The Many Faces of Torsion

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WARNING: STILL UNDER CONSTRUCTION

At least six definitions of "torsion" appear in the physics literature complicating the utilization and understanding of the concept. In this article, the differences and similarities of the various definitions are compared, and their relationships to spaces of absolute parallelism with applications to irreversible thermodynamics, coherent structures in plasmas and fluids, and general relativity are described.

I. Introduction

In this article the concept of torsion and torsion fields will be investigated. There are at least six different definitions of torsion to be found in the literature. They include:

The Frenet Torsion of a space curve.

The Frenet Helicity Torsion of a Bernoulli fluid flow.

The Torsion described by the anti-symmetric components of a Cartan right connection.

The Torsion described by the anti-symmetric components of a Cartan left connection.

The Cartan Torsion 2-forms.

The Topological Torsion of a 1-form.

Each of these types of torsion will be defined and discussed in that which follows. All are related in one way or another to the idea of Cartan's Repere Mobile, which was developed from the Frenet-Serret theory of singly parameterized space curves. At a point p on a space curve C in 3 dimensional euclidean space, a basis frame of orthonormal vectors (unit tangent, normal and binormal) can be constructed by metric and differential processes. As the point p moves along the space curve, the basis vectors so defined are not constant, but change direction (and magnitude if not normalized) as the parameter of the curve varies. The functions that define the components of these unit basis vectors form a matrix of differentiable functions, $[F]$. This matrix is defined as the Repere Mobile. The method (in 3 dimensions) generates 3 intrinsic parameters (arc length, s, Frenet curvature, k, and Frenet torsion, τ) that characterize the space curve. The Frenet torsion cannot be zero for space curves that are intrinsically 3 dimensional and can not be confined to a plane. This is the first clue that torsion has something to do with topological dimension 3 or greater. The details of the method are explained below.

From a physical point of view torsion also has numerous manifestations. First consider a string attached to the middle of a toothpick placed at each end of the string. Place the string on a overhead projector table to display the stable image of the string and its endpoint toothpicks. The string lies flat on the projection surface. Next place a similar string and toothpick combination on the overhead projector, but this time twist the string through several turns before placing it flat on the projector surface. The projected images appear to be congruent. However, carefully pick each string by using one toothpick as a suspension point. The opposite toothpick on one of the strings rotates, and the other does not. One of the strings had torsion, the other did not. For those old enough to have constructed rubber band powered model airplanes with propellors, another topological effect of torsion is observed as the propellor rubber band is wound more and more. At a certain point the rubber band starts to "knot" or "bead". As more and more energy is put into the twisted rubber band, these quantized knots continue to appear, until the entire length of the rubber band is "knotted". Then as the process continues a second layer of knots appears. I know of no mathematical analysis of this observable effect of torsion.

As a second example, consider a piece of thick-walled rubber tubing, It has a natural elastic state as a cylinder which will support parallel generators as long straight lines on the outer surface of the tube. Assume that there are 4 such generators painted on the tube, 90 degrees apart. Next bend the elastic tube into a circle and fasten the ends together such that the end of each generator is matched (A sawed off bolt inserted into each end of the tube will due nicely as a fastener.) The bending operation has introduced curvature (stress energy) into the system, but each of the generators (now curved) form a ring which is parallel to a plane and to the other rings formed by the other generators. In fact, the bent (internally stressed) ring will lie flat on this plane. The stressed tube has curvature, but it is torsion free. Next assemble the tube into a circle, but this time rotate the tube through some multiple of pi/2 before attaching the ends together. The generators are no longer parallel to a plane, and moreover, the assembled structure will not lie flat on a plane. If the twist is 90 degrees the four generators reduce to ond single closed curve. If the twist is 180 degrees, the generators reduce to 2 linked space curves. There is more than one stable state for the twisted configuration. In one state the tube encircles the oblate axis of the assembly, once, and in another state the tube encircles the oblate axis twice. The twisted assembly has both curvature and torsion stress-energy. Riemannian spaces with curvature have been associated with the stress-energy of matter, but Riemannian spaces are torsion free. As such physical systems have torsion they are not describable by a classic Riemannain geomtry. An objective of torsion theories is to include the stress energy due to torsion into the analysis.

Although the Frenet-Serret theory is generated from a single parameter mapping s into a euclidean space of three variables $\{x, y, z\}$, the ideas can be extended to multiple parameter mappings. One and two parameter mappings into higher dimensional spaces always lead to integrable systems of ordinary differential equations. The first occurrence of non-integrability occurs for 3 parameters. The explicit mapping from one set of (coordinate) variables $\{y^a\}$ to another set of variables $\{x^k\}$ in terms of differentiable functions ϕ^k

$$\phi : y^a \Rightarrow x^k = \phi^k(y^a), \tag{1}$$

induces a linear relationship between the coordinate differentials,

$$d\phi : |dy^a\rangle \Rightarrow \left|dx^k\right\rangle = \left[\partial\phi^k(y^b)/\partial y^a\right] \; |dy^a\rangle \tag{2}$$

Working backwards, the differential equations (2) are said to be integrable to exactness. They have a unique solution (1) whose differentials reproduce the differential equations. For the integrable situations, the Jacobian matrix of functions

$$\left[F_a^k(y^b)\right] = \left[\partial\phi^k(y^b)/\partial y^a)\right] \tag{3}$$

can play the role of a basis set for a vector space, at least on subspaces of (y^b) where the determinant of the Jacobian matrix does not vanish. In four dimensions, the elements of the basis frame are vector columns of four components where each column vector is presumed to transform as a contravariant tensor. The set of four columns are often described as tetrads. While the Frenet theory (using metrical constraints) developed the basis frame in terms of an orthonormal set, the Jacobian mapping provides neither a normalized nor an orthogonal basis frame.

Once a basis frame, $[F]$, is established, it is often possible to construct other objects in the theory, called connections, $[C]$. The connections are matrices of differential 1-forms, that linearly connect differentials of the basis vectors to linear combinations of the basis elements. The idea can be interpreted as one of closure, where the differential of the set remains within the set and does not create something that is outside the set. On the domain of support of the determinant of the basis frame, it is possible to construct the elements of the connection by one differential process, and other, algebraic, processes. Recall that the domain of support implies that an inverse matrix of functions, $[G]$, can be determined algebraically, such that

$$[F] \circ [G] = [I]\,. \tag{4}$$

Differentiation of this matrix equation leads to a linear relationship between the differentials of the functions that define the basis frame, and the functions themselves. As

$$d\,[F] \circ [G] + [F] \circ d\,[G] = [0]\,, \tag{5}$$

post multiplication by $[F]$ yields either

$$d[F] - [F] \circ [C] = 0 \quad or \quad d[F] + [\Delta] \circ [F] = 0. \tag{6}$$

where $[C] = -d\,[G] \circ [F]$ and $[\Delta] = [F] \circ d[G]$.

The matrix $[C]$ is defined as the right Cartan matrix of connection 1-forms. The matrix $[\Delta]$ is defined as the left Cartan matrix of connection

1-forms. The matrix elements of the right Cartan connection matrix, $[C]$, are differential 1-forms, $C^a_{bc}dy^c$. For a holonomic mapping of an integrable differential system, the coefficients of C^a_{bc} are symmetric: $C^a_{bc} - C^a_{cb} \Rightarrow 0$. In a more general case, the anti-symmetric components are not zero and permit the definition of the (right Cartan affine) torsion coefficients, $T^a_{bc} = C^a_{bc} - C^a_{cb} \neq 0$. The integrable holonomic system is said to be free of affine torsion, $T^a_{bc} = 0$.

However, note that although the concepts developed above started from a set of differentials that were uniquely integrable, the method of deriving a connection depends only on the fact that there is at each parametric point of a domain a matrix of functions $[F]$ with a non-zero determinant. The last three equations above are applicable even though the differential equations defined by the basis frame are not integrable. For such non-integrable basis frames, the matrix elements of the right Cartan matrix have a certain asymmetry such that the affine Torsion components do not vanish.

$$T^a_{bc} = C^a_{bc} - C^a_{cb} \neq 0. \tag{7}$$

The details of this method with examples are described below. The key idea is that affine torsion is associated with situations where the differential equations are not integrable to exactness.

The anti-symmetric parts of the right Cartan connection $[C]$ are possibly the most common of the definitions of torsion used in current field theories. Although Eddington, in 1921, used the idea of anti-symmetric connection coefficients in attempts to unify gravity and electromagnetism, and Cartan about the same time developed the theory of spaces with torsion, and communicated these ideas with Einstein over the next 10 or more years, not much was accomplished with the concept of torsion. Schroedinger in the 30's thought that the inclusion of torsion would infuse new blood into general relativity, but his work stimulated very little response, relatively speaking. Just before WWII, L. Brillouin wrote (1938):

> "If one does not admit the symmetry of the (connection) coefficients, C^a_{bc} (in the notation above) , one obtains the twisted spaces of Cartan, spaces which scarcely have been used in physics to the present, but which seem to be called to an important role."

