ELECTROMAGNETIC WAVES IN THE VACUUM WHICH HAVE TORSION AND SPIN.

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Abstract. Wave solutions to the homogeneous Maxwell equations have been found that are not transverse, exhibit both torsion and spin, and for which the second Poincare invariant $\mathbf{E} \circ \mathbf{B} \neq 0$.

1. Introduction, the Domain of Classical Electromagnetism

The classic definition of the electromagnetic vacuum is a domain of space-time $\{x, y, z, t\}$ which supports both the Maxwell-Faraday equations,

$$curl\ \mathbf{E} + \partial\mathbf{B}/\partial t = 0, \quad div\ \mathbf{B} = 0, \tag{1}$$

and the Maxwell-Ampere equations,

$$curl\ \mathbf{H} - \partial\mathbf{D}/\partial t = \mathbf{J}, \quad div\ \mathbf{D} = \rho, \tag{2}$$

subject to the vacuum constraints that the charge-current densities are zero,

$$\mathbf{J} = 0, \quad \rho = 0, \tag{3}$$

and that the field excitations, **D** and **H**, are linearly connected to the field intensities, **E** and **B**, [1] by means of the Lorentz (homogeneous and isotropic) constitutive relations:

$$\mathbf{D} = \varepsilon\mathbf{E}, \quad \mathbf{B} = \mu\mathbf{H}. \tag{4}$$

The vacuum constraint that the charge current densities are zero implies that the solutions to the homogeneous Maxwell equations also satisfy the vector wave equation, typically of the form

$$grad\ div\ \mathbf{B} - curl\, curl\, \mathbf{B} - \varepsilon\mu\partial^2\mathbf{B}/\partial t^2 = 0. \tag{5}$$

The constant wave phase velocity, c, is taken to be

$$c^2 = 1/\varepsilon\mu \tag{6}$$

In addition, it is subsumed that the classic Maxwell electromagnetic system is constrained by the additional topological statement that the field intensities are deducible from a system of potentials, $[\mathbf{A},\phi]$:

$$\mathbf{B} = curl\,\mathbf{A},\quad \mathbf{E} = -grad\,\phi - \partial\mathbf{A}/\partial t. \tag{6}$$

This constraint topologically excludes domains of support for the covariant field intensities, **E** and **B**, which are compact domains without a boundary. It is this constraint that distinguishes classical electromagnetism from Yang Mills theories.

Besides the charge current 4-vector density, $[\mathbf{J},\rho]$, whose integral over any closed 3 dimensional manifold is a deformation invariant of the Maxwell system, there exist two other algebraic combinations of the fields and potentials that can lead to similar topological quantities. These objects are the Spin 4 vector, or current, defined as

$$\mathbf{S}_4 = [\mathbf{A}\times\mathbf{H} + \mathbf{D}\phi, \mathbf{A}\circ\mathbf{D}] \equiv [\mathbf{S},\sigma]. \tag{7}$$

and the Torsion 4 vector defined as

$$\mathbf{T}_4 = [\mathbf{E}\times\mathbf{A} + \mathbf{B}\phi, \mathbf{A}\circ\mathbf{B}] \equiv [\mathbf{T},h]. \tag{8}$$

The derivation of these 4-component fields and their topological implications is developed in section 3, below. The 4-divergence of these 4-component vectors leads to the Poincare projective invariants of the Maxwell system:

$$Poincare\ 1 = div_3(\mathbf{A}\times\mathbf{H} + \mathbf{D}\phi) + \partial(\mathbf{A}\circ\mathbf{E})/\partial t = (\mathbf{B}\circ\mathbf{H} - \mathbf{D}\circ\mathbf{E}) - (\mathbf{A}\circ\mathbf{J} - \rho\phi) \tag{9}$$

$$Poincare\ 2 = div_3(\mathbf{E}\times\mathbf{A} + \mathbf{B}\phi) + \partial(\mathbf{A}\circ\mathbf{B})/\partial t = -2\mathbf{E}\circ\mathbf{B} \tag{10}$$

Most of the time-dependent vacuum solutions to the Maxwell system of equations, which are found in the historic literature, typically are composed of simple ”globally transverse” or TEM waves, where not only the first and the second Poincare invariants are identically zero, but also the Torsion and the Spin vectors, as defined above, are identically zero. These constraints define what is meant by a TEM wave, for when both the Torsion and the Spin vectors vanish, then the vector potential, **A**, is orthogonal to both the **D** and the **B** fields. As a consequence, the vector potential for such systems is in the direction of the momentum flow, $\mathbf{D}\times\mathbf{B}$, of the radiative field. A simple TEM wave permits the global existence of a smooth surface (of constant phase) without self intersections or defects, a surface whose tangent manifold contains the fields **D** and **B** (and therefore in the vacuum, **E** and **H**).

