The Many Faces of Torsion

R. M. Kiehn

Emeritus, Physics Dept., Univ. Houston
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http://www.cartan.pair.com
rkiehn2352@aol.com

Abstract

At least six definitions of "torsion" appear in the physics literature complicating the utilization and understanding of the concept. In this article, the differences and similarities of the various definitions are compared, and their relationships to spaces of absolute parallelism with applications to irreversible thermodynamics, coherent structures in plasmas and fluids, and general relativity are described.

1. Introduction

In this article the concept of torsion and torsion fields will be investigated. There are at least six different definitions of torsion to be found in the literature. They include:

1. The Frenet Torsion of a 3D space curve describing the dynamics of a particle.

$$2\tau_{Frenet\_Torsion} = t \circ curl\, t - n \circ curl\, n - b \circ curl\, b, \tag{1.1}$$

$$\text{in terms of orthonormal Basis} : [\text{tangent, normal , binormal}] \tag{1.2}$$

2. The Helicity (Torsion) of a fluid flow or a plasma (deformations).

$$helicity = V \circ curl\, V, \tag{1.3}$$

$$V = \text{Flow Velocity} \tag{1.4}$$

3. The Cartan Torsion 2-forms in terms of the left Cartan connection.

$$\Sigma_{Vector\_of\_Car\tan\_torsion\_2-forms} = d\left|\sigma^k\right\rangle - [\mathbb{C}_{left}]\hat{}\left|\sigma^k\right\rangle \tag{1.5}$$

4. The "Affine Torsion" 2-forms in terms of the Right Cartan connection (not equal to the Cartan Torsion 2-forms)

$$\Sigma_{Vector\_of\_Affine\_torsion\_2-forms} = [\mathbb{C}_{right}]\hat{}\left|\sigma^k\right\rangle \tag{1.6}$$

5. The Topological Torsion of a 1-form.

$$\text{Topological Torsion } = \sigma\hat{}d\sigma \tag{1.7}$$

6. The Torsion of elastic deformation defined by the map of closed 1-forms:

$$\begin{aligned} \text{Elastic} &: \text{Torsion Map} && (1.8)\\ X &= -kyz && (1.9)\\ Y &= +kxz && (1.10)\\ Z &= kz && (1.11) \end{aligned}$$

$$\text{Jacobian matrix of theTorsion Map}\begin{bmatrix} 0 & -(kz) & -ky \\ (kz) & 0 & +kx \\ 0 & 0 & k \end{bmatrix} \tag{1.12}$$

Each of these definitions have similarities as well as domains of agreement, but they are not precisely equivalent under all situations.

Parallel displacement of a vector along a path in a Riemannian space does not necessarily preserve the direction of the vector, if the Riemannian curvature is non-zero. The parallel transport of a vector around a closed path in a Riemannian space may yield a vector with different direction, but with the *same length*, when the transport path returns to the origin. Eddington, in 1921 [1], became interested in a suggestion of Weyl [2] whereby displacements of a vector may not only change the direction, but could also change "length". Thereby length could depend upon the displacement path, and could be "non-integrable". The Weyl idea generalized the concept of the Riemannian curvature tensor to admit certain symmetric components which would accomodate the idea that displacement would not preserve length. To quote Eddington: "..it is not possible to compare lengths (except zero-length) at different places, for the result of the comparison will depend upon the route taken in bringing the two lengths into juxtaposition". Eddington then introduced a definite but arbitrary "gauge-system" where the "logarithmic change" in length was equal to a linear differential 1-form.

$$\delta l/l = \delta(\log l) = \kappa_\mu dx^\mu \neq d(\log l). \tag{1.13}$$

It is to be noted that $\delta(\log l)$ is a symbol for a 1-form, and is NOT equal to a perfect differential. In otherwords, the exterior derivative of the "gauge-system", $\kappa_\mu dx^\mu$, is not zero. Eddington states that the conclusion that the vector κ_μ represents the covariant 4-vector of electromagnetism is irresistable. Eddington goes further to demonstrate that the "gauge-system" can be related to a conformal mapping where by

$$g'_{\mu\nu} \Leftarrow e^\varphi g_{\mu\nu} \text{ is related to } \kappa'_\mu dx^\mu \Leftarrow \kappa_\mu dx^\mu + d\varphi. \tag{1.14}$$

Using the idea that the length of a displaced vector may not be constant along a path, Eddington then reconstructed the idea of infinitesimal parallelism (which, at the time, was defined by a linear connection) and came to the conclusion that the (arbitrary) linear connection C^k_{mn} could be anti-symmetric in the lower two indices. Eisenhart then introduced the path dependence in terms of a chiral (plus or minus) connection where the chiral effect was associated with the anti-symmetry in the connection: $C^k_{[mn]} \neq 0$. In Riemannian geometry, the axiom of a metric leads to a Christoffel connection Γ^k_{mn} which is always symmetric: $\Gamma^k_{[mn]} = 0$.

The possibility of an anti-symmetry in the lower two indices of a connection has come to be known as "affine torsion". Such a description is unfortunate; the concept of a connection with antisymmetry is valid for vector basis frames that are not elements of the transitive (without fixed points) Affine group of 13 parameters in 4D. Vector basis frames of the intransitive 13 parameter group (where the origin is a fixed point) also admit anti-symmetry in the connection coefficients.

To distinguish between the two 13 parameter matrix groups, the name P-Affine will be reserved for the transitive group, and the word W-Affine will be used for the intransitive affine group. Both groups can have anti-symmetric components, but the anti-symmetric coefficients that lead to a vector of "affine torsion" two forms is not the same for the two groups. In addition there is a 10 parameter subgroup of both 13 parameter groups which historically was defined as the "Affine group with a fixed point [Turnbull], This subgroup will be called the "F-Affine" group.

This subgroup is a common subgroup of both the P-Affine group and the W-affine group.

Consider a simple[1] example of the P-Affine group written as:

$$\left[\mathbb{F}_{P\_Affine}\right] = \begin{bmatrix} 1 & 0 & 0 & -V^x \\ 0 & 1 & 0 & -V^y \\ 0 & 0 & 1 & -V^z \\ 0 & 0 & 0 & \pm\,\psi \end{bmatrix}, \tag{1.15}$$

$$\left[\mathbb{F}_{P\_Affine}\right] \circ |dy\rangle = \left| \begin{matrix} \sigma^x \\ \sigma^x \\ \sigma^x \\ \pm\,\sigma^t \end{matrix} \right\rangle = \left| \begin{matrix} dx - V^x dt \\ dy - V^y dt \\ dz - V^z dt \\ \pm\,\psi dt \end{matrix} \right\rangle \tag{1.16}$$

with a vector of "P-Affine Torsion 2-forms"

$$\left|\Sigma_{P-Affine\_torsion\_2-forms}\right\rangle = \left[\mathbb{C}_{P\_Affine\_right}\right]\hat{}\ |dy^m\rangle \tag{1.17}$$

$$\left|\Sigma_{P-Affine\_torsion\_2-forms}\right\rangle = \mathbb{C}\hat{}\ |dy^m\rangle = \left| \begin{matrix} -d(V^x)\hat{}d(t) + V^x d(\ln\psi)\hat{}d(t)) \\ -d(V^y)\hat{}d(t) + V^y d(\ln\psi)\hat{}d(t)) \\ -d(V^z)\hat{}d(t) + V^z d(\ln\psi)\hat{}d(t)) \\ \pm d(\ln\psi)\hat{}d(t) \end{matrix} \right\rangle \tag{1.18}$$

$$\neq \left|\Sigma_{Car\tan\_torsion}\right\rangle \tag{1.19}$$

Compare this to the simple example of a W-Affine group,

$$\left[\mathbb{F}_{W-Affine}\right] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ A_x & A_y & A_z & \pm\phi \end{bmatrix}, \tag{1.20}$$

$$\left[\mathbb{F}_{W\_Affine}\right] \circ |dy\rangle = \left| \begin{matrix} \sigma^x \\ \sigma^x \\ \sigma^x \\ \pm\,\sigma^t \end{matrix} \right\rangle = \left| \begin{matrix} dx \\ dy \\ dz \\ \pm\,(Action) \end{matrix} \right\rangle \tag{1.21}$$

$$\text{1-form of potentials} \ : \ Action = A_x dx + A_x dx + A_x dx \pm \phi dt. \tag{1.22}$$

[1]The Simple matrix format could be generalized to include a 3x3 antisymetric spatial rotational component, and 3x3 spatial dilation.

with a vector of "W-Affine 2-forms",

$$\left|\Sigma_{W-Affine\_torsion\_2-forms}\right\rangle = \left[\mathbb{C}_{W-Affine\_right}\right]\hat{}\ \left|dy^m\right\rangle \tag{1.23}$$

$$\left|\Sigma_{W-Affine\_torsion\_2-forms}\right\rangle = \left|\begin{matrix} 0 \\ 0 \\ 0 \\ \pm F/\phi \end{matrix}\right\rangle = \left|\begin{matrix} 0 \\ 0 \\ 0 \\ \pm d(Action)/\phi \end{matrix}\right\rangle \tag{1.24}$$

$$\neq \left|\Sigma_{Car\tan\_torsion}\right\rangle \tag{1.25}$$

Compare this to the simple example of a "Affine group with a fixed point" = "F-Affine" group,

$$[\mathbb{F}_{F-Affine}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \pm\psi \end{bmatrix}. \tag{1.26}$$

with a vector of "F-Affine" 2-forms,

$$\left|\Sigma_{F-Affine\_torsion\_2-forms}\right\rangle = \left[\mathbb{C}_{F-Affine\_right}\right]\hat{}\ \left|dy^m\right\rangle \tag{1.27}$$

$$\left|\Sigma_{F\_Affine\_torsion\_2\_forms}\right\rangle = \left|\begin{matrix} 0 \\ 0 \\ 0 \\ \pm d(\ln\psi)\hat{}dt \end{matrix}\right\rangle \tag{1.28}$$

$$\neq \left|\Sigma_{Car\tan\_torsion}\right\rangle \tag{1.29}$$

Note that the timelike anti-symmetries of the P-Affine group and the W-Affine group (the 4th component of the torsion 2-forms) are alway associated with a 2-form composed of two perfect differentials, $\pm d(\ln\psi)\hat{}d(t)$. The Pfaff topological dimension for σ^t is therefor 2. This result implies that the time-like 1-form, σ^t is integrable, and thermodynamically is reversible. The other 1-forms , (σ^x , σ^y , σ^z), depend upon "acceleration" in the P-Affine case, and are exact differentials in the W-Affine case.