During and just after WWII Kondo in Japan and Bilby in the UK developed an application of the concept of torsion to the analysis of dislocations in solids. However, it took the hype of elementary particles, quantum gravity and string theory to rejuvenate interest in torsion. On the other hand, in 1997, Chandia and Zanelli (hep-th/97081380) wrote (apparently disregarding the dislocation theories)

> "Despite many years of research and a host of scattered and suggestive results, torsion has remained a curiosity in differential geometry which seems to have no consequences for the real world."

The attempts of the last twenty five years have led to many statements which do not apply outside of the now forgotten constraints that were used to generate them. They are statements which now have become propaganda. Examples are "the source of torsion is spin" and "the torsion field does not propagate" are not universally true, and are valid only relative to rather severe constraints made by their originators. On the otherhand, researchers "measuring anomalies" have grasped for the straws of "torsion" to explain their results, giving, in many cases, the concept of torsion somewhat of a "bad name".

Throughout the many attempts to incorporate torsion into physical theories, it is almost universally true that a certain fundamental concept is ignored or forgotten: that fact is that the torsion of an uniquely integrable system is zero; hence torsion is something that should be associated with lack of unique integrability. Besides the notion of "affine" torsion, there are field theories that use alternate definitions of torsion (such the Cartan Torsion 2-forms, and topological torsion) to be described below.

Successive exterior differentiation of the equations defining parallel transport (6) lead to what are called Cartan's first equations of structure, and Cartan's matrix of Curvature 2-forms, $[\Theta]$.

$$d[F]\circ[C]+[F]\circ d[C]=[F]\{[C]\hat{}[C]+d[C]\}=[F]\circ[\Theta] \tag{8}$$

Factoring out the basis frame leads to

$$\text{Cartan's}\ \ 1^{st}\ equations\ of\ structure:\{[C]\hat{}[C]+d[C]\}=[\Theta] \tag{9}$$

Further exterior differentiations of the equations of structure lead to what are called the Bianchi identities, but that is not of immediate utility to the purpose of this article.

Cartan's 1^{st} equations of structure were available to differential geometers through the metric theories of tensor calculus that were developed after Riemann's disclosures. The metric tensor itself is a suitable Frame Field $[F]$ and can be utilized to generate a Cartan Matrix of Connection Coefficients. In this case, the Cartan connection is identical to the Christoffel connection which is torsion free. All Riemannian spaces (which support a symmetric Frame field defined as the metric tensor) have a metric induced connection that is (affine) torsion free.

However there are frame fields that are not symmetric, and yet support connections. It is these anti-symmetric Frame Fields that may support non-zero "affine" torsion. Suppose that a Frame Field exists, or can be constructed by some reasonable argument, and leads to a system of differential equations. It is not known that the system can be generated from an integral map. It is not obvious that the map of perfect differentials on the initial domain y^a map uniquely into perfect differentials on the domain x^k. For example, given $[F]$, it is not obvious that the Frame matrix maps perfect differentials dy^a into perfect exact differentials, $\mathrm{d}x^k$. Let's study those cases where

$$[F_a^k] \circ |dy^a\rangle \Rightarrow \left|\sigma^k\right\rangle \tag{10}$$

such that the components of the vector $\left|\sigma^k\right\rangle$ are formally linear combinations of the dy^a. It is not obvious (nor true) that each component of the RHS can be represented by a perfect differential. If the RHS objects are perfect differentials, then $d(\left|\sigma^k\right\rangle) = 0$ and it follows that

$$d[F_a^k]\hat{}\ |dy^a\rangle = [F_b^k] \circ [C_{ac}^b] dy^c\hat{}dy^a \Rightarrow d\left|\sigma^k\right\rangle = 0. \tag{11}$$

Such constraints require that

$$C_{bc}^a - C_{cb}^a \Rightarrow 0 \tag{12}$$

Hence the Cartan matrix is free from affine torsion, when each component of $\left|\sigma^k\right\rangle$ is an exact single differential of a function.

The statement is also true if the vector of 1-forms, $\left|\sigma^k\right\rangle$, is closed, but not exact, as would be the case if the domain was not simply connected. As an example of such a closed but not exact situation, suppose the frame matrix was such that

$$\sigma^k = dy^k + \gamma_{ij}^k \{y^i dy^j - y^j dy^i)/\{(y^i)^2 + (y^j)^2\} \tag{13}$$

Then the harmonic terms with constant coefficients γ_{ij}^k have zero exterior derivative, and are therefor non exact but closed. In fluid dynamics language, each harmonic term can be associated with a circulation integral related to rotation, but the vorticity generated by each harmonic term is zero. An example of such a situation occurs in the map to spherical coordinates (see below). The bottom line is that such complications still yield a right Cartan matrix that is torsion free, but introduce "internal" degrees of freedom associated with each coordinate pairing. In 3 dimensions, the number of rotation degrees of freedom is 3; in 4 dimensions the number of rotation degrees of freedom is 6; in general, $n(n-1)/2$. (The comparison of these terms to linking integrals, angular momentum and spin orbit coupling will be deferred.)

If the vector of 1-forms, $\left|\sigma^k\right\rangle$, is not closed, $d\left|\sigma^k\right\rangle \neq 0$, then there exist anti-symmetric components to the Cartan matrix in the sense that $C_{bc}^a - C_{cb}^a \neq 0$. That is, if the Frame matrix maps perfect differentials into linear combinations of differentials which are not closed, the Frame matrix will generate "affine" torsion coefficients. (Recall that the anti-symmetry being discussed is *not* the matrix anti-symmetry, which is defined as $C_{bc}^a - C_{ac}^b$.). The difficulty is that there are certain classes of problems where the form σ^k is not closed, but admits an integrating factor, such that the product is closed, and there are cases where an integrating factor does not exist. The integrating factor does not exist if the topological torsion of the form σ^k is not zero. The topological torsion of each σ^k is defined as

$$Topo\log ical\ Torsion\ of\ each\ \sigma^k : \sigma^k\hat{}d\sigma^k \neq 0 \tag{14}$$

When the topological torsion 3-form vanishes, the Frobenius integrability theorem is satisfied and an integrating factor exists. When the topological torsion is not zero, there is no integrating factor, and the criteria for a symmetric torsion free connection is not satisfied. Examples of frame fields for which the topological torsion of one or more of the σ^k are given below for the Hopf map and the Instanton map.

The components of the vector of 1-forms constructed from a frame matrix acting on a set of perfect differentials forms a set of n-biens (vierbeins in 4D). The original concept of "n-beins" was defined in terms of a local coordinate system of geodesics, where, as in the Frenet analysis, the "n-beins" or base vectors were metrically defined as unit orthonormal vectors. Herein, the term anholonomic vierbein is used to describe anholonomic representations (linear differential forms) of coordinate differentials. No metric is required at this stage. The anholonomic vierbeins need not be normalized nor do they need to be orthogonal. They can be used in projective spaces where a metric does not exist.

A complication arises from the fact that there are many ways to construct vierbeins from the matrix $[F_a^k]$, its inverse, $[G_j^b]$, or the induced symmetric metric frame $[g] = [F]^{transpose}[\eta][F]$. For example, if the position vector $\mathbf{r}$ on y^a is mapped via frame matrix to a vector on X^k on x^k, then $X^k = [F] \circ |y^a\rangle$. Differentiation on both sides leads to

$$dX^k = d[F] \circ |y^a\rangle + [F] \circ d|y^a\rangle = [F]\{[C] \circ |y^a\rangle + d|y^a\rangle\} \tag{15}$$

If the frame matrix $[F]$ is constructed as an element of the orthonormal group, then the Cartan matrix is an antisymmetric matrix of differential forms. In 3D, the anti-symmetric components of the Cartan connection for a vector $\mathbf{\Omega}ds$ such that $[C/ds] \circ |y^a\rangle = \mathbf{\Omega} \times \mathbf{r}$. Division by ds leads to the classic "relative" velocity formula,

$$\mathbf{V} = [F]\{\mathbf{\Omega} \times \mathbf{r} + \mathbf{v}\} \tag{16}$$

A second differentiation leads to the Acceleration formula.

$$dV^k = [F]\{[C]\,\hat{}\,[C/ds] \circ |\mathbf{y}^a\rangle + 2[C/ds] \circ |\mathbf{v}^a\rangle + d[C/ds]\hat{}\,|v^a\rangle + d|\mathbf{v}^a\rangle\} \tag{17}$$

Symbolic substitution leads to the expression

$$\mathbf{A} = [F]\{\mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r}) + 2(\mathbf{\Omega} \times \mathbf{v}) + d\mathbf{\Omega}/ds \times \mathbf{v} + \mathbf{a}\} \tag{18}$$

For the orthonormal frame, it is apparent that the formula involves the centripetal acceleration, the Coriolis acceleration, terms due to rotational acceleration, and finally the local acceleration. The point to be made is that a *similar* 4 part decomposition occurs for Frame fields that are NOT orthonormal.