There exist solutions to the Maxwell system in the vacuum for which either the Torsion 4-vector, or the Spin 4-vector, or both, are not identically zero. Such solutions are by definition not globally transverse. A TE wave is defined as a wave for which the Spin vector vanishes, but the Torsion vector does not, and a TM wave is defined as a wave for which the Torsion vector vanishes, but the Spin vector does not. If the 4-divergence of either the Torsion or the Spin 4-vector vanishes, then it is possible to construct deformation invariant topological quantities in terms of the deRham period integrals, by integrating either the Torsion 4-vector or the Spin 4-vector over a closed 3-dimensional submanifold. When the first Poincare invariant vanishes,

$$Poincare\ 1 = (\mathbf{B} \circ \mathbf{H} - \mathbf{D} \circ \mathbf{E}) - (\mathbf{A} \circ \mathbf{J} - \rho\phi) \Rightarrow 0,$$

the deRham period integral constructed from the Spin 4-vector defines a topological quantity called Spin. When the second Poincare invariant vanishes,

$$Poincare\ 2 = -2\mathbf{E} \circ \mathbf{B} \Rightarrow 0,$$

the deRham period integral constructed form the Torsion 4-vector defines a topological quantity called Torsion, or Helicity. These two topological quantities are usually distinct.

It is the purpose of this article to display exact time dependent radiative vacuum solutions to the above system of equations which are not TEM, TE, or TM waves. In particular non-transverse wave solutions to the vacuum Maxwell equations will be presented for which $\mathbf{E} \circ \mathbf{B} \neq 0$. Such field configurations have been associated with time reversal symmetry breaking in certain Lagrangian field theories[2] and, more recently, with thermodynamic irreversibility [3].

In addition it will be demonstrated that there exist superpositions of non-transverse vacuum waves for which the Torsion 4-vector vanishes identically, but the Spin 4 vector does not, and yet both Poincare invariants are zero. For such superposed solutions, the Spin topological quantity is non-zero and is conserved as an evolutionary invariant. Moreover, in the example to be displayed, the Poynting vector of radiative energy flow is proportional to the spatial components of the Spin 4-vector:

$$\mathbf{E} \times \mathbf{H} \approx factor \bullet (\mathbf{A} \times \mathbf{H} + \mathbf{D}\phi) = factor \bullet \mathbf{S}.$$

These special solutions are transverse magnetic, $h = \mathbf{A} \circ \mathbf{B} = 0$, but are not transverse electric, $\sigma = \mathbf{A} \circ \mathbf{D} \neq 0$. The topological quantity of Spin is quantized to the integers, and the energy flux is proportional to the Spin current. Therefore, these special vacuum solutions appear to give classical credence to the Planck concept of a photon as an object of unit spin, with radiated energy proportional to the spin.

2. Example Radiative Vacuum Solutions which are not transverse.

2.1 A First Example.

As an example of a radiative solution which is not a simple transverse wave, consider system of potentials given by the equations

$$\mathbf{A} = [+y, -x, -ct]/\lambda^4\ , \quad \phi = cz/\lambda^4, \quad where\ \lambda^2 = -c^2t^2 + x^2 + y^2 + z^2.$$

Compute the field intensities and find that

$$\mathbf{E} = [-2(cty - xz), +2(ctx + yz), -(c^2t^2 + x^2 + y^2 - z^2)]2c/\lambda^6$$

and

$$\mathbf{B} = [-2(cty + xz), +2(ctx - yz), +(c^2t^2 + x^2 + y^2 - z^2)]2/\lambda^6.$$

These equations for the electromagnetic field intensities satisfy the Maxwell-Faraday equations (1).