The timelike symmetries of the W-Affine group however can be of Pfaff dimension 4, which implies that σ^t may not be integrable, could be path dependent and could be thermodynamically irreversible. The concept of a smooth rotation can lead to thermodynamic irreversibility, in the W-affine case, but when composed from two segments of the P-affine group, then σ^t is reversible. This effect was noticed in another manner in vol 1 [VOLUME1].

In the 10 parameter F-Affine subgroup, the system still supports "affine torsion" if the function ψ has a spatial dependence. In fact, for the example above, the antisymmetries of the connection for the intransitive W-Affine group are proportional to the components of of the exterior differential of a 1-form of Action; i.e., the antisymmetries can be put into correspondence with $F = dA$, the format of the electromagnetic field 2-form.

> *It is remarkable that, for the W-Affine group of Basis Frames, the coefficients of "W-Affine torsion" of the Right Cartan matrix of connection 1-forms are directly related to the structural format of the electromagnetic field 2-form, $F = dA$.*

The **Wave-Affine group** of Basis Frames has a connection with W-Affine Torsion 2-forms that are abstractly related to the covariant forces per unit charge of electromagnetic theory. The **Particle Affine group** of Basis Frames has a connection with P-Affine Torsion 2-forms which are directly related to contravariant Accelerations.

However, Riemannian spaces, defined as metric spaces with an associated Christoffel connection, are "affine torsion free". Spaces with an arbitrary connection which is symmetric in the lower indices are also "affine torsion free".

In the mid 20's, Eisenhart, Cartan, and Schouten were actively interested in the concept of "affine torsion", and its relationship to the notion of parallelism and path dependence of vector transport. Cartan, about the same time as Eddington, developed the theory of spaces with torsion and communicated many of the ideas (and extensions of "parallelism" to what became known as "absolute parallelism") to Einstein over the next 10 or more years. However the concept of torsion did not lead to much in the way of practical application. Schroedinger in the 30's thought that the inclusion of "affine torsion" would infuse new blood into general relativity, but his work stimulated very little response, relatively speaking. Just before WWII, L. Brillouin wrote :

> "*If one does not admit the symmetry of the (connection) coefficients, C^a_{bc}, one obtains the twisted spaces of Cartan, spaces which scarcely have been used in physics to the present, but which seem to be called to an important role.*" (1938)

During and just after WWII Kondo in Japan, and Bilby in the UK, developed an application of the concept of "affine torsion" to the analysis of dislocations in solids. However, it took the hype of elementary particles, quantum gravity and string theory to rejuvenate interest in torsion. On the other hand, in 1997, Chandia and Zanelli (hep-th/97081380) wrote (apparently disregarding the dislocation theories of Kondo and Bilby)

> *"Despite many years of research and a host of scattered and suggestive results, torsion has remained a curiosity in differential geometry which seems to have no consequences for the real world."*

The attempts of the last twenty five years have led to statements (in physical applications), which do not apply outside of the now forgotten constraints that were used to generate them. The differences between the various types of "affine torison" have been forgotten. They are now statements which have become propaganda. Examples of such current statements are "the source of torsion is spin" and "the torsion field does not propagate". These statements are not universally true, and are valid only relative to rather severe constraints placed on the mathematical systems studied by their originators. For example, It would seem obvious that the "W-Affine torsion, which is generated by the 2-form of an electromagentic like Action, should be capable of propagation. On the otherhand, researchers "measuring anomalies" have grasped for the straws of "torsion" to explain their results, giving the concept of torsion somewhat of a "bad name".

Throughout the many attempts to incorporate torsion into physical theories, it is almost universally true that a certain fundamental concept is ignored or forgotten: that fact is that the ubiquitous "affine" torsion is zero for any evolutionary process which is uniquely integrable and can be represented by a diffeomorphism; hence it would appear torsion is something that should be associated with lack of unique integrability and/or multiple solutions, and irreversibility, and defect regions of a domain where diffeomorphic properties fail.

There is another problem in that the literature presents several different, but related definitions of "torsion". One finds "Frenet torsion", "Affine torsion" of a connection, Cartan-Eisenhart left and right torsion, "Anholonomic torsion", Torsion 2-forms, Topological torsion, etc. In this article an attempt will be made to clarify the differences between these various definitions. Much can be learned by first studying "integrable" systems and their evolution, where the evolution is described by a map from a variety M to a variety N. First, the diffeomorphic class (dim M=dim N) will be studied, where both the map, its differential, the inverse map, and its differential exist, and are given. Note that the diffeomorphic map is at the heart of the concepts of tensor analysis, where tensors are defined in terms of the Jacobian matrix generated by such diffeomorphisms. Yet, as mentioned above, affine torsion is excluded from such evolutionary processes. Such systems are said to be "integrable" in the sense that the functional maps are given, and that the induced differential equations (which are to be integrated) have been given solutions from the data of the mapping.

The mapping generates topological problems when the dimension of M and the dimension of N are not the same. There are two important non-diffeomorphic cases to consider. The first defines a parametric map as an immersion from a space M to N, where dim M $<$ dim N. The "immersion" requires that the rank of the Jacobian matrix is equal to the dimension of M. The second case is projective "submersion", where the Jacobian of the mapping is of dimension N, and dim M $>$ dim N. This submersive mapping can be used to obtain an understanding of fiber bundles.

2. Frenet Torsion of a Space Curve

One of the earliest concepts of torsion came about through the classical Frenet-Serret analysis of a space curve in 3D. The concept is based on a map (a contravariant position vector, $\mathbf{R}(t)$) from a single parameter, t, (in physics called time) to a point, p, on the curve, C, in a space of N=3 dimensions. The method assumes the topology of kinematic perfection, in that the differential or Pfaffian expression, $d\mathbf{R}(t) - \mathbf{V}(t)dt$ is constrained to be zero, without fluctuations, on the domain of interest. In addition, the classical method presumes a geometrical constraint of isometry, where by it is possible, by reparameterization, $t \Rightarrow s$, to define a unit tangent vector $\mathbf{t}(s)$ everywhere along the curve. This (Lagrangian) technique will be developed below, and compared to a different (Eulerian) method which does not depend upon the concept of kinematic perfection, or normalization. This latter (more hydrodynamic) method presumes that a vector field, $\mathbf{V}(x, y, z)$ is specified on the domain, but does not claim apriori that the three Pfaffian constraints, $d\mathbf{R}(t) - \mathbf{V}(t)dt = \Delta\mathbf{X}$, are without fluctuations ($\Delta\mathbf{X} \neq 0$). The method appears to be applicable to problems which involve deformation and dissipative phenomena, and can be extended to include the topological evolution of fields. Although the topology of the field is not subsumed to be constrained by either a set of kinematic or a dynamic neighborhood conditions free from fluctuations, the topological constraint of continuous (but perhaps dissipative and hence irreversible) evolution relative to a single parameter of time does impose an integrability condition, which, surprisingly, constrains the fluctuation domain of the dynamics, but does not constrain the fluctuation domain of the kinematics.

At first, a space curve will be considered to be a 1D submanifold in Euclidean 3D space. Recall that in the theory of subspaces of Euclidean 3 dimensions there are two main methods of analysis. One method involves the parametric version of a surface as a 2 parameter map into the 3D Euclidean space, and the parametric version of a space curve as 1 parameter map into the 3D euclidean space. The other method involves the implicit version of a surface as represented by the null set of a single scalar function, and the implicit version of a space curve as the intersection of two implicit surfaces. The parametric method may be used to describe orientable or non-orientable 2D submanifolds of one component. The implicit method may be used to describe orientable 2D submanifolds of several components. Each of these methods will be used below to define various types of torsion and curvature.