In addition to the basis frame at the point p, it is sometimes necessary to define a position vector from an origin, or from a perspective point in projective geometry, to the point p. The basis frame may be used to define the position vector $|y^a\rangle$ (or the definition of the "origin" from a point p) and the differential position vector, $|dy^a\rangle$, in terms of possibly non-exact differential 1-forms called anholonomic vierbeins, $|\sigma^a\rangle$,(in 4 dimensions). The key ingredient of the vierbein concept is that of the action of the reciprocal matrix $[G]$ on the coordinate differentials. (It is also possible to describe a (different) metric set of anholonomic vierbeins - see below).

The identity,

$$|dy^a\rangle = [F] \circ [G] \circ |dy^a\rangle = [F] \circ |\sigma^a\rangle \tag{19}$$

defines a set of anholonomic vierbein(s) as

$$|\sigma^a\rangle = [G] \circ |dy^a\rangle \tag{20}$$

In a general setting it is not at all clear that the exterior differential of the anholonomic vierbeins is zero, as it would be if they were integrable to exactness. Exterior differentiation of the definition (20) leads to the equation,

$$d[F] \circ |\sigma^a\rangle + [F] \circ d\,|\sigma^a\rangle = [F]\{[C] \circ |\sigma^a\rangle + d\,|\sigma^a\rangle\} = [F] \circ |\Sigma^a\rangle\,. \tag{21}$$

Factoring out the basis frame leads to the format,

$$\mathrm{Car\,tan's}\;\; 2^{nd}\; equations\; of\; structure : \{[C] \circ |\sigma^a\rangle + d\,|\sigma^a\rangle\} = |\Sigma^a\rangle \tag{22}$$

which is known as "Cartan's Second Structural Equation". The vector of two forms $|\Sigma^a\rangle$ are defined as Cartan's Torsion 2-forms. The relationship of this definition of torsion and the previous definitions of torsion is presented in detail below.

The first equations of structure depend upon the basis system, and the second equations of structure relate to the definition of a origin, or a (non transitive) point of "rotation" or "focus". Interweaved with these ideas is the concept of topological torsion, which is related to the concept of non-existence of a unique integral equivalent to a system of differential equations.

The Frenet theory is constrained by both a metric (euclidean) idea and an orthogonality idea. The two concepts are separate and can be individually relaxed. In the more general situation, the question arises as to how the basis frame, $[F]$, is to be selected or constructed, and how various basis frames can be classified. The most important feature of the Cartan right connection is that it defines the differential of any basis vector in terms of linear combinations of the original basis vectors. The concept is one of closure. The basis frames so generated can be classified by their group structure, forming equivalence classes often called gauge groups by physicists. However, the choice of a gauge group is arbitrary, and to insist that the gauge group is preserved under evolution is a constraint that may be interesting, but is a constraint that is not necessary.

Methods of constructing the Frame Field, the connection, and the vierbiens will be explained below. The bottom line is that care must be taken to define what type of torsion is being employed in physical applications. Each of the six definitions of torsion given above will be examined in detail, in hopes of clarifying intuitive and perhaps prejudice positions about torsion fields.

II. Frenet Torsion of a Space Curve

The classical Frenet analysis of a space curve is based on a map (a contravariant position vector, $\mathbf{R}(t)$) from a single parameter, t, (in physics called time) to a point, p, on the curve, C, in a space of N=3 dimensions. The method assumes the topology of kinematic perfection, in that the differential or Pfaffian expression, $d\mathbf{R}(t) - \mathbf{V}(t)dt$ is constrained to be zero, without fluctuations, on the domain of interest. In addition, the classical method presumes a geometrical constraint of isometry, where by it is possible, by reparameterization, $t \Rightarrow s$, to define a unit tangent vector $\mathbf{t}(s)$ everywhere along the curve. This (Lagrangian) technique will be developed below, and compared to a different (Eulerian) method which does not depend upon the concept of kinematic perfection, or normalization. This latter (more hydrodynamic) method presumes that a vector field, $\mathbf{V}(x, y, z)$ is specified on the domain, but does not claim a priori that the three Pfaffian constraints, $d\mathbf{R}(t) - \mathbf{V}(t)dt = \Delta\mathbf{X}$, are without fluctuations ($\Delta\mathbf{X} \neq 0$). The method appears to be applicable to problems which involve deformation and dissipative phenomena, and can be extended to include the topological evolution of fields. Although the topology of the field is not subsumed to be constrained by either a set of kinematic or a dynamic neighborhood conditions free from fluctuations, the topological constraint of continuous (but perhaps dissipative and hence irreversible) evolution relative to a single parameter of time does impose an integrability condition, which, surprisingly, constrains the fluctuation domain of the dynamics, but does not constrain the fluctuation domain of the kinematics.

A. Isometry vs. Deformation

Latent in the development of the classic Frenet theory is the concept of isometry, in which a neighborhood constraint is assumed over the domain of interest. That is, isometries define that subset of all possible transformations which preserve "size". Recall that isometries include rigid body motions which preserve both size and shape, but isometries permit changes of shape induced by bending without deformations associated with the concepts of stretching, compression or torsion twisting. An isometric transformation preserves the inner or DOT product of vectors, and, more generally, the distance between any pair of points. The DOT or scalar product permits the geometric definition of the two distinct ideas: orthogonality and normalization. "Size" is usually defined in terms of the distance concept. Recall that such constraints on evolutionary processes lead to the notion of the covariant derivative of tensor calculus, which is defined to preserve the line element.

However, physical systems admit a wider class of evolutionary process. Of special interest to this article are those evolutionary processes in which deformations are permitted. Such processes are impossible to describe by covariant transplantation processes. However, Cartan invented the concept of the Lie derivative which can be applied to non-isometric processes of deformation, and in particular to topologically changing processes which admit irreversible dissipation. An objective of this article is to extend the Frenet Cartan theory of curves (based on intrinsic geometrical properties which are invariants of isometries) to a Cartan-Frenet theory of fields (in which the invariants are topological properties, or invariants of deformations). The transplantation law to be utilized is the Lie derivative, a process that admits domain deformation. In addition, the Grassmann product is used to define the notion of orthogonality. Normality invariance of a scalar product is replaced by the measure invariance of a pseudo-scalar or density invariant. The fundamental invariant is presumed to be a measure, not a metric.

B. Fluctuations

To preview what is to be discussed, realize that the concept of an isometry is an equivalent class of mappings that preserve length. (Recall that the definition of a covariant derivative was based on preserving the infinitesimal distance between a pair of points). Once the constraint of isometry is relaxed, then the notion of "invariant" distance can be extended to include variations or fluctuations in the distance between pairs of points. Herein, these fluctuations will be treated in terms of deformation processes. In particular, the limiting processes that are often used to define the kinematic concept of velocity and the dynamic concept of force will not be assumed to constrain the system topology, a priori. Instead, the kinematic concept of velocity will be viewed as a constraint on the neighborhoods, and the constraints are not presumed to be exact in an isometric sense. That is,

$$d\mathbf{R}-\mathbf{V}dt=\Delta\mathbf{X}, \quad with\ \Delta\mathbf{X}\ \neq 0\ identically\ , and \tag{23}$$

$$d\mathbf{p}-\mathbf{f}dt=\Delta\mathbf{p},\ with\ \Delta\mathbf{p}\neq 0\ identically. \tag{24}$$

The fluctuations, $\Delta\mathbf{p},\Delta\mathbf{X}$, are vectors of 1-forms whose arguments are not necessarily closed nor integrable. The now classic Langevin method presumes that the fluctuation 1-forms are functions of time alone. Such an assumption is not employed herein, and is a point of departure from the Langevin developments.

C. A classical Frenet immersion. The motion of a particle

The usual derivation of the Frenet equations for the moving trihedron (Mobile Repere) subsumes an immersion from a 1 dimensional manifold of time, t, into the space $\{x,y,z\}$ over which is created a vector field, $\mathbf{V}(t)$. The methods implicitly depend upon the existence of an invariant inner or DOT product to build a basis frame, which starts from the concept of a unit tangent vector, $\mathbf{T}$. The idea of a 1-dimensional immersion effectively constrains the space kinematically such that

$$d\mathbf{R}(t)-\mathbf{V}(t)dt=0 \tag{25}$$

without fluctuations, where $\mathbf{R}(t)$ is the classic position vector from the origin. In terms of a Euclidean inner product or norm on $\{x, y, z\}$, the unit tangent field can be defined as $\mathbf{T}(t) = \mathbf{V}(t)/(\mathbf{V} \circ \mathbf{V})^{1/2}$such that,

$$\mathbf{T} = [\mathbf{T}_x(t), \mathbf{T}_x(t), \mathbf{T}_x(t)]. \tag{26}$$

The usual assumption is that there exists a parameter, s, defined to be the arclength, such that

$$d\mathbf{R}(s) - \mathbf{T}(s)ds = 0 \; without \; fluctuation. \tag{27}$$

This result follows directly from the assumed mapping. Multiply the equation above by $\mathbf{T}(s)$ to yield the differential 1-form (in the euclidean space $\mathbf{T}(s) = \mathbf{t}(s)$)

$$ds = \mathbf{t}_x dx + \mathbf{t}_y dy + \mathbf{t}_z dz. \tag{28}$$

Substitution of the definition of $\mathbf{t}(t) = \mathbf{V}(t)/(\mathbf{V} \circ \mathbf{V})^{1/2}$ and the expressions for $d\mathbf{R}(t)$ without fluctuations, yields the formula:

$$ds - (\mathbf{V} \circ \mathbf{V})^{1/2} dt = 0 \; without \; fluctuation. \tag{29}$$

This viewpoint is that of a classic "particle", where $\mathbf{V}(t)$ defines the kinematic trajectory of some infinitesimal "point" particle. Successive derivatives of the position vector with respect to the parameter of time can be converted by the Gramm-Schmidt process into a set of orthonormal base vectors, $\mathbf{T}(s), \mathbf{N}(s), \mathbf{B}(s)$. The derivatives of these base vectors with respect to arclength generate the classic Frenet equations:

$$d\mathbf{T}(s) = \mathbf{N}(s) \cdot k ds \tag{30}$$

$$d\mathbf{N}(s) = -\mathbf{T}(s) \cdot k ds + \mathbf{B}(s) \cdot \tau ds \tag{31}$$

$$d\mathbf{B}(s) = -\mathbf{N}(s) \cdot \tau ds \tag{32}$$

where $k(s)$ is the Frenet curvature, $\tau(s)$ is the Frenet torsion, and s is the arclength of the space curve.

