UK, *Epistemics of Local and Global in Mathematics and Physics* ; section VI p.336. Sponsored by the British Society for the Philosophy of Science (BSPS).

2 G de Rham, *Variétés Differentiables* (Paris, 1955). Its definitions of **pair and impair differential forms are in one to one correspondence with Schouten's 'ordinary' tensors and 'W'-tensors in ref.3.**

3 J A Schouten, *Tensor Analysis for Physicists* (Oxford University Press 1951). Dover reprint 1974.(see ref.2)

4 J B Barbour and H Pfister, editors of a volume entitled; *Mach's Principle*, based on a 1993 Tübingen Conference (Birkhäuser, Boston 1995) Einstein studies volume 6.

5 As general accessible source see C H Misner, K S Thorne, J A Wheeler, *Gravitation* (Freeman, San Fransisco 1973) For the original source in translation compare: E Mach, *The Science of Mechanics* (La Salle, Illinois 1960).

6 E Schroedinger, Ann. der Phys.,**77**, 325 (1925) English translation in ref.4.

7 T E Phipps jr, Essays in Physics **3**, 414 (1990)