Almost all concepts of curvature are related in one way or another to the idea of Cartan's Repere Mobile, which was developed from the Frenet-Serret theory of singly parameterized space curves. The construction of the Repere Mobile for a space curve starts with a parametric map of t into the Euclidean position vector, $\{\mathbf{x}^k(t)\}$. The position vector is a vector from a given (fixed) point, or origin, to a "moving" point p on the space curve C. At $t=0$, the map defines a particular point in the space $\mathbf{x}^k$, ususally defined as the "initial" conditions. Differential processes may be used to construct the kinematic tangent vector to the given space curve, defined as the Velocity vector in physics,

$$d\mathbf{x}^k(t)/dt = \mathbf{V}^k(t). \tag{2.1}$$

There exist another parameters, s, such that a choice $t = t(s)$, rescales the Velocity vector direction field at each point s along the curve such that :

$$\begin{aligned} \mathbf{V}^k(t) &\Rightarrow \mathbf{T}^k(s) = \mathbf{V}^k(t(s))/\lambda(s), && (2.2)\\ d\mathbf{x}^k(s)/ds &= \mathbf{T}^k(s). && (2.3) \end{aligned}$$

It is classic to specialize the new parameter, s, such that $\mathbf{T}^k(s)$ is of unit norm at all parametric points s along the space curve, C. Then, differential and algebraic processes are used to construct three orthonormal vectors $[\mathbf{T}, \mathbf{N}, \mathbf{B}]$ (unit tanget, normal and binormal direction fields) to form a basis Frame $[F] = [\mathbf{T}, \mathbf{N}, \mathbf{B}]$ at all points s along the curve at which the determinant of the Frame matrix $[F]$ is not zero.

A metric is subsumed on the euclidean 3-space, and a new parameter is defined as the arc length s.

$$d\mathbf{x}^k - \mathbf{t}^k ds = 0 \tag{2.4}$$

$$\left|\mathbf{t}^k(t)\right\rangle = (\partial t/\partial s) \left|\partial x^k(t)/\partial t\right\rangle \tag{2.5}$$

At a point p on a space curve C in 3 dimensional euclidean space, a basis frame of orthonormal vectors (unit tanget, normal and binormal) can be constructed by metric and differential processes. As the point p moves along the space curve, the basis vectors so defined are not constant, but change direction as the parameter s of the curve varies. The functions that define the components of these unit basis vectors form a matrix of differentiable functions, $[\mathbb{F}]$. This matrix of functions is defined as the Repere Mobile. The method generates 3 intrinsic parameters (arc length, s, Frenet curvature, k, and Frenet torsion, τ) that characterize the space curve. The Frenet torsion cannot be zero for space curves that are intrinsically 3 dimensional and can not be confined to a plane. The details of the method are explained below.

Although the Frenet-Serret theory is generated from a single parameter mapping s into a euclidean space of three variables {x,y,z}, the ideas can be extended to multiple parameter mappings. One and two parameter mappings into higher dimensional spaces always lead to integrable systems of ordinary differential equations. The first occurence of non-integrability occurs for 3 parameters. The explicit mapping from a set of parameters $\{s^a\}$ to another set of (coordinate) variables $\{x^k\}$ can be given in terms of differentiable functions ϕ^k

$$\phi : s^a \Rightarrow x^k = \phi^k(s^a), \tag{2.6}$$

induces a linear relationship between the coordinate differentials,

$$d\phi : |ds^a\rangle \Rightarrow \left|dx^k\right\rangle = \left[\partial\phi^k(s^b)/\partial s^a)\right] \ |ds^a\rangle \tag{2.7}$$

Working backwards, the differential equations (2.7) are said to be integrable to exactness. They have a unique solution (2.6) whose differentials reproduce the differential equations. For the integrable situations, the Jacobian matrix of functions

$$\left[F_a^k(s^b)\right] = \left[\partial\phi^k(s^b)/\partial s^a)\right] \tag{2.8}$$

can play the role of a basis set for a vector space, at least on subspaces of (s^b) where the determinant of the Jacobian matrix does not vanish. In four dimensions, the elements of the basis frame are vector columns of four components where each column vector is presumed to transform as a contravariant tensor. The set of four columns are often described as tetrads. While the Frenet theory (using metrical constraints) developed the basis frame in terms of an orthonormal set, the Jacobian mapping provides neither a normalized nor an orthogonal basis frame.

Extend the parametric idea to a map from a set of coordinate variables Y^a to a set of coordinate variables $\{X^k\}$. The coordinate variables can be partitioned into subspace coordinates and parameters, $\{Y^a\} = \{y^m; s^\sigma\}$ for the domain or initial state, and another set of coordinate variables and parameters, $\{X^k\} = \{x^k; t^\rho\}$ defined on the target or range or final state. If there exist C1 functional relationships between the final and intial states, such that

$$\phi : X^k = \phi^k(Y^a) \tag{2.9}$$

then there exists a linear relationship among the differentials:

$$d\phi : |dY^a\rangle \Rightarrow \left|dX^k\right\rangle = \left[\partial\phi^k(Y^b)/\partial Y^a)\right] \circ |dY^a\rangle = \left[F_a^k(Y^b)\right] \circ |dY^a\rangle . \tag{2.10}$$

Now it has been of interest to physics and mathematics to extend this idea by assuming that there exists a linear mapping on the column vector array of differentials of the initial state into a column vector of 1-forms on the final state:

$$\left[F_a^k(Y^b)\right] \circ |dY^a\rangle \Rightarrow \left|\sigma^k(Y^b, dY^b)\right\rangle . \tag{2.11}$$

The 1-forms $\sigma^k(Y^b, dY^b)$ are linear combinations of the differentials dY^b. The matrix of mapping functions on the initial state, $\left[F_a^k(Y^b)\right]$, will be described as a Basis Frame of functions on the domain where the determinant of basis frame matrix does not vanish. The 1-forms, $\left|\omega^k(Y^b, dY^b)\right\rangle$,so generated need not be exact differentials, nor need they be closed differentials. If the 1-forms are all closed then there does not exist a unique primitive set of mapping functions, ϕ, that defined the coordate variables X^k as functions, $\phi^k(Y^a)$ of Y^a. The linear system is said to be not integable.

The rows of the Basis Frame may be considered to be covariant vectors on the initial state; they are well defined functions on the initial state. On the other hand, the columns of the Basis Frame may be considered to be contravaraint vectors on the final state. These column vectors are not well defined on the final state as the arguements of the functions that make up the vector components are on the initial state; these vectors are well defined on the final state only if the inverse mapping is available. The more interesting cases in physics are when the inverse mapping does not exist.

Once a Basis Frame $[\mathbb{F}]$ is selected, and the domain is limited to that set of points for which the determinant does not vanish, then the Basis Frame $[\mathbb{F}]$ becomes a representative of the General Linear Group. An inverse matrix $[G]$ exists. It is then possible to derive a matrix of differential 1-forms that exhibit differential closure for the Basis Frame. That is the differential of every column is a linear combination of the columns of the Basis Frame. The matrix of differentials forms what is called a linear *connection*.

The connections are matrices of differential 1-forms, that linearly connect differentials of the basis vectors to linear combinations of the basis elements. The idea is one of closure, where the differential of the set does not create something that is outside the set. On the domain of support of the determinant of the basis frame, it is possible to construct the elements of the connection by one differential process, and other, algebraic, processes. Recall that the domain of support implies that an inverse matrix of functions, $[\mathbb{G}]$, can be determined algebraically, such that

$$[\mathbb{F}] \circ [\mathbb{G}] = [\mathbb{I}]\,. \tag{2.12}$$

Differentiation of this matrix equation leads to a linear relationship between the differentials of the functions that define the basis frame, and the functions themselves. As

$$d\,[\mathbb{F}] \circ [\mathbb{G}] + [\mathbb{F}] \circ d\,[\mathbb{G}] = [0]\,, \tag{2.13}$$

post multiplication by $[\mathbb{F}]$ yields either

$$d[\mathbb{F}] - [\mathbb{F}] \circ [\mathbb{C}_{right}] = 0 \quad or \quad d[\mathbb{F}] + [\mathbb{C}_{left}] \circ [\mathbb{F}] = 0. \tag{2.14}$$

where

$$[\mathbb{C}_{right}] = -d\,[\mathbb{G}] \circ [\mathbb{F}] \qquad and \qquad [\mathbb{C}_{left}] = [\mathbb{F}] \circ d[\mathbb{G}].$$

The matrix $[\mathbb{C}_{right}]$ is defined as the right Cartan matrix of connection 1-forms. The matrix $[\mathbb{C}_{left}]$ is defined as the left Cartan matrix of connection 1-forms. The matrix elements of the right Cartan connection matrix, $[\mathbb{C}_{right}]]$, are differential 1-forms, $C^a_{bc}dy^c$. For a holonomic mapping of an integrable differential system, the coefficients of C^a_{bc} are symmetric: $C^a_{bc} - C^a_{cb} \Rightarrow 0$. In a more general case, the anti-symmetric components are not zero and define the (right Cartan affine) torsion coefficients, $C^a_{[bc]} = C^a_{bc} - C^a_{cb} \neq 0$. The integrable holonomic system is said to be free of affine torsion, $C^a_{[bc]} = 0$.

However, note that although the concepts developed above started from a set of differentials that were uniquely integrable, the method depends only on the fact that there exists, at each parametric point of a domain, a matrix of functions $[\mathbb{F}]$ with a non-zero determinant. The last three equations above that are used to define a connection are applicable even though the differential equations defined by the basis frame are not integrable. For such non-integrable basis frames, the matrix elements of the right Cartan matrix have a certain asymmetry such that the affine Torsion components do not vanish.

$$C^a_{[bc]} = C^a_{bc} - C^a_{cb} \neq 0. \tag{2.15}$$

The details of this method with examples are described below. The key idea is that affine torsion is associated with situations where the differential equations are not integrable to exactness.

Consider a differentiable domain $\{y\}$ that supports an invertible Frame matrix, $[F(y)]$; then it is true that

$$[F^k_a(y)] \circ |dy^a\rangle \Rightarrow |\sigma^k\rangle .$$

However, the vector of 1-forms $|\sigma^k\rangle$ need not be composed of exact differentials. One or more of the 1-forms that make up the elements of $|\sigma^k\rangle$ may not be integrable. Each of the 1-forms that make up $|\sigma^k\rangle$ are either exact, closed, integrable or not integrable.