Recall that Cartan was the champion of the Repere Mobile, of which the Frenet basis frame, $[\mathbf{F}] = [\ \mathbf{T}(s), \mathbf{N}(s), \mathbf{B}(s)]$ is the classic example. The derivatives of the basis vectors are presumed to be closed globally, which implies that the basis frame is an element of some group constraint placed upon the base space. (This group is often called the gauge group, but it should be considered as a topological constraint on the domain.). For the Frenet system, the group is the SO3, the normalized orthogonal group in three dimensions. As will be described later, the group constraint permits the exterior derivatives of the basis frame to be linearly connected to the elements of the basis frame itself. Symbolically,

$$d[\mathbf{F}] = [\mathbf{F}] \circ [\mathbf{C}] \tag{33}$$

where $[\mathbf{C}]$ is the Cartan matrix of connection 1-forms. For the Frenet system,

$$[\mathbf{C}] = \begin{bmatrix} 0 & kds & 0 \\ -kds & 0 & \tau ds \\ 0 & -\tau ds & 0 \end{bmatrix} \tag{34}$$

The entire Frenet construction is based on a "scalar" product concept preserving the unit norm of each basis vector. This assumption constrains the Frenet development such that the basis frame $[\mathbf{F}]$ must be an element of the orthonormal group. The assumption that $[\mathbf{F}]$ is orthonormal forces the Cartan matrix to be anti-symmetric. The classical Frenet analysis of the space curve is restricted therefor to isometric transformations! Later on, the development will be in terms of a projective basis, where by going to 1 dimension higher, the concept of an invariant inner product on N space is not needed.

It is a classic problem of vector analysis in euclidean 3 space to presume that the position vector is a given function of time, t, and then to compute the three parameters of the Frenet theory [Brand].

A most illustrative example is given by the twisted cubic,

$$\mathbf{R}(t) = \{2t, t^2, t^3/3\}, \tag{35}$$

for which

$$d\mathbf{R}(t) = \mathbf{V}(t)dt = \{2, 2t, t^2\}dt, \tag{36}$$

such that

$$(\mathbf{V}(t)\circ\mathbf{V}(t))^{1/2} = \sqrt{4 + 4t^2 + t^4} = 2 + t^2 \tag{37}$$

It follows that

$$s(t) = 2t + t^3/3 + constant, \tag{38}$$

$$k(t) = |\mathbf{V}(t) \times \mathbf{A}(t)| / (\mathbf{V}(t) \circ \mathbf{V}(t)^{3/2} = 2/(2+t)^2 \tag{39}$$

and

$$\tau(t) = d\mathbf{A}(t)/dt \circ \mathbf{V}(t) \times \mathbf{A}(t) / \{k^2 (\mathbf{V} \circ \mathbf{V})^3\} = 2/(2+t^2)^2. \tag{40}$$

Note that if the functions that define the position vector are less that cubic in time, the Frenet torsion coefficient vanishes. The twisted cubic permits the differential arc-length, ds, to be integrated in terms of the parameter, t. Note that if the velocity field is linear in t, then the Frenet torsion vanishes. If the motion is torsion free then the space curve resides in a plane.

TO BE ADDED 1. Darboux vector and the left Cartan matrix for a Frenet system

D. Intrinsic Space Curves

An interesting result of the Frenet analysis is that when the three parameters s, k, *and* τ have been computed, these functions are independent from the particular coordinates used to describe the space curve. These parameters are intrinsic to the space curve. Space curves that admit the same single values of Frenet curvature, k and Frenet torsion, τ, as functions of s, are congruent. The idea is that the variables of arc length, Frenet curvature and Frenet torsion may be used as "intrinsic" coordinates to describe a space curve, very much like the Cayley-Hamilton invariants of matrix theory.

Planar Examples: Coordinates in the plane are curvature k and arclength s

1. The Space Curve is a Euclidean Straight line:

The equivalent in the k, s plane is a straight line along the s axis, $k = 0$

2. The Space Curve is a Euclidean circle:

The equivalent in the k, s plane is a straight line parallel to the s axis. $k = s^0 = 1$

3. The Space Curve is a Logarithmic spiral:
   The equivalent in the k, s plane is a hyperbola: $k = s^{-1}$
4. The Space Curve is a Spiral:
   The equivalent in the k, s plane is a straight line. $k = s$.
5. The Space Curve is a Cornu spiral:
   The equivalent in the k, s plane is a quadratic $k = s^2$
6. The Space Curve is a Mushroom spiral:
   The equivalent in the k, s plane is a cubic $k = s^3$
7. The Space Curve emulates a Rayliegh-Taylor instability:
   The equivalent in the k, s plane is a harmonic $k = \cos^2(s)$
8. The Space Curve emulates a Kelvin-Helmholtz instability:
   The equivalent in the k, s plane is a harmonic $k = \cos^2(s)/\sin(s)$

III. The Frenet Helicity Torsion of a Bernoulli flow.

The Frenet theory of a space curve is useful for describing the "Lagrangian" evolution of a point particle, but for a fluid a bit more must be involved. The "particle" evolves along a curve, where in a fluid one is interested in evolution and variation in three spatial directions, not just one. In particular, for an Eulerian view of a fluid, the constraint of kinematic perfection without fluctuations must be abandoned, appearing as a useful result only in special cases (streamline flows). The fluid is also a compressible, deformable media, so the metric constraint of isometry in the Frenet theory of a particle also must be relaxed. A three dimensional variety will be subsumed at first.

In particular, the differential arc length (ds in the Frenet analysis) is no longer an exact differential. To prevent misinterpretation, the inexact differential of arc length will be defined by the symbol σ for the 1-form with coefficients from a covariant vector field, $\mathbf{t}(x, y, z)$:

$$\sigma = \mathbf{t}_x dx + \mathbf{t}_y dy + \mathbf{t}_z dz. \tag{41}$$

The fluid system is presumed to support a volume element, Ω, with density ρ such that

$$\Omega = \rho dx\hat{}dy\hat{}dz \tag{42}$$

Note that this volume element is zero if the constraints of kinematic perfection are presumed to be valid. Substitution of $dx = V^x dt$, $dy = V^y dt$, $dz = V^z dt$ into the formula for Ω yields zero.

Next consider a vector field $\mathbf{T}(x, y, z)$ such that the 2-form density T exists as

$$T = i(\mathbf{T})\Omega = \rho T^x dy\hat{}dz - \rho T^y dx\hat{}dz + \rho T^z dx\hat{}dy. \tag{43}$$

Constrain this vector field such that 3-form $a\hat{}T$ is the unit volume element:

$$\sigma\hat{}T = \{t_x T^x + t_y T^y + t_z T^z\} \cdot \Omega = 1 \cdot \Omega \tag{44}$$

This constraint is similar (but not exactly equal) to the normalization constraint of the classical Frenet method.