Next compute the field excitations using the Lorentz vacuum constitutive relations. Substitute these fields into the Maxwell-Ampere equations (2) and determine that $\mathbf{J} = 0$ and $\rho = 0$. The solutions presented therefore satisfy the homogeneous Maxwell equations without charge currents, and are therefore acceptable vacuum solutions. The electromagnetic fields are also solutions to the vector wave equation.

The Spin current density for this first non-transverse wave example is evaluated as:

$$\mathbf{S}_4 = [x(3\lambda^2 - 4y^2 - 4x^2), y(3\lambda^2 - 4y^2 - 4x^2), z(\lambda^2 - 4y^2 - 4x^2), t(\lambda^2 - 4y^2 - 4x^2)]2/\lambda^{10}$$

The Torsion current may be evaluated as

$$\mathbf{T}_4 = -[x, y, z, t]2c/\lambda^8.$$

The 4-divergence of the Spin current is zero. Hence the first Poincare invariant vanishes. As the interaction energy density, $(\mathbf{A} \circ \mathbf{J} - \rho\phi)$, for the vacuum is zero, the vanishing of the first Poincare invariant implies that the magnetic energy density is exactly equal to the electric energy density, typical of oscillator systems. However, the divergence of the Torsion vector is not zero, and the second Poincare invariant has the non-zero value:

$$\mathit{Poincare}\ 2 = -2c\mathbf{E} \circ \mathbf{B} = 8c/\lambda^8.$$

These solutions are not simple transverse waves for both $\mathbf{A} \circ \mathbf{B} \neq 0$, and $\mathbf{A} \circ \mathbf{D} \neq 0$. Note that the physical units of the second Poincare invariant are that of an energy density multiplied by an impedance (ohms).

However, as the first Poincare invariant is zero it is possible to construct a deformation invariant in terms of the deRham period integral over a closed 3 dimensional submanifold:

$$\mathit{Spin} = \iiint_{closed} \{S_x dy\hat{}dz\hat{}dt - S_y dx\hat{}dz\hat{}dt + S_z dx\hat{}dy\hat{}dt - \sigma dx\hat{}dy\hat{}dz\}.$$

As shown in section 3 below, this object is a topological quantity (the ratio of all of its possible values form rational fractions) that is a relative integral invariant of an arbitrary infinitesimal group of motions.

2.2 A Second Example.

It is to be noted that the example solution given above is but one of a class of vacuum wave solutions that have similar non transverse properties. For a second example, consider the fields that can be constructed from the potentials,

$$\mathbf{A} = [+ct, -z, +y]/\lambda^4, \quad \phi = cx/\lambda^4, \quad \mathit{where}\ \lambda^2 = -c^2t^2 + x^2 + y^2 + z^2.$$

These potentials will generate the field intensities

$$\mathbf{E} = [+(-c^2t^2 + x^2 - y^2 - z^2), +2(ctz + yx), -2(cty - zx)]2c/\lambda^6$$

and

$$\mathbf{B} = [+(-c^2t^2 + x^2 - y^2 - z^2), +2(-ctz + yx), +2(cty + zx)]2/\lambda^6.$$

As before, these fields satisfy the Maxwell-Faraday equations, and the associated excitations satisfy the Maxwell-Ampere equations without producing a charge current 4-vector. However, it follows by direct computation that the second Poincare invariant, and the Torsion 4-vector are of opposite signs to the values computed for the first example:

$$\mathbf{E} \circ \mathbf{B} = +4c/\lambda^8 \quad \textit{and} \quad \mathbf{A} \circ \mathbf{B} = +2ct/\lambda^8 \ .$$

2.3 Superposition of the two examples.

When the two examples are combined by addition or subtraction, the resulting wave is transverse magnetic, but not transverse electric. Not only does the second Poincare invariant vanish under superposition, but so also does the Torsion 4 vector. Conversely, the examples above show that there can exist transverse magnetic waves which can be decomposed into two non-transverse waves. A notable feature of the superposed solutions is that the Spin 4 vector does not vanish, hence the example superposition is a wave that is not transverse electric. Hoever, for the examples above and their superposition, the first Poincare invariant vanishes, which implies that the Spin remains a conserved topological quantity for the superposition. The spin current density for the combined examples is given by the formula:

$$\begin{aligned}\mathbf{S}_4 = {} & [-2cx(y+ct)^2, cy(y+ct)(x^2 - y^2 + z^2 - 2cty - c^2