1. Exact means that the differential equations there exists a map ϕ from $\{y^a\}$ to $\{x^k\}$ such that the Frame $[F^k_a(y)]$ is the Jacobian matrix of the map. The differential equations, $|dx^k\rangle = |\sigma^k\rangle$ are said to be integrable. If follows that

$$\begin{aligned} d\,|\sigma^k\rangle &= d\,|dx^k\rangle \Rightarrow d[F^k_a(y)] \circ |dy^a\rangle = [F^k_a(y)] \circ [C^a_{bc} dy^c]\,\hat{}\,|dy^a\rangle \\ &\Rightarrow [F^k_a(y)] \circ |C^a_{[bc]} dy^c \hat{}\, dy^b\rangle = 0. \end{aligned}$$

Hence, if all of the induced 1-forms $|\sigma^k\rangle$ are exact, there is NO affine torsion, as right Cartan connection matrix elements $C^a_{[bc]}$ must vanish. The domain has no singularities.

2. Closed means that the exterior derivative of the 1-forms $|\sigma^k\rangle$ vanish. As in case 1 above, such spaces are free from Affine torsion. $d\,|\sigma^k\rangle = 0$. However, the

1-forms need not be exact, globally. There may not exist globally a single unique function on y whose total differential will represent the 1-form σ^k. In hydrodynamics, the analogue is that the 1-form σ^k has zero vorticity, but finite circulation. In electrodynamics it would be said that there exists a Bohm-Aharanov effect. In wave propagation, the phase admits a dislocation singularity.

3. Integrable means that the topological torsion must vanish. For any particular element σ^k, there exist the possibility that $d\sigma^k$ exists, but the 3-form $\sigma^k\hat{\ }d\sigma^k = 0$. The 3-form $\sigma^k\hat{\ }d\sigma^k$ is defined as the *topological torsion* of the 1-form σ^k. It is necessary and sufficient that if the 1-form σ^k admits an integrating factor, then the topological torsion 3-form must vanish. The integrability concept is usually defined as the Frobenius integrability theorem. When the Frobenius theorem is satisfied, then there are the possibilities that the 1-forms that compose $\left|\sigma^k\right\rangle$ admit integrating factors, $\lambda(y^a)$, such that $\lambda(y^a)\sigma^k$ is exact (over, perhaps, a limited domain. The individual 1-forms that make up $\left|\sigma^k\right\rangle$ are not closed:

$$d\left|\sigma^k\right\rangle = [F^k_a(y)] \circ \left|C^a_{[bc]}dy^c\hat{\ }dy^b\right\rangle \neq 0.$$

The anti-symmetric parts of the right Cartan connection $[C^a_{[bc]}]$ are not zero, and the non-zero symbols $C^a_{[bc]}$ define the components of the "Affine Torsion" tensor. Integrable but not closed systems admit "affine torsion", but such types of integrable "affine torsion" can be eradicated by a new choice of the Basis Frame. The concept is that such realizations of "affine torsion" are not irreducible.

In fact, if the 3-form of Topological Torsion of a particular 1-form, σ^k, vanishes , $\sigma^k\hat{\ }d\sigma^k = 0$, then integrating factors $\lambda^{(k)}$ exist such that by $d(\lambda^{(k)}\sigma^k) \Rightarrow 0$. If it is true that the topological torsion of each σ^k vanishes, then it is possible to multiply the original Basis Frame $[\mathbb{F}(y)]$ by a diagonal matrix of integrating factors, $\lambda^{(k)}(y)$, such that the modified Basis Frame $[\widehat{\mathbb{F}}(y)$ has zero "affine torsion". Moreover, all components σ^k which have zero topological torsion can transformed away by simply moultiplying the original basis frame by a diagonal basis frame consisting of those appropriate affine torsion" can be "transformed away" unless the components of $\left|\sigma^k\right\rangle$

$$\begin{aligned}
\text{If } [\widehat{\mathbb{F}}(y)] &= [(\lambda^{(k)}(y))F^k_a(y)] \\
[\widehat{\mathbb{F}}(y)] \circ |dy^a\rangle &= [\lambda^{(k)}(y)F^k_a(y)] \circ |dy^a\rangle = \left|\lambda^{(k)}\sigma^k\right\rangle \text{ such that} \\
d[\widehat{\mathbb{F}}(y)] \circ |dy^a\rangle &= d\left|\lambda^{(k)}\sigma^k\right\rangle \Rightarrow 0, \text{ if } \lambda^{(k)} \text{ is an integrating factor.}
\end{aligned}$$

This result implies that the new frame $\left[\lambda^k(y^e)\delta^k_m\right] \circ [F^m_a(y)] = [\widehat{F}^k_a(y)]$ is **free** from "Affine torsion" as

$$[\widehat{F}^k_a(y)] \circ |dy^a\rangle = \left|\widehat{\sigma}^k\right\rangle, \tag{2.16}$$

$$d\left|\widehat{\sigma}^k\right\rangle = d\{\left[\lambda^k(y^e)\delta^k_m\right] \circ |dy^m\rangle = 0. \tag{2.17}$$

. The domain of invertibility of the new frame can be significantly different from that of the old frame, as it depends on the determinant of diagonal array of integrating factors, which are functions of (x,y,z,t). The affine torsion is reducible, but a class of defect structures appear, compensating for the (perhaps) smaller domain of definition for the Frame matrix.

4. Non Integrable means that the vector of 3-forms $\left|\sigma^k\hat{}d\sigma^k\right\rangle \neq 0$. In this case, the Frame admits irreducible affine torsion.

Successive exterior differentiation of the equations defining parallel transport (2.14) lead to what are called Cartan's first equations of structure, and Cartan's matrix of Curvature 2-forms, $[\Theta]$.

$$d[\mathbb{F}] \circ [\mathbb{C}] + [\mathbb{F}] \circ d[\mathbb{C}] = [\mathbb{F}]\{[\mathbb{C}]\hat{}[\mathbb{C}] + d[\mathbb{C}]\} = [\mathbb{F}] \circ [\Theta] \tag{2.18}$$

Factoring out the basis frame leads to

$$\text{Cartan's } 1^{st} \text{ equations of structure} : \{[\mathbb{C}]\hat{}[\mathbb{C}] + d[\mathbb{C}]\} = [\Theta] \tag{2.19}$$

Further exterior differentiations of the equations of structure lead to what are called the Bianchi identities, but that is not of immediate utility.

Cartan's 1^{st} equations of structure were available to differential geometers through the metric theories of tensor calculus that were developed after Riemann's disclosures. The metric tensor itself is a suitable Frame Field $[F]$ and can be utilized to generate a Cartan Matrix of Connection Coefficients. In this case, the Cartan connection is identical to the Christoffel connection. All Riemannian spaces (which support a symmetric Frame field defined as the metric tensor) have a metric induced connection that is (affine) torsion free.

However there are frame fields that are not symmetric, and yet support connections. It is these anti-symmetric Frame Fields that may support non-zero "affine" torsion. Suppose that a Frame Field exists, but it is not known that the system was generated from an integral map. It is not obvious that the map of perfect differentials on the intitial domain y^a map into perfect differentials on the domain x^k. For example, given $[F]$, it is not obvious that the Frame matrix maps perfect differentials dy^a into perfect exact differentials. To cover those cases, write

$$[F_a^k] \circ |dy^a\rangle \Rightarrow \left|\sigma^k\right\rangle \tag{2.20}$$

If the RHS objects are perfect differentials, then $d(\left|\sigma^k\right\rangle) = 0$ and it follows that

$$d[F_a^k]\hat{}\ |dy^a\rangle = [F_b^k] \circ [C_{ac}^b] dy^c\hat{}dy^a \Rightarrow d\left|\sigma^k\right\rangle = 0. \tag{2.21}$$

Such constraints require that

$$C_{bc}^a - C_{cb}^a \Rightarrow 0 \tag{2.22}$$

Hence the Cartan matrix is free from affine torsion, when the RHS is an exact single differential of a function. The statement is also true if the vector of 1-forms, $\left|\sigma^k\right\rangle$, is closed, and not exact. If the vector of 1-forms, $\left|\sigma^k\right\rangle$, is not closed, $d\left|\sigma^k\right\rangle \neq 0$, then there exist anti-symmetric components to the Cartan matrix in the sense that $C_{bc}^a - C_{cb}^a \neq 0$. That is, if the Frame matrix maps perfect differentials into differentials which are not closed, the Frame matrix will also generate "affine" torsion coefficients. (Recall that the anti-symmetry being discussed is not the matrix anti-symmetry, which is defined as $C_{bc}^a - C_{ac}^b$.)

Special cases of linear independent columns of functions that make up a vector Basis Frame go by other names. Sometimes these matrix arrays of functions are called tetrads and some times they are called veilbeins (vierbeins in 4D) in the literature. The idea that Basis Frame operating on a vector of perfect exact differentials creates a vector array $\left|\sigma^k\right\rangle$ of exterior differential 1-forms is a different concept. The vector array of one forms is not the same as a matrix array of functions. The set of 1-forms $\left|\sigma^k\right\rangle$ may be exact, closed, or integrable or non-integrable. There no conventional name for the vector array of 1-forms. I will use the term "Basis Frame" or "Repere Mobile" for the matrix of Basis Functions $[\mathbb{F}]$, and will not use the words tetrads or vierbeins for such structures. I will use the word "Vierbein 1-forms" for the set $\left|\sigma^k\right\rangle$.