If $\mathbf{T}$ and $\mathbf{t}$ are defined as before, $\mathbf{t} = \mathbf{T} = \mathbf{V}/(V_x V^x + V_y V^y + V_z V^z)^{1/2}$, the construction for the arclength 1-form, σ, is almost identical to that used for the Frenet analysis of a point particle. However, there is one major difference. The Eulerian viewpoint specifies the components of velocity as functions of three parameters, $\{x, y, z\}$, and not of the single parameter, t. (Time dependent flows will be discussed later by using a 4 dimensional variety). The arclength is no longer, necessarily, a perfect differential, nor is it necessarily closed, $d\sigma \neq 0$. In fact for a given velocity field, $\mathbf{V}$, it is necessary to compute the class, or Pfaff dimension, of the differential of arclength, σ. There are three classes in three dimensions (herein vorticity is defined as the curl of the unit tangent field, $\mathbf{t}$) :

Pfaff Dimension 1	$\sigma \neq 0, d\sigma = 0$	Potential flow, vorticity $= 0$
Pfaff Dimension 2	$d\sigma \neq 0, \sigma\hat{}d\sigma = 0$	vorticity $\perp$ velocity, helicity $= 0$
Pfaff Dimension 3	$\sigma\hat{}d\sigma \neq 0$	Beltrami component, helicity $\neq 0$

(45)

In principle, the Pfaff dimension defines the minimum number of functions required to describe the differential form in the sense of a submersion (recall that the Frenet particle approach was based upon an immersion.) The key point is that in the Frenet theory, Frenet torsion implied that the space curve was immersed in three dimensions and could not be mapped to a two dimensional set. In the fluid, when $\sigma\hat{}d\sigma \neq 0$ then the minimum domain for the tangent field is three dimensional.

$$\text{Frenet theory torsion, } \tau \neq 0 \supset \text{three dimensions}$$
$$\text{Topological Torsion, } \sigma\hat{}d\sigma \neq 0, \supset \text{three dimensions.}$$

For the Frenet case, the vanishing of the Frenet torsion implies that the tangent to the curve resides *in* a plane surface. When the topological torsion for the cotangent field vanishes, the cotangent vector is *orthogonal to* a surface (not necessarily a plane).

It will be demonstrated below that the 3-form of topological torsion (in three dimensions) is given by the expression

$$\sigma\hat{}d\sigma = \mathbf{t} \cdot curl(\mathbf{t})\, \Omega = \{\mathbf{V} \cdot curl(\mathbf{V})/(\mathbf{V} \cdot \mathbf{V})\}\Omega \tag{46}$$

When the 1-form σ of arc length is integrable, the topological torsion vanishes. The function $\{\mathbf{V} \cdot curl(\mathbf{V})/(\mathbf{V} \cdot \mathbf{V})\}$ is defined as the Helicity density in hydrodynamics. The Helicity function (equivalent to the topological torsion function in 3 D) is closely related to the Frenet Torsion function of the previous section, but it is not precisely the same as Frenet Torsion without further constraints.

From the field (fluid) point of view, it is possible to define a covariant field in terms of the functions that makeup a given contravariant Vector field, $\mathbf{V}(x, y, z)$.

$$\mathbf{t}(x, y, z) = \{\mathbf{t}_x, \mathbf{t}_y, \mathbf{t}_z\} = \{V^x, V^y, V^z\}/\psi^{n/p}, \tag{47}$$

where $\psi^{n/p}(x, y, z)$ is a scaling function yet to be specified, but will be one of the possible Holder norms,

$$\psi^{n/p}(x, y, z) = \{a_x(V^x)^p + a_y(V^y)^p + a_z(V^z)^p\}^{n/p}. \tag{48}$$

If the domain is isotropic, a special choice for the Holder norm is the quadratic form, p=2,n=1

$$\psi^{n/p}(x, y, z) \Rightarrow \{(V^x)^2 + (V^y)^2 + (V^z)^2\}^{1/2} = \{\phi\}^{1/2}. \tag{49}$$

This classic choice will be presumed for this article. The features of other norms will be described elsewhere.

Use the scaled covariant field $\mathbf{t}(x, y, z)$to create a differential 1-form of arclength σ (σ is not exact, nor even integrable in the sense of Frobenius):

$$\sigma = \mathbf{t}_x dx + \mathbf{t}_y dy + \mathbf{t}_z dz. \tag{50}$$

The domain of interest is presumed to support a measure N-form, $\Omega = \rho dx\hat{}dy\hat{}dz$ in 3 dimensions. Use the velocity direction field, $\mathbf{V} = [V^x, V^y, V^z]$, to construct a rescaled tangent vector, $\mathbf{T} = \mathbf{V}/\{\phi\}^{1/2}$, and then use this rescaled vector to construct the adjoint N-1 form (density), T, such that

$$T = i(\mathbf{T})\Omega = \rho\{T^x dy\hat{}dz - T^x dx\hat{}dz + T^z dx\hat{}dy\}. \tag{51}$$

(In the more general case, the two scaling functions of the cotangent and tangent vectors need not be the same.) It follows, however, that the 3-form $s\hat{}T$ creates the N=3 measure, or Grassmann norm,

$$\sigma\hat{}T = \rho\{(V^x)^2 + (V^y)^2 + (V^z)^2\}/(\phi)\ dx\hat{}dy\hat{}dx = \Omega. \tag{52}$$

In other words, choose the exponents such that the zero form $i(\mathbf{T})\sigma = 1$.

If the measure Ω is an invariant with respect to $\mathbf{V}$, then the Lie derivative of Ω with respect to $\mathbf{V}$ must vanish.

$$L_{(\mathbf{V})}\Omega = d(i(\mathbf{V})\Omega) = \{div(\rho\mathbf{V})\}\Omega \tag{53}$$

In other words for an invariant measure relative to the direction field $\mathbf{V}$, the divergence of $\rho\mathbf{V}$ must vanish. recall that given $\mathbf{V}$, there is a unique direction field, $\mathbf{W}$, with an infinite number "integrating" factors, ρ, such that the current, $\mathbf{J} = \boldsymbol{\rho}\mathbf{W}$ is divergence free.

The contravariant scaled field $\mathbf{T}$ plays the role of a reciprocal vector to the covariant, $\mathbf{t}$, but the "Grassmann norm", $\sigma\hat{}T$, as a pseudoscalar density, is independent from metric, and does not transform as a scalar under functional substitution. The transformational properties of all p-forms are well defined in terms of the Jacobian pull-back, and therefor this concept of a "Grassman norm" is an idea free from metric or connection constraints.

The Lie derivative of the action, σ, with respect to the reparameterized Velocity field, $\mathbf{T}$, (the convective derivative of σ along $\mathbf{T}$) becomes:

$$L_{(\mathbf{T})}\sigma = i(\mathbf{T})d\sigma + d(i(\mathbf{T})\sigma) = i(\mathbf{T})d\sigma + 0 = k(x, y, z)\ n, \tag{54}$$

and is transverse (orthogonal) to the direction field. That is, the Lie derivative creates a new 1-form, n, which satisfies the equation, $i(\mathbf{T})n = 0$. It follows that the Grassman product also vanishes:

$$n\hat{}T = (T^x n_x + T^y n_y + T^z n_z)\ \rho dx\hat{}dy\hat{}dz = \{i(\mathbf{T})n\}\ \Omega = 0 \cdot\ \Omega = 0 \tag{55}$$

The Lie derivative process is the analog of the Frenet procedure where differentiation of the unit tangent vector creates the normal vector. The result permits the creation of the N-1 form $N = i(\mathbf{N})\Omega$,and the normal vector from the normalization constraint: $n\hat{}N = 1\Omega$.

In three dimensions there exist 3 functionally independent 1-forms, which are named $\{\sigma, n, b\}$. In three dimensions, b may be constructed from the components of the N-1 = 3 form created from the product $\sigma\hat{}n = i(\mathbf{B})\Omega$. This construction is analog to the Gibbs cross product construction in Frenet theory. The scaling components of b are deduced from the equation, $b\hat{}\sigma\hat{}n = 1\Omega$. It is also possible in 3 dimensions to decompose every 2-form, such as $d\sigma$ into three parts:

$$d\sigma = A_\sigma(\sigma\hat{}n) + B_\sigma(n\hat{}b) + C_\sigma(b\hat{}\sigma) \tag{56}$$

The coefficients are evaluate from the 3-forms

$$\sigma\hat{}d\sigma = B_\sigma(\sigma\hat{}n\hat{}b) = B_\sigma\Omega \tag{57}$$

$$n\hat{}d\sigma = C_\sigma(\sigma\hat{}n\hat{}b) = C_\sigma\Omega \tag{58}$$

$$b\hat{}d\sigma = A_\sigma(\sigma\hat{}n\hat{}b) = A_\sigma\Omega \tag{59}$$

The Lie derivative of the arclength with respect to the direction fields $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ become

$$L_{(\mathbf{T})}\sigma = i(\mathbf{T})d\sigma = A_\sigma n - C_\sigma b \tag{60}$$

$$L_{(\mathbf{N})}\sigma = i(\mathbf{N})d\sigma = -A_\sigma\sigma + B_\sigma b \tag{61}$$

$$L_{(\mathbf{B})}\sigma = i(\mathbf{B})d\sigma = -B_\sigma n + C_\sigma\sigma \tag{62}$$

Note that the Lie derivatives are with respect to 3 different direction fields, where in the Frenet analysis, the differentiations were performed relative to the same direction (second and third order differentiations.)