The Basis Frames need not be normalized nor do they need to be orthogonal. They can be used in projective spaces where a metric does not exist. There are two ways to construct Vierbein 1-forms: the first method creates a vector array of 1-forms by multiplying a vector of exact differentials by the Basis Frame matrix

$[F^k_a]$. A second method creates a different vector array of 1-forms by multiplying a vector of perfect differentials the inverse frame matrix, $[G^b_j]$.

For example, if the position vector $\mathbf{r}$ on y^a is mapped via frame matrix to a vector on X^k on x^k, then $X^k = [F] \circ |y^a\rangle$. Differentiation on both sides leads to

$$dX^k = d\,[\mathbb{F}] \circ |y^a\rangle + [\mathbb{F}] \circ d\,|y^a\rangle = [\mathbb{F}]\,\{[\mathbb{C}] \circ |y^a\rangle + d\,|y^a\rangle\} \tag{2.23}$$

If the frame matrix $[\mathbb{F}]$ is constructed as an element of the orthonormal group, then the Cartan matrix is an antisymmetric matrix of differential forms. In 3D, the anti-symmetric components of the Cartan connection for a vector $\mathbf{\Omega}ds$ such that $[\mathbb{C}/ds] \circ |y^a\rangle = \mathbf{\Omega} \times \mathbf{r}$. Division by ds leads to the classic formula,

$$\mathbf{V} = [\mathbb{F}]\,\{\mathbf{\Omega} \times \mathbf{r} + \mathbf{v}\} \tag{2.24}$$

A second application leads to the Acceleration formula.

$$dV^k = [\mathbb{F}]\,\{[\mathbb{C}]\,\hat{}\,[\mathbb{C}/ds] \circ |y^a\rangle + 2\,[\mathbb{C}/ds] \circ |v^a\rangle + d[\mathbb{C}/ds]\hat{}\,|v^a\rangle + d\,|v^a\rangle\} \tag{2.25}$$

Substitution leads to the expression

$$\mathbf{A} = [\mathbb{F}]\,\{\mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r}) + 2(\mathbf{\Omega} \times \mathbf{v}) + d\mathbf{\Omega}/ds \times \mathbf{v} + \mathbf{a}\} \tag{2.26}$$

For the orthonormal frame, it is apparent that the formula involves the centripetal acceleration, the Coriolis acceleration, terms due to rotational acceleration, and finally the local acceleration. The point to be made is that a similar 4 part decomposition occurs for Frame fields that are NOT orthonormal.

In addition to the basis frame at the point p, it is sometimes necessary to define a position vector from an origin, or from a perspective point in projective geometry, to the point p. The basis frame may be used to define the position vector $|y^a\rangle$ (or the definition of the "origin" from a point p) and the differential position vector, $|dy^a\rangle$, in terms of possibly non-exact differential 1-forms $|\varpi^a\rangle$,(in 4 dimensions). The key ingredient of the vierbein concept is that of the action of the Reciprocal matrix $[G]$ on the coordinate differentials.

The identity,

$$|dy^a\rangle = [F] \circ [G] \circ |dy^a\rangle = [F] \circ |\omega^a\rangle \tag{2.27}$$

defines the vector array of 1-forms:

$$|\varpi^a\rangle = [\mathbb{G}] \circ |dy^a\rangle \tag{2.28}$$

In a general setting it is not at all clear that the exterior differentiation of the vector array of 1-forms $|\varpi^a\rangle$ is zero, as it would be if the ϖ^a were integrable to exactness. Exterior differential of the definition (2.28) leads to the equation,

$$d[\mathbb{F}] \circ |\varpi^a\rangle + [\mathbb{F}] \circ d\,|\varpi^a\rangle = [\mathbb{F}]\{[\mathbb{C}] \circ |\varpi^a\rangle + d\,|\varpi^a\rangle\} = [\mathbb{F}] \circ |\Sigma^a\rangle\,. \tag{2.29}$$

Factoring out the basis frame leads to the equation

$$\text{Cartan's }\ 2^{nd} \text{ equations of structure} : \{[\mathbb{C}] \circ |\varpi^a\rangle + d\,|\varpi^a\rangle\} = |\Sigma^a(\varpi)\rangle \tag{2.30}$$

which is known as "Cartan's Second Structural Equation". The vector of two forms $|\Sigma^a\rangle$ are defined as Cartan's Torsion 2-forms. The relationship of this definition of torsion and the previous defintions of torsion is presented in detail below.

Successive exterior differentiation of the second equation of structure will generate (abstractly) the equivalent to Coriolis accelerations and centrifugal force found in, or relative to, rotating coordinate systems in classical mechanics. The first equations of structure depend upon the basis system, and the second equations of structure relate to the definition of a origin. Interweaved with these ideas is the concept of topological torsion, which is related to the concept of non-existence of a unique integral equivalent to a system of differential equations.

The Frenet theory is constrained by both a metric (euclidean) idea and an orthogonality idea. The two concepts are separate and can be individually relaxed. In the more general situation, the question arises as to how the basis frame, $[F]$, is to be selected or constructed, and how various basis frames can be classified. The most important feature of the Cartan right connection is that it defines the differential of any basis vector in terms of linear combinations of the original basis vectors. The concept is one of closure. The basis frames so generated can be classified by their group structure, forming equivalence classes often called gauge groups by physicists. However, the choice of a gauge group is arbitrary, and to insist that the gauge group is preserved under evolution is a constraint that may be interesting, but is a constraint that is not necessary.

Methods of constructing the Frame Field, the connection, and the vector of vierbein 1-forms will be explained below. Each of the six definitions of torsion given above will be examined in detail, in hopes of clarifying intuitive and perhaps prejudice positions about torsion fields.

3. Frenet Torsion of a Space Curve

The classical Frenet analysis of a space curve is based on a map (a contravariant position vector, $\mathbf{R}(t)$) from a single parameter, t, (in physics called time) to a point, p, on the curve, C, in a space of N=3 dimensions. The method assumes the topology of kinematic perfection, in that the differential or Pfaffian expression, $d\mathbf{R}(t) - \mathbf{V}(t)dt$ is constrained to be zero, without fluctuations, on the domain of interest. In addition, the classical method presumes a geometrical constraint of isometry, where by it is possible, by reparameterization, $t \Rightarrow s$, to define a unit tangent vector $\mathbf{t}(s)$ everywhere along the curve. This (Lagrangian) technique will be developed below, and compared to a different (Eulerian) method which does not depend upon the concept of kinematic perfection, or normalization. This latter (more hydrodynamic) method presumes that a vector field, $\mathbf{V}(x, y, z)$ is specified on the domain, but does not claim apriori that the three Pfaffian constraints, $d\mathbf{R}(t) - \mathbf{V}(t)dt = \Delta\mathbf{X}$, are without fluctuations ($\Delta\mathbf{X} \neq 0$). The method appears to be applicable to problems which involve deformation and dissipative phenomena, and can be extended to include the topological evolution of fields. Although the topology of the field is not subsumed to be constrained by either a set of kinematic or a dynamic neighborhood conditions free from fluctuations, the topological constraint of continuous (but perhaps dissipative and hence irreversible) evolution relative to a single parameter of time does impose an integrability condition, which, surprisingly, constrains the fluctuation domain of the dynamics, but does not constrain the fluctuation domain of the kinematics.

3.1. Isometry vs. Deformation

Latent in the development of the classic Frenet theory is the concept of isometry, in which a neighborhood constraint is assumed over the domain of interest. That is, isometries define that subset of all possible transformations which preserve "size". Recall that isometries include rigid body motions which preserve both size and shape, but isometries permit changes of shape induced by bending without deformations associated with the concepts of stretching, compression or torsion twisting. An isometric transformation preserves the inner or DOT product of vectors, and, more generally, the distance between any pair of points. The DOT or scalar product permits the geometric definition of the two distinct ideas: orthogonality and normalization. "Size" is usually defined in terms of the distance concept. Recall that such constraints on evolutionary processes lead to the notion of the covariant derivative of tensor calculus, which is defined to preserve the line element.

However, physical systems admit a wider class of evolutionary process. Of special interest to this arcticle are those evolutionary processes in which deformations are permitted. Such processes are impossible to describe by covariant transplantation processes. However, Cartan invented the concept of the Lie derivative which can be applied to non-isometric processes of deformation, and in particular to topologically changing processes which admit irreversible dissipation. An objective of this article is to extend the Frenet Cartan theory of curves (based on intrinsic geometrical properties which are invariants of isometries) to a Cartan-Frenet theory of fields (in which the invariants are topological properties, or invariants of deformations). The transplantion law to be utilized is the Lie derivative, a process that admits domain deformation. In addition, the Grassman product is used to define the notion of orthogonality. Normality invariance of a scalar product is replaced by the measure invariance of a pseudo-scalar or density invariant. The fundamental invariant is presumed to be a measure, not a metric.