The components of the 1-form n have an interesting interpretation in 3 dimensions, because the components of the 2-form, $d\sigma$, can be put into correspondence with the Gibbs curl of the covariant field, $\mathbf{t}$. In fact, the curl of $\mathbf{t}$ generates the "Darboux vector" of the field:

$$Darboux\ vector\ field : curl\ \mathbf{t} = \{-1/2\ \nabla\phi \times \mathbf{V} + \phi\ curl\ \mathbf{V}\}/\phi^{3/2} \tag{63}$$

In classic Frenet theory, the dot product of the Darboux vector and the unit tangent field defines the torsion coefficient, $\tau = \mathbf{t} \circ curl\ \mathbf{t} = \mathbf{V} \circ curl\ \mathbf{V}/\phi = \mathbf{V} \circ curl\ \mathbf{V}/(\mathbf{V} \circ \mathbf{V})$, and is equivalent to the three form density coefficient of the topological torsion created from the 1-form of arc length:

$$\sigma\hat{}d\sigma = \tau\ \rho dx\hat{}dy\hat{}dz = (\mathbf{t} \circ curl\ \mathbf{t})\ \rho dx\hat{}dy\hat{}dz \tag{64}$$

The topological torsion can always be evaluated for any 1-form on any domain, and is equal to the Frenet torsion of a fiber for the special constraints of normalization that correspond to the orthonormal group.

The N-1 form density, N, adjoint to the 1 form n is defined by the Grassman equation,

$$n\hat{}N = \rho dx\hat{}dy\hat{}dx = \Omega, \tag{65}$$

and has a representation as the contravariant vector $\mathbf{N}$, such that $N = i(\mathbf{N})\Omega$. The direction field of $\mathbf{N}$ is presumed to be proportional to the components of n but scaled by the factor, $\gamma_{\mathbf{N}}$, such that $\gamma_{\mathbf{N}}\{\{n_x^2 + n_y^2 + n_z^2\} = 1$

A third contravariant vector is produced in 3D by the Gibbs product of $\mathbf{V}$ and $\mathbf{N}$, and more generally in terms of the exterior product:

$$B = i(\mathbf{V} \times \mathbf{N})\rho dx\hat{}dy\hat{}dx = \sigma\hat{}n. \tag{66}$$

The "hydrodynamic Frenet equations are deduced by forming the convective Lie derivative of 1-form of action arclength, σ, with respect to the three "orthogonal directional fields $\mathbf{T}, \mathbf{N}, \mathbf{B}$.

$$L_{(\mathbf{V})}\sigma = k\,n. \tag{67}$$

$$L_{(\mathbf{N})}\sigma = -k\,\sigma + \varpi\,b. \tag{68}$$

$$L_{(\mathbf{B})}\sigma = -\varpi\,n. \tag{69}$$

ϖ is defined as the abnormality of the field,

$$\varpi = \mathbf{V} \circ curl\mathbf{V}/(\mathbf{V} \circ \mathbf{V}) = \mathbf{t} \circ curl\mathbf{t}. \tag{70}$$

Note that the abnormality, ϖ, may differ from the Frenet torsion, τ, for the vectors

These results are to be compared with the usual Frenet analysis in which successive derivatives in the same direction of the Tangent field, $\mathbf{T}$, are use to generate the classical Frenet structure. Successive Lie derivatives of the action, a, in the same direction of the tangent field , $\mathbf{T}$, field do not yield the Frenet results.

The three 1-forms, σ, n, b, form a natural volume element and a basis of 1-forms on the space {x,y,z}.

$$\sigma\hat{}n\hat{}b = \rho dx\hat{}dy\hat{}dz = \Omega. \tag{71}$$

The fundamental Grassman relations are:

$$\sigma\hat{}T = \Omega \quad \sigma\hat{}N = 0 \; \sigma\hat{}B = 0 \tag{72}$$

$$n\hat{}T = 0 \quad n\hat{}N = \Omega \quad n\hat{}B = 0 \tag{73}$$

$$b\hat{}T = 0 \quad b\hat{}N = 0 \quad b\hat{}B = \Omega \tag{74}$$

where $\Omega = \rho dx\hat{}dy\hat{}dz$.

The concept of orthogonality becomes the idea of measure zero! By forming the Lie derivative of the Grassman relations, a set of necessary conditions can be derived that preserve the Grassman norm relative to the Lie derivative. These equations are written below in 3-vector form, but it should be remembered that these equations are all pseudo scalars in which the density factor, ρ, has been suppressed.

$$L_{(\mathbf{T})}\{\sigma\hat{}N\} = 0 = div\mathbf{N} + \mathbf{T} \circ curl\mathbf{B} + \mathbf{N} \circ (curl\mathbf{T} \times \mathbf{T}) \tag{75}$$

$$L_{(\mathbf{T})}\{\sigma\hat{}B\} = 0 = div\mathbf{B} - \mathbf{T} \circ curl\mathbf{N} + \mathbf{B} \circ (curl\mathbf{T} \times \mathbf{T}) \tag{76}$$

$$L_{(\mathbf{N})}\{n\hat{}B\} = 0 = div\mathbf{B} + \mathbf{N} \circ curl\mathbf{T} + \mathbf{B} \circ (curl\mathbf{N} \times \mathbf{N}) \tag{77}$$

$$L_{(\mathbf{N})}\{n\hat{}T\} = 0 = div\mathbf{T} - \mathbf{N} \circ curl\mathbf{B} + \mathbf{T} \circ (curl\mathbf{N} \times \mathbf{N}) \tag{78}$$

$$L_{(\mathbf{B})}\{b\hat{}T\} = 0 = div\mathbf{T} + \mathbf{B} \circ curl\mathbf{N} + \mathbf{T} \circ (curl\mathbf{B} \times \mathbf{B}) \tag{79}$$

$$L_{(\mathbf{B})}\{b\hat{}N\} = 0 = div\mathbf{N} - \mathbf{B} \circ curl\mathbf{T} + \mathbf{N} \circ (curl\mathbf{B} \times \mathbf{B}) \tag{80}$$

The equations above are similar to the integrability conditions that lead to the Codazzi equations in the isometric setting. However, the set of transformations (Vector fields V) that satisfy the above constraints admit deformations and topological change.

In Frenet theory, when the Frenet torsion, τ, vanishes, the Tangent vector and the Normal vector reside in a plane. Hence the Binormal vector is representable by a gradient field and is integrable. The analog for the Eulerian fluid would be the requirement that

$$b\hat{}db = \tau\Omega \tag{81}$$

IV. The Torsion described by the anti-symmetric components of a right Cartan Connection.

In this section the focus is on the equation

$$d[\mathbf{F}] = [\mathbf{F}] \circ [\mathbf{C}] \tag{82}$$

where given a frame field of functions, $[\mathbf{F}]$ and an inverse $[\mathbf{G}]$ the right Cartan connection can be constructed in two ways:

$$[C] = -d\,[G] \circ [F] = [G] \circ d[F] \tag{83}$$

The Cartan (right matrix) of connection 1-forms implies that the differential of any column basis vector is a linear combination of all column basis vectors of the set.

$$d \left| \begin{matrix} e_a^1 \\ e_a^2 \\ e_a^3 \\ e_a^4 \end{matrix} \right\rangle = \left| \begin{matrix} e_1^1 \\ e_1^2 \\ e_1^3 \\ e_1^4 \end{matrix} \right\rangle C_{ac}^1 dy^c + \left| \begin{matrix} e_2^1 \\ e_2^2 \\ e_2^3 \\ e_2^4 \end{matrix} \right\rangle C_{ac}^2 dy^c + \left| \begin{matrix} e_3^1 \\ e_3^2 \\ e_3^3 \\ e_3^4 \end{matrix} \right\rangle C_{ac}^3 dy^c + \left| \begin{matrix} e_4^1 \\ e_4^2 \\ e_4^3 \\ e_4^4 \end{matrix} \right\rangle C_{ac}^4 dy^c \tag{84}$$

This is a concept of closure, a concept which is inherent in much of Cartan's work. The formulation is related to a "passive" interpretation of the action of the total differential on any basis column vector. The "affine" torsion of the Frame field is defined as the anti-symmetric combination

$$T^a_{bc} = C^a_{bc} - C^a_{cb} \neq 0. \tag{85}$$

The Frame matrix can be associated with a map into a euclidean space or a map from a euclidean space. If the differentials of the domain are presumed to be exact and differentials of the range are presumed to be closed (exterior differential is zero) then the right Cartan matrix relative to the frame is torsion free. If the range space differentials are exact then the right Cartan matrix is the same as the Christoffel connection for the induced metric on the domain space. If the map is to a euclidean space, then the right Cartan matrix is symmetric in the lower two indices. In the other hand if the map is from a euclidean space, the connection on the range space is not equal to the Christoffel connection, for the metric is constant on the range, and the Christoffel symbols are zero.

To cement the ideas it useful to give several examples. The first example will be for a map from spherical coordinates to euclidean coordinates in 3 space. The Frame matrix will be constructed from the Jacobian of the mapping. The second example will consider the map from a euclidean 3 space to spherical coordinates. Again the Frame matrix can be deduced easily. The third example will construct the frame matrix from a given 1-form of Action on a 4 dimensional variety. The fourth example will be based on the Hopf map where one of the vierbeins is not integrable but the other 3 vierbeins are integrable. The fifth example will correspond to the instanton system, where 3 of the vierbeins are not integrable, but one is integrable. Maple programs are available to do the computations.