3.2. Fluctuations

To preview what is to be discussed, realize that the concept of an isometry is an equivalent class of mappings that preserve length. (Recall that the definition of a covariant derivative was based on preserving the infinitesimal distance between a pair of points). Once the constraint of isometry is relaxed, then the notion of "invariant" distance can be extended to incude variations or fluctuations in the distance between pairs of points. Herein, these fluctuations will be treated in terms of deformation processes. In particular, the limiting processes that are often used to define the kinematic concept of velocity and the dynamic concept of force will not be assumed to constrain the system topology, apriori. Instead, the kinematic concept of velocity will be viewed as a constraint on the neighborhoods, and the constraints are not presumed to be exact in an isometric sense. That is,

$$d\mathbf{R} - \mathbf{V}dt = \Delta\mathbf{X}, \quad with\ \Delta\mathbf{X}\ \neq 0\ identically\ , and \tag{3.1}$$

$$d\mathbf{p} - \mathbf{f}dt = \Delta\mathbf{p},\ with\ \Delta\mathbf{p} \neq 0\ identically. \tag{3.2}$$

The fluctuations, $\Delta\mathbf{p}, \Delta\mathbf{X}$, are vectors of 1-forms whose arguments are not necessarily closed nor integrable. The now classic Langevin method presumes that the fluctuation 1-forms are functions of time alone. Such an assumption is not employed herein, and is a point of departure from the Langevin developments.

3.3. A classical Frenet immersion. The motion of a particle

The usual derivation of the Frenet equations for the moving trihedron (Mobile Repere) subsumes an immersion from a 1 dimensional manifold of time, t, into the space $\{x, y, z\}$ over which is created a vector field, $\mathbf{V}(t)$. The methods implicity depend upon the existence of an invariant inner or DOT product to build a basis frame, which starts from the concept of a unit tangent vector, $\mathbf{T}$. The idea of a 1-dimensional immersion effectively constrains the space kinematically such that

$$d\mathbf{R}(t) - \mathbf{V}(t)dt = 0 \tag{3.3}$$

without fluctuations, where $\mathbf{R}(t)$ is the classic position vector from the origin. In terms of a Euclidean inner product or norm on $\{x, y, z\}$, the unit tangent field can be defined as $\mathbf{T}(t) = \mathbf{V}(t)/(\mathbf{V} \circ \mathbf{V})^{1/2}$such that,

$$\mathbf{T} = [\mathbf{T}_x(t), \mathbf{T}_x(t), \mathbf{T}_x(t)]. \tag{3.4}$$

The usual assumption is that there exists a parameter, s, defined to be the arclength, such that

$$d\mathbf{R}(s) - \mathbf{T}(s)ds = 0\ without\ fluctuation. \tag{3.5}$$

This result follows directly from the assumed mapping. Multiply the equation above by $\mathbf{T}(s)$ to yield the differential 1-form (in the euclidean space $\mathbf{T}(s) = \mathbf{t}(s)$)

$$ds = \mathbf{t}_x dx + \mathbf{t}_y dy + \mathbf{t}_z dz. \tag{3.6}$$

Substitution of the definition of $\mathbf{t}(t) = \mathbf{V}(t)/(\mathbf{V} \circ \mathbf{V})^{1/2}$ and the expressions for $d\mathbf{R}(t)$ without fluctuations, yields the formula:

$$ds - (\mathbf{V} \circ \mathbf{V})^{1/2}dt = 0\ without\ fluctuation. \tag{3.7}$$

This viewpoint is that of a classic "particle", where $\mathbf{V}(t)$ defines the kinematic trajectory of some infinitesimal "point" particle. Successive derivatives of the position vector with respect to the parameter of time can be converted by the Gramm-Schmidt process into a set of orthonormal base vectors, $\mathbf{T}(s), \mathbf{N}(s), \mathbf{B}(s)$. The derivatives of these base vectors with respect to arclength generate the classic Frenet equations:

$$d\mathbf{T}(s) = \mathbf{N}(s) \cdot kds \tag{3.8}$$

$$d\mathbf{N}(s) = -\mathbf{T}(s) \cdot kds + \mathbf{B}(s) \cdot \tau ds \tag{3.9}$$

$$d\mathbf{B}(s) = -\mathbf{N}(s) \cdot \tau ds \tag{3.10}$$

where $k(s)$ is the Frenet curvature, $\tau(s)$ is the Frenet torsion, and s is the arclength of the space curve.

Recall that Cartan was the champion of the Repere Mobile, of which the Frenet basis frame, $[\mathbf{F}] = [\ \mathbf{T}(s), \mathbf{N}(s), \mathbf{B}(s)]$ is the classic example. The derivatives of the basis vectors are presumed to be closed globally, which implies that the basis frame is an element of some group constraint placed upon the base space. (This group is often called the gauge group, but it should be considered as a topological constraint on the domain.). For the Frenet system, the group is the SO3, the normalized orthogonal group in three dimensions. As will be described later, the group constraint permits the exterior derivatives of the basis frame to be linearly connected to the elements of the basis frame itself. Symbolically,

$$d[\mathbf{F}] = [\mathbf{F}] \circ [\mathbf{C}] \tag{3.11}$$

where $[\mathbf{C}]$ is the Cartan matrix of connection 1-forms. For the Frenet system,

$$[\mathbf{C}] = \begin{bmatrix} 0 & kds & 0 \\ -kds & 0 & \tau ds \\ 0 & -\tau ds & 0 \end{bmatrix} \tag{3.12}$$

The entire Frenet construction is based on a "scalar" product concept preserving the unit norm of each basis vector. This assumption constrains the Frenet development such that the basis frame $[\mathbf{F}]$ must be an element of the orthonormal group. The assumption that $[\mathbf{F}]$ is orthonormal forces the Cartan matrix to be anti-symmetric. The classical Frenet analysis of the space curve is restricted therefor to isometric transformations! Later on, the development will be in terms of a projective basis, where by going to 1 dimension higher, the concept of an invariant inner product on N space is not needed.

It is a classic problem of vector analysis in euclidean 3 space to presume that the position vector is a given function of time, t, and then to compute the three parameters of the Frenet theory [Brand].

A most illustrative example is given by the twisted cubic,

$$\mathbf{R}(t) = \{2t, t^2, t^3/3\}, \tag{3.13}$$

for which

$$d\mathbf{R}(t) = \mathbf{V}(t)dt = \{2, 2t, t^2\}dt, \tag{3.14}$$

such that

$$(\mathbf{V}(t)\circ\mathbf{V}(t))^{1/2} = \sqrt{4 + 4t^2 + t^4} = 2 + t^2 \tag{3.15}$$

It follows that

$$s(t) = 2t + t^3/3 + constant, \tag{3.16}$$

$$Frenet\ Curvature: \ \ k(t) = |\mathbf{V}(t) \times \mathbf{A}(t)| / (\mathbf{V}(t)\circ\mathbf{V}(t)^{3/2} = 2/(2 + t^2)^2, \tag{3.17}$$

and

$$Frenet\ Torsion: \ \tau(t) = d\mathbf{A}(t)/dt \circ \mathbf{V}(t) \times \mathbf{A}(t)/\{k^2(\mathbf{V} \circ \mathbf{V})^3\} = 2/(2 + t^2)^2. \tag{3.18}$$

As the ratio of torsion to curvature is a constant, the space curve defines a helix. Note that if the functions that define the position vector are less than cubic in time, the Frenet torsion coefficient vanishes. The twisted cubic example permits the differential arc-length, ds, to be integrated in terms of the parameter, t. Note that if the velocity field is linear in t, then the Frenet torsion vanishes. If the motion is torsion free then the space curve resides in a plane.

The unit vectors become:

$$
\begin{aligned}
\mathbf{T}(t) &= \{2, 2t, t^2\}/(2+t^2), && (3.19)\\
\mathbf{N}(t) &= \{-2t, 2-t^2, 2t\}/(2+t^2), && (3.20)\\
\mathbf{B}(t) &= \{t^2, -2t, 2\}/(2+t^2) && (3.21)
\end{aligned}
$$

Note that if the functions that define the position vector are less that cubic in time, the Frenet torsion coefficient vanishes. The twisted cubic permits the differential arc-length, ds, to be integrated in terms of the parameter, t. Note that if the velocity field is linear in t, then the Frenet torison vanishes. If the motion is torsion free then the space curve resides in a plane.

TO BE ADDED 1. Darboux vector and the left Cartan matrix

3.4. Intrinsic Space Curves

An interesting result of the Frenet analysis is that when the three parameters $s,\ k,\ and\ \tau$ have been computed, these functions are independent from the particular coordinates used to describe the space curve. These parameters are intrinsic to the space curve. Space curves that admit the same single values of Frenet curvature, k and Frenet torsion, τ, as functions of s, are congruent. The idea is that the variables of arc length, Frenet curvature and Frenet torsion may be used as "intrinsic" coordinates to describe a space curve, very much like the Cayley-Hamilton invariants of matrix theory.

Planar Examples: Coordinates in the plane are curvature k and arclength s

1. The Space Curve is a Euclidean Straight line:
   The equivalent in the k, s plane is a straight line along the s axis, $k = 0$
2. The Space Curve is a Euclidean circle:
   The equivalent in the k, s plane is a straight line parallel to the s axis. $k = s^0 = 1$
3. The Space Curve is a Logarithmic spiral:
   The equivalent in the k, s plane is a hyperbola: $k = s^{-1}$
4. The Space Curve is a Spiral:
   The equivalent in the k, s plane is a straight line. $k = s$.
5. The Space Curve is a Cornu spiral:
   The equivalent in the k, s plane is a quadratic $k = s^2$
6. The Space Curve is a Mushroom spiral:

The equivalent in the k, s plane is a cubic $k = s^3$

7. The Space Curve emulates a Rayliegh-Taylor instability:
   The equivalent in the k, s plane is a harmonic $k = \cos^2(s)$
8. The Space Curve emulates a Kelvin-Helmholtz instability:
   The equivalent in the k, s plane is a harmonic $k = \cos^2(s)/\sin(s)$

4. The Frenet Helicity Torsion of a Bernoulli flow.

The Frenet theory of a space curve is useful for describing the "Lagrangian" evolution of a point particle, but for a fluid a bit more must be involved. The "particle" evolves along a curve, where in a fluid one is interested in evolution and variation in three spatial directions, not just one. In particular, for an Eulerian view of a fluid, the constraint of kinematic perfection without fluctuations must be abandoned, appearing as a useful result only in special cases (streamline flows). The fluid is also a compressible, deformable media, so the metric constraint of isometry in the Frenet theory of a particle also must be relaxed. A three dimensional variety will be subsumed at first.

In particular, the differential arc length (ds in the Frenet analysis) is no longer an exact differential. To prevent misinterpretation, the inexact differential of arc length will be defined by the symbol σ for the 1-form with coefficients from a

covariant vector field, $\mathbf{t}(x, y, z)$:

$$\sigma = \mathbf{t}_x dx + \mathbf{t}_y dy + \mathbf{t}_z dz. \tag{4.1}$$

The fluid system is presumed to support a volume element, Ω, with density ρ such that

$$\Omega = \rho dx\hat{}dy\hat{}dz \tag{4.2}$$

Note that this volume element is zero if the constraints of kinematic perfection are presumed to be valid. Substitution of $dx = V^x dt$, $dy = V^y dt$, $dz = V^z dt$ into the formula for Ω yields zero.

Next consider a vector field $\mathbf{T}(x, y, z)$ such that the 2-form density T exists as

$$T = i(\mathbf{T})\Omega = \rho T^x dy\hat{}dz - \rho T^y dx\hat{}dz + \rho T^z dx\hat{}dy. \tag{4.3}$$

Constrain this vector field such that 3-form $a\hat{}T$ is the unit volume element:

$$\sigma\hat{\ }T = \{t_x T^x + t_y T^y + t_z T^z\} \cdot \Omega = 1 \cdot \Omega \tag{4.4}$$

This constraint is similar (but not exactly equal) to the normalization constraint of the classical Frenet method.

If $\mathbf{T}$ and $\mathbf{t}$ are defined as before, $\mathbf{t} = \mathbf{T} = \mathbf{V}/(V_x V^x + V_y V^y + V_z V^z)^{1/2}$, the construction for the arclength 1-form, σ, is almost identical to that used for the Frenet analysis of a point particle. However, there is one major difference. The Eulerian viewpoint specifies the components of velocity as functions of three parameters, $\{x, y, z\}$, and not of the single parameter, t. (Time dependent flows will be discussed later by using a 4 dimensional variety). The arclength is no longer, necessarily, a perfect differential, nor is it necessarily closed, $d\sigma \neq 0$. In fact for a given velocity field, $\mathbf{V}$, it is necessary to compute the class, or Pfaff dimension, of the differential of arclength, σ. There are three classes in three dimensions (herein vorticity is defined as the curl of the unit tangent field, $\mathbf{t}$) :

$$\begin{array}{lcl}
\text{Pfaff Dimension 1} & \sigma \neq 0, d\sigma = 0 & \text{Potential flow, vorticity } = 0 \\
\text{Pfaff Dimension 2} & d\sigma \neq 0, \sigma\hat{\ }d\sigma = 0 & \text{vorticity} \perp \text{velocity, helicity } = 0 \\
\text{Pfaff Dimension 3} & \sigma\hat{\ }d\sigma \neq 0 & \text{Beltrami component, helicity} \neq 0
\end{array} \tag{4.5}$$

In principle, the Pfaff dimension defines the minimum number of functions required to describe the differential form in the sense of a submersion (recall that the Frenet particle approach was based upon an immersion.) The key point is that in the Frenet theory, Frenet torsion implied that the space curve was immersed in three dimensions and could not be mapped to a two dimensional set. In the fluid, when $\sigma\hat{\ }d\sigma \neq 0$ then the minimum domain for the tangent field is three dimensional.

$$\text{Frenet theory torsion, } \tau \neq 0 \supset \text{three dimensions}$$
$$\text{Topological Torsion, } \sigma\hat{\ }d\sigma \neq 0, \supset \text{three dimensions.}$$

For the Frenet case, the vanishing of the Frenet torsion implies that the tangent to the curve resides *in* a plane surface. When the topological torsion for the cotangent field vanishes, the cotangent vector is *orthogonal to* a surface (not necessarily a plane).

It will be demonstrated below that the 3-form of topological torsion (in three dimensions) is given by the expression

$$\sigma\hat{}d\sigma = \mathbf{t}\cdot curl(\mathbf{t})\ \Omega = \{\mathbf{V}\cdot curl(\mathbf{V})/(\mathbf{V}\cdot\mathbf{V})\}\Omega \tag{4.6}$$

When the 1-form σ of arc length is integrable, the topological torsion vanishes. The function $\{\mathbf{V}\cdot curl(\mathbf{V})/(\mathbf{V}\cdot\mathbf{V})\}$ is defined as the Helicity density in hydrodynamics. The Helicity function (equivalent to the topological torsion function in 3 D) is closely related to the Frenet Torsion function of the previous section, but it is not precisely the same as Frenet Torsion without further constraints.

From the field (fluid) point of view, it is possible to define a covariant field in terms of the functions that makeup a given contravariant Vector field, $\mathbf{V}(x, y, z)$.

$$\mathbf{t}(x, y, z) = \{\mathbf{t}_x, \mathbf{t}_y, \mathbf{t}_z\} = \{V^x, V^y, V^z\}/\psi^{n/p}, \tag{4.7}$$

where $\psi^{n/p}(x, y, z)$ is a scaling function yet to be specified, but will be one of the possible Holder norms,

$$\psi^{n/p}(x, y, z) = \{a_x(V^x)^p + a_y(V^y)^p + a_z(V^z)^p\}^{n/p}. \tag{4.8}$$

If the domain is isotropic, a special choice for the Holder norm is the quadratic form, p=2,n=1

$$\psi^{n/p}(x, y, z) \Rightarrow \{(V^x)^2 + (V^y)^2 + (V^z)^2\}^{1/2} = \{\phi\}^{1/2}. \tag{4.9}$$

This classic choice will be presumed for this article. The features of other norms will be described elsewhere.

Use the scaled covariant field $\mathbf{t}(x, y, z)$to create a differential 1-form of arclength σ (σ is not exact, nor even integrable in the sense of Frobenius):

$$\sigma = \mathbf{t}_x dx + \mathbf{t}_y dy + \mathbf{t}_z dz. \tag{4.10}$$

The domain of interest is presumed to support a measure N-form, $\Omega = \rho dx\hat{}dy\hat{}dz$ in 3 dimensions. Use the velocity direction field, $\mathbf{V} = [V^x, V^y, V^z]$, to construct a rescaled tangent vector, $\mathbf{T} = \mathbf{V}/\{\phi\}^{1/2}$, and then use this rescaled vector to construct the adjoint N-1 form (density), T, such that

$$T = i(\mathbf{T})\Omega = \rho\{T^x dy\hat{}dz - T^x dx\hat{}dz + T^z dx\hat{}dy\}. \tag{4.11}$$

(In the more general case, the two scaling functions of the cotangent and tangent vectors need not be the same.) It follows, however, that the 3-form $s\hat{}T$ creates the N=3 measure, or Grassman norm,

$$\sigma\hat{}T = \rho\{(V^x)^2 + (V^y)^2 + (V^z)^2\}/(\phi)\ dx\hat{}dy\hat{}dx = \Omega. \tag{4.12}$$

In otherwords, choose the exponents such that the zero form $i(\mathbf{T})\sigma = 1$.

If the measure Ω is an invariant with respect to $\mathbf{V}$, then the Lie derivative of Ω with respect to $\mathbf{V}$ must vanish.

$$L_{(\mathbf{V})}\Omega = d(i(\mathbf{V})\Omega) = \{div(\rho\mathbf{V})\}\Omega \tag{4.13}$$

In otherwords for an invariant measure relative to the direction field $\mathbf{V}$, the divergence of $\rho\mathbf{V}$ must vanish. recall that given $\mathbf{V}$, there is a unique direction field, $\mathbf{W}$, with an infinite number "integrating" factors, ρ, such that the current, $\mathbf{J} = \boldsymbol{\rho}\mathbf{W}$ is divergence free.

The contravariant scaled field $\mathbf{T}$ plays the role of a reciprocal vector to the covariant, $\mathbf{t}$, but the "Grassman norm", $\sigma\hat{}T$, as a pseudoscalar density, is independent from metric, and does not transform as a scalar under functional substitution. The transformational properties of all p-forms are well defined in terms of the Jacobian pull-back, and therefor this concept of a "Grassman norm" is an idea free from metric or connection constraints.

The Lie derivative of the action, σ, with respect to the reparameterized Velocity field, $\mathbf{T}$, (the convective derivative of σ along $\mathbf{T}$) becomes:

$$L_{(\mathbf{T})}\sigma = i(\mathbf{T})d\sigma + d(i(\mathbf{T})\sigma) = i(\mathbf{T})d\sigma + 0 = k(x, y, z)\ n, \tag{4.14}$$

and is transverse (orthogonal) to the direction field. That is, the Lie derivative creates a new 1-form, n, which satisfies the equation, $i(\mathbf{T})n = 0$. It follows that the Grassman product also vanishes:

$$n\hat{}T = (T^x n_x + T^y n_y + T^z n_z)\ \rho dx\hat{}dy\hat{}dz = \{i(\mathbf{T})n\}\ \Omega = 0 \cdot\ \Omega = 0 \tag{4.15}$$

The Lie derivative process is the analog of the Frenet procedure where differentiation of the unit tangent vector creates the normal vector. The result permits the creation of the N-1 form $N = i(\mathbf{N})\Omega$,and the normal vector from the normalization constraint: $n\hat{}N = 1\Omega$.