1. Spherical Coordinates to Euclidean 3 Space.

An example is given by the map ϕ from $\xi^a = \{r, \theta, \varphi\}$ to $x^k = \{x, y, z\}$ given in terms of spherical coordinates :

$$\phi : \{r, \theta, \varphi\} \Rightarrow \{x, y, z\} = \{rsin(\varphi)cos(\theta), rsin(\varphi)sin(\theta), rcos(\varphi)\} \tag{86}$$

The induced differential map is given by the expression

$$d\phi \quad : \quad |d\xi^a\rangle \Rightarrow |dx^k\rangle = \left[\partial\phi^k(\xi^b)/\partial\xi^a\right] \circ |d\xi^a\rangle \tag{87}$$

$$\left|\begin{array}{c} dx \\ dy \\ dz \end{array}\right\rangle = \begin{array}{ccc} sin(\varphi)cos(\theta) & -r\sin(\varphi)\sin(\theta) & r\cos(\varphi)\cos(\theta) \\ sin(\varphi)sin(\theta) & r\sin(\varphi)\cos(\theta) & r\cos(\varphi)\sin(\theta) \\ cos(\varphi) & 0 & -r\sin(\varphi) \end{array} \circ \left|\begin{array}{c} dr \\ d\theta \\ d\varphi \end{array}\right\rangle \tag{88}$$

The matrix of partial differentials is the Jacobian matrix of functions with arguments on the initial state with independent variables, $\{r, \theta, \varphi\}$. No metric and no domain of support has been specified. In that which follows the domain of support is defined as that set of values ξ^b on the initial state, where the Jacobian determinant does not vanish ($r^2 \sin(\theta) \neq 0$). The Jacobian matrix can be viewed as a matrix of contravariant vectors (on the final state, x^k) in columns, and can be used as a basis frame (with arguments on the initial state ξ^b) on the domain of support (where $\det\left[\partial\phi^k(\xi^b)/\partial\xi^a\right] \neq 0$). That is, assume the basis frame is given by a set of contravariant columns with row index k and column index a and with arguments on ξ^b :

$$\left[F_a^k\right] = \left[\partial\phi^k(\xi^b)/\partial\xi^a\right] = \begin{array}{ccc} sin(\varphi)cos(\theta) & -r\sin(\varphi)\sin(\theta) & r\cos(\varphi)\cos(\theta) \\ sin(\varphi)sin(\theta) & r\sin(\varphi)\cos(\theta) & r\cos(\varphi)\sin(\theta) \\ cos(\varphi) & 0 & -r\sin(\varphi) \end{array} \tag{89}$$

As yet there has been no metric imposed upon the space, but even without specification of a metric it is possible to use the general formulas given above to define a Cartan connection

$$d\,[F] = [F] \circ [C] = [F] \circ [-d\,[G] \circ [F]] = [F] \circ [[G] \circ d\,[F]] \tag{90}$$

The right Cartan matrix becomes:

$$[C] = \begin{array}{ccc} 0 & -rd(\theta)(\sin\varphi)^2 & -rd(\varphi) \\ d(\theta)/r & d(r)/r + \cos\varphi d(\varphi)/\sin\varphi & \cos\varphi d(\theta)/\sin\varphi \\ d(\varphi)/r & -\cos\varphi\sin\varphi d(\theta) & d(r)/r \end{array} \tag{91}$$

The individual components to the connection C^b_{ac} can be read off from the matrix above to yield

$$\begin{array}{ccc} 0 & C^1_{22} = -r(\sin\varphi)^2 & C^1_{33} = -r \\ C^2_{12} = 1/r & C^2_{21} = 1/r,\ C^2_{23} = \cos\varphi/\sin\varphi & C^2_{32} = \cos\varphi/\sin\varphi \\ C^3_{13} = 1/r & C^3_{22} = -\cos\varphi/\sin\varphi & C^3_{31} = 1/r \end{array} \tag{92}$$

It is apparent that

$$T^a_{bc} = C^a_{bc} - C^a_{cb} \neq 0, \tag{93}$$

hence the Frame is torsion free (as expected for an integrable map).

In classical tensor analysis, the concept of an affine connection is associated with the (right) Cartan matrix (ref. L. Brand) as follows: (Remember, all the functions have arguments ξ^c of the initial domain of definition.)

$$\begin{aligned} d\left[F^k_a\right] &= \left[F^k_b\right] \circ \left[C^b_{ac} d\xi^c\right] = \left[F^k_b\right] \circ \left[\left[G^b_j\right] \circ d\left[F^j_a\right]\right] && (94) \\ &= \left[F^k_b\right] \circ \left[\left[G^b_j\right] \circ \left[\{\partial^2 \phi^j(\xi^m)/\partial\xi^c \partial\xi^a\} d\xi^c\right]\right]. && (95) \end{aligned}$$

As the system is integrable and (assumed) twice differentiable, it follows that the coefficient functions of the connection are symmetric

$$C^b_{ac} = C^b_{ca} = \left[G^b_j\right] \circ \left[\{\partial^2 \phi^j(\xi^m)/\partial\xi^c \partial\xi^a\} d\xi^c\right]. \tag{96}$$

The assumption that the order of partial derivatives is not important eliminates any lower index antisymmetry in the Cartan connection coefficients.

If one computes the pullback metric g_{ab} on the initial domain $\{\xi^c\}$ induced by the quadratic form on the final state, $\eta_{jk} dx^j dx^k$

$$[g_{ab}(\xi^c)] = \left[F^j_a\right] \circ \left[\eta_{jk}\right] \circ \left[F^k_b\right], \tag{97}$$

and then uses the classic Christoffel formulas for deriving a connection from a metric,

$$Christoffel : \{^{\,b}_{ac}\} = g^{be}\{\partial g_{ce}/\partial\xi^a + \partial g_{ea}/\partial\xi^c - \partial g_{ac}/\partial\xi^e\}. \tag{98}$$

it follows that, for a Jacobian basis frame, the Cartan connection is the same as the Christoffel connection, and the connection is (affine) torsion free:

$$If \;\; \left[F^k_a\right] = \left[\partial\phi^k(\xi^b)/\partial\xi^a\right], \, then \; C^b_{ac} = \{^{\,b}_{ac}\}. \tag{99}$$

For later comparison to Shipov's ideas note that in this integrable case,

$$C^b_{ac} - \{^{\,b}_{ac}\} = 0. \tag{100}$$

2. Euclidean 3 space to Spherical Coordinates.

The map from a Euclidean space to the space of spherical coordinates yields much different results. As the metric on the domain of definition is a set of constants, the Christoffel symbols must vanish. It is possible to generate a Frame matrix from the differentials of the coordinate mappings. It is to be expected that the Affine torsion of the right Cartan matrix is zero. Howeve, the components of the right Cartan connection are not zero. Hence, on the intitial domain, the difference between the Cartan connection and the Christoffel connection is equal to the Cartan connection, and is not equal to zero as in the previous example.

3. The Frame matrix generated from a 1-form of Action

later

4. The Frame matrix associated with the Hopf map.

later

5. The Frame matrix associated with the Instanton map.

later

A. The Cartan Torsion 2-forms.

later

B. The Frame field and Topological Torsion of a 1-form.

On a n+1 dimensional variety that supports a differential 1-form A it is possible to construct algebraically a Frame field that consists of n vectors orthogonal to the components of A. If the 1-form was a perfect differential, its components represent the "normal" field, $\mathbf{n}$, to a surface, and the n orthogonal vectors are called "tangent" vectors $\mathbf{e}$ to the surface. Keep this concept (of implicit surfaces) in mind, but now expand the idea to situations where the 1-form is not a perfect differential.

The objective is to construct the projective frame field from a given 1-form, and call attention to the fact that two types of torsion defects (both rotational and translational) can be generated on a projective manifold. Although the affine translational torsion has a growing literature, the projective rotational torsion has been ignored. Yet, the suggestion of this article is that rotational torsion, intuitively, seems to be of more importance for hydrodynamic situations.

Consider a 1-form of Action on a 4 dimensional domain of definition given by the expression,

$$\mathcal{A} = \{v_k(x, y, z, t)dx^k - cdt\}/\phi(x, y, z, t)$$

At any point p of the domain, there exists 3 vectors $\mathbf{e}_m$ of four components that are orthogonally transversal to the form in the sense that $i(\mathbf{e}_m)\mathcal{A} = 0$. These vectors (to within an arbitrary factor) may be used as column vectors of a basis frame at the point p. The coefficient functions of the one form itself (to within an arbitrary factor) form the 4 elements of a basis frame at the point p. Following the work of H. Flanders, a useful but not unique choice for a basis set at the point p is given by the expression,

$$\mathbb{F} = [\mathbf{e}_k, \mathbf{n}] = \mathbb{F} = \begin{bmatrix} 1 & 0 & 0 & -v_x/\phi \\ 0 & 1 & 0 & -v_y/\phi \\ 0 & 0 & c & -v_z/\phi \\ v_x/c & v_y/c & v_{/cz} & +c/\phi \end{bmatrix}.$$

The determinant of this matrix is equal $\det \mathbb{F} = (c^2 + A_x^2 + A_y^2 + A_z^2)/c\phi$, which is never zero for bounded coefficients. Hence this basis frame has an inverse almost everywhere.

The existence of the inverse matrix, of course, guarantees that the differential of every basis vector is decomposable into a linear combination of the original elements of the basis set, independent of the partition. The matrix of coefficients of this linear expansion defines the (right) Cartan matrix of connection 1-forms, $\mathbb{C}_r$,

$$d\mathbb{F} = \mathbb{F} \circ \{d\mathbb{F} \circ \mathbb{F}^{-1}\} = \mathbb{F} \circ \{-d\mathbb{F}^{-1} \circ \mathbb{F}\} = \mathbb{F} \circ \mathbb{C}_r$$

over the domain of support for the basis frame (where $\mathbb{F}^{-1}$ exists).

It is convenient to partition the (arbitrary) basis frame $\mathbb{F}$ in terms of the *associated* (horizontal, interior, coordinate or transversal) vectors, $\mathbf{e}_k$, and the *adjoint* (normal, exterior, parametric or vertical) field, $\mathbf{n}_p$,

$$\mathbb{F} = [\mathbf{e}_k, \mathbf{n}] = [\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{n}].$$

The corresponding Cartan matrix has the partition,

$$d\mathbb{F} = d\begin{bmatrix} e_1^1 & e_2^1 & ... & n^1 \\ e_1^2 & e_2^2 & ... & n^2 \\ ... & ... & ... & ... \\ e_1^n & e_2^n & ... & n^n \end{bmatrix} = \mathbb{F}\circ\mathbb{C} = \mathbb{F}\circ\begin{bmatrix} \Gamma_1^1 & \Gamma_2^1 & ... & \gamma^1 \\ \Gamma_1^2 & \Gamma_2^2 & ... & \gamma^2 \\ ... & ... & ... & ... \\ h_1 & h_2 & ... & \Omega \end{bmatrix}$$

The Cartan matrix, $\mathbb{C}$, is a matrix of differential 1-forms which can be evaluated explicitly from the functions that make up the basis frame if they admit first partial derivatives. Moreover, the differential of the position vector can be expanded in terms of the same basis frame and a set of Pfaffian 1-forms:

$$d\mathbf{R} = \mathbb{I}\circ\left|\begin{matrix} d\mathbf{x} \\ ... \\ d\mathbf{t} \end{matrix}\right\rangle = \mathbb{F}\circ\mathbb{F}^{-1}\circ\left|\begin{matrix} d\mathbf{x} \\ ... \\ d\mathbf{t} \end{matrix}\right\rangle = \mathbb{F}\circ\left|\begin{matrix} \boldsymbol{\sigma} \\ ... \\ \boldsymbol{\omega} \end{matrix}\right\rangle,$$

where the vector $\left|\begin{matrix} \boldsymbol{\sigma} \\ ... \\ \boldsymbol{\omega} \end{matrix}\right\rangle$ is a (4 component) vector of 1-forms that can be computed explicitly.

By the Poincare lemma, it follows that

$$dd\mathbb{F} = d\mathbb{F}\hat{}\mathbb{C} + \mathbb{F}\hat{}d\mathbb{C} = \mathbb{F}\circ\{\mathbb{C}\hat{}\mathbb{C}+d\mathbb{C}\} = 0,$$

and

$$dd\mathbf{R} = d\mathbb{F}\hat{}\left|\begin{matrix} \boldsymbol{\sigma} \\ \boldsymbol{\omega} \end{matrix}\right\rangle + \mathbb{F}\circ\left|\begin{matrix} d\boldsymbol{\sigma} \\ d\boldsymbol{\omega} \end{matrix}\right\rangle = \mathbb{F}\circ\{\mathbb{C}\hat{}\left|\begin{matrix} \boldsymbol{\sigma} \\ \boldsymbol{\omega} \end{matrix}\right\rangle + \left|\begin{matrix} d\boldsymbol{\sigma} \\ d\boldsymbol{\omega} \end{matrix}\right\rangle\} = 0.$$

These equations indicate that the Cartan curvature 2-forms and the Cartan torsion 2-forms vanish for the specified Frame.

As the Frame matrix and the Cartan matrix are partitioned relative to the tangent (or interior) vectors $\mathbf{e}$ and the normal (or exterior) vectors, $\mathbf{n}$, the

Poincare lemma breaks up into linearly independent factors, each of which must vanish. The results are:

$$dd\mathbf{R} = \mathbf{e}\{d\,|\boldsymbol{\sigma}\rangle + [\boldsymbol{\Gamma}]\hat{}\,|\boldsymbol{\sigma}\rangle - \omega\hat{}\,|\boldsymbol{\gamma}\rangle\} + \mathbf{n}\{d\omega + \Omega\hat{}\omega + \langle\mathbf{h}|\,\hat{}\,|\boldsymbol{\sigma}\rangle\} = 0$$

$$dd\mathbf{e} = \mathbf{e}\{d[\boldsymbol{\Gamma}] + [\boldsymbol{\Gamma}]\hat{}[\boldsymbol{\Gamma}] + |\boldsymbol{\gamma}\rangle\ \hat{}\ \langle\mathbf{h}|\} + \mathbf{n}\{d\,\langle\mathbf{h}| + \Omega\hat{}\,\langle\mathbf{h}| + \langle\mathbf{h}|\,\hat{}[\boldsymbol{\Gamma}]\} = 0$$

$$dd\mathbf{n} = \mathbf{e}\{d\,|\boldsymbol{\gamma}\rangle + [\boldsymbol{\Gamma}]\hat{}\,|\boldsymbol{\gamma}\rangle - \Omega\hat{}\,|\boldsymbol{\gamma}\rangle\} + \mathbf{n}\{d\Omega + \Omega\hat{}\Omega + \langle\mathbf{h}|\,\hat{}\,|\boldsymbol{\gamma}\rangle\} = 0$$

By reasons of linear independence, each of the curly bracket factors must vanish, leading to the results on the interior domain (coefficients of $\mathbf{e}$):

$$\begin{aligned} d\,|\boldsymbol{\sigma}\rangle + [\boldsymbol{\Gamma}]\hat{}\,|\boldsymbol{\sigma}\rangle &= \omega\hat{}\,|\boldsymbol{\gamma}\rangle \equiv |\boldsymbol{\Sigma}\rangle = \left|\begin{array}{l} \omega\hat{}\gamma^1 \\ \omega\hat{}\gamma^2 \\ \omega\hat{}\gamma^3 \end{array}\right\rangle \\ |\boldsymbol{\Sigma}\rangle &= \text{the interior torsion vector of dislocation 2-forms.} \end{aligned}$$

$$\begin{aligned} d[\boldsymbol{\Gamma}] + [\boldsymbol{\Gamma}]\hat{}[\boldsymbol{\Gamma}] &= -\,|\boldsymbol{\gamma}\rangle\ \hat{}\ \langle\mathbf{h}| \equiv [\boldsymbol{\Theta}] = \left[\begin{array}{ccc} \gamma^1\hat{}h_1 & \gamma^1\hat{}h_2 & \gamma^1\hat{}h_3 \\ \gamma^2\hat{}h_1 & \gamma^2\hat{}h_2 & \gamma^2\hat{}h_3 \\ \gamma^3\hat{}h_1 & \gamma^3\hat{}h_2 & \gamma^3\hat{}h_3 \end{array}\right] \\ [\boldsymbol{\Theta}] &= \text{the matrix of interior curvature 2-forms} \end{aligned}$$

$$\begin{aligned} d\,|\boldsymbol{\gamma}\rangle + [\boldsymbol{\Gamma}]\hat{}\,|\boldsymbol{\gamma}\rangle &= \Omega\hat{}\,|\boldsymbol{\gamma}\rangle \equiv |\boldsymbol{\Psi}\rangle = \left|\begin{array}{l} \Omega\hat{}\gamma^1 \\ \Omega\hat{}\gamma^2 \\ \Omega\hat{}\gamma^3 \end{array}\right\rangle \\ |\boldsymbol{\Psi}\rangle &= \text{the exterior torsion vector of disclination 2-forms.} \end{aligned}$$

The first two equations are precisely Cartan's equations of structure (on an affine domain). It is the last equation of exterior disclination 2-forms, $d\,|\boldsymbol{\gamma}\rangle + [\boldsymbol{\Gamma}]\hat{}\,|\boldsymbol{\gamma}\rangle = |\boldsymbol{\Psi}\rangle$, that appears to be a new equation of structure valid on a projective domain, when $\Omega \neq 0$. Further exterior differentiations lead to the Bianchi identities.

(later)

It is important to note that the method of construction leads to a Frame field such that the Cartan curvature 2-forms and the Cartan Torsion 2-forms both vanish. Such spaces are defined as spaces of absolute parallelism. The example demonstrates that an arbitrary 1-form of class 4, will generate an A4 space of the type studied by Shipov. The curvature of the subspaces and the torsion of the subspaces is not (necessarily) zero.

(Maple programs for computing these things will be attached later.)