In three dimensions there exist 3 functionally independent 1-forms, which are named $\{\sigma, n, b\}$. In three dimensions, b may be constructed from the components of the N-1 = 3 form created from the product $\sigma\hat{}n = i(\mathbf{B})\Omega$. This construction is analog to the Gibbs cross product construction in Frenet theory. The scaling components of b are deduced from the equation, $b\hat{}\sigma\hat{}n = 1\Omega$. It is also possible in 3 dimensions to decompose every 2-form, such as $d\sigma$ into three parts:

$$d\sigma = A_\sigma(\sigma\hat{}n) + B_\sigma(n\hat{}b) + C_\sigma(b\hat{}\sigma) \tag{4.16}$$

The coefficients are evaluate from the 3-forms

$$\sigma\hat{}d\sigma = B_\sigma(\sigma\hat{}n\hat{}b) = B_\sigma\Omega \tag{4.17}$$

$$n\hat{}d\sigma = C_\sigma(\sigma\hat{}n\hat{}b) = C_\sigma\Omega \tag{4.18}$$

$$b\hat{}d\sigma = A_\sigma(\sigma\hat{}n\hat{}b) = A_\sigma\Omega \tag{4.19}$$

The Lie derivative of the arclength with respect to the direction fields $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ become

$$L_{(\mathbf{T})}\sigma = i(\mathbf{T})d\sigma = A_\sigma n - C_\sigma b \tag{4.20}$$

$$L_{(\mathbf{N})}\sigma = i(\mathbf{N})d\sigma = -A_\sigma\sigma + B_\sigma b \tag{4.21}$$

$$L_{(\mathbf{B})}\sigma = i(\mathbf{B})d\sigma = -B_\sigma n + C_\sigma\sigma \tag{4.22}$$

Note that the Lie derivatives are with respect to 3 different direction fields, where in the Frenet analysis, the differentiations were performed relative to the same direction (second and third order differentiations.)

The components of the 1-form n have an interesting interpretation in 3 dimensions, because the components of the 2-form, $d\sigma$, can be put into correspondence with the Gibbs curl of the covariant field, $\mathbf{t}$. In fact, the curl of $\mathbf{t}$ generates the "Darboux vector" of the field:

$$Darboux\ vector\ field : curl\ \mathbf{t} = \{-1/2\ \nabla\phi \times \mathbf{V} + \phi\ curl\ \mathbf{V}\}/\phi^{3/2} \tag{4.23}$$

In classic Frenet theory, the dot product of the Darboux vector and the unit tangent field defines the torsion coefficent, $\tau = \mathbf{t}\circ curl\ \mathbf{t} = \mathbf{V}\circ curl\ \mathbf{V}/\phi = \mathbf{V}\circ curl\ \mathbf{V}/(\mathbf{V} \circ \mathbf{V})$, and is equivalent to the three form density coefficient of the topological torsion created from the 1-form of arc length:

$$\sigma\hat{}d\sigma = \tau\ \rho dx\hat{}dy\hat{}dz = (\mathbf{t}\circ curl\ \mathbf{t})\ \rho dx\hat{}dy\hat{}dz \tag{4.24}$$

The topological torsion can always be evaluated for any 1-form on any domain, and is equal to the Frenet torsion of a fiber for the special constraints of normalization that correspond to the orthonormal group.

The N-1 form density, N, adjoint to the 1 form n is defined by the Grassman equation,

$$n\hat{}N = \rho dx\hat{}dy\hat{}dx = \Omega, \tag{4.25}$$

and has a representation as the contravariant vector $\mathbf{N}$, such that $N = i(\mathbf{N})\Omega$. The direction field of $\mathbf{N}$ is presumed to be proportional to the components of n but scaled by the factor, $\gamma_{\mathbf{N}}$, such that $\gamma_{\mathbf{N}}\{\{n_x^2 + n_y^2 + n_z^2\} = 1$

A third contravariant vector is produced in 3D by the Gibbs product of $\mathbf{V}$ and $\mathbf{N}$, and more generally in terms of the exterior product:

$$B = i(\mathbf{V}\times\mathbf{N})\rho dx\hat{}dy\hat{}dx = \sigma\hat{}n. \tag{4.26}$$

The "hydrodynamic Frenet equations are deduced by forming the convective Lie derivative of 1-form of action arclength, σ, with respect to the three "orthogonal directional fields $\mathbf{T},\mathbf{N},\mathbf{B}$.

$$L_{(\mathbf{V})}\sigma = k\,n. \tag{4.27}$$

$$L_{(\mathbf{N})}\sigma = -k\,\sigma + \varpi\,b. \tag{4.28}$$

$$L_{(\mathbf{B})}\sigma = -\varpi\,n. \tag{4.29}$$

ϖ is defined as the abnormality of the field,

$$\varpi = \mathbf{V}\circ curl\mathbf{V}/(\mathbf{V}\circ\mathbf{V}) = \mathbf{t}\circ curl\mathbf{t}. \tag{4.30}$$

Note that the abnormality, ϖ, may differ from the Frenet torsion, τ, for the vectors

These results are to be compared with the usual Frenet analysis in which successive derivatives in the same direction of the Tangent field, $\mathbf{T}$, are use to generate the classical Frenet structure. Successive Lie derivatives of the action, a, in the same direction of the tangent field , $\mathbf{T}$, field do not yield the Frenet results.

The three 1-forms, σ, n, b, form a natural volume element and a basis of 1-forms on the space {x,y,z}.

$$\sigma\hat{}n\hat{}b = \rho dx\hat{}dy\hat{}dz = \Omega. \tag{4.31}$$

The fundamental Grassman relations are:

$$\sigma\hat{}T = \Omega \;\; \sigma\hat{}N = 0 \; \sigma\hat{}B = 0 \tag{4.32}$$

$$n\hat{}T = 0 \;\; n\hat{}N = \Omega \quad n\hat{}B = 0 \tag{4.33}$$

$$b\hat{}T = 0 \;\; b\hat{}N = 0 \;\; b\hat{}B = \Omega \tag{4.34}$$

where $\Omega = \rho dx\hat{}dy\hat{}dz$.

The concept of orthogonality becomes the idea of measure zero! By forming the Lie derivative of the Grassman relations, a set of neccesary conditions can be derived that preserve the Grassman norm relative to the Lie derivative. These equations are written below in 3-vector form, but it should be remembered that these equations are all pseudo scalars in which the density factor, ρ, has been supressed.

$$L_{(\mathbf{T})}\{\sigma\hat{}N\} = 0 = div\mathbf{N} + \mathbf{T} \circ curl\mathbf{B} + \mathbf{N} \circ (curl\mathbf{T} \times \mathbf{T}) \tag{4.35}$$

$$L_{(\mathbf{T})}\{\sigma\hat{}B\} = 0 = div\mathbf{B} - \mathbf{T} \circ curl\mathbf{N} + \mathbf{B} \circ (curl\mathbf{T} \times \mathbf{T}) \tag{4.36}$$

$$L_{(\mathbf{N})}\{n\hat{}B\} = 0 = div\mathbf{B} + \mathbf{N} \circ curl\mathbf{T} + \mathbf{B} \circ (curl\mathbf{N} \times \mathbf{N}) \tag{4.37}$$

$$L_{(\mathbf{N})}\{n\hat{}T\} = 0 = div\mathbf{T} - \mathbf{N} \circ curl\mathbf{B} + \mathbf{T} \circ (curl\mathbf{N} \times \mathbf{N}) \tag{4.38}$$

$$L_{(\mathbf{B})}\{b\hat{}T\} = 0 = div\mathbf{T} + \mathbf{B} \circ curl\mathbf{N} + \mathbf{T} \circ (curl\mathbf{B} \times \mathbf{B}) \tag{4.39}$$

$$L_{(\mathbf{B})}\{b\hat{}N\} = 0 = div\mathbf{N} - \mathbf{B} \circ curl\mathbf{T} + \mathbf{N} \circ (curl\mathbf{B} \times \mathbf{B}) \tag{4.40}$$

The equations above are similar to the integrability conditions that lead to the Codazzi equations in the isometric setting. However, the set of transformations (Vector fields V) that satisfy the above constraints admit deformations and topological change.

In Frenet theory, when the Frenet torsion, τ, vanishes, the Tangent vector and the Normal vector reside in a plane. Hence the Binormal vector is representable by a gradient field and is integrable. The analog for the eulerian fluid would be the requirement that

$$b\hat{}db = -\tau\Omega \tag{4.41}$$

The general relationship between Frenet torsion and topological torsion is that

$$\begin{aligned} 2\tau_{Frenet} &= t \circ curl\, t - n \circ curl\, n - b \circ curl\, b && (4.42) \\ &= t\hat{}dt - n\hat{}dn - b\hat{}db && (4.43) \end{aligned}$$

4.1. Part II (to follow soon)

4.2. The Torsion described by the anti-symmetric components of a right Cartan Connection.

4.3. The Torsion described by the anti-symmetric components of a left Cartan Connection.

4.4. The Cartan Torsion 2-forms.

4.5. The Topological Torsion of a 1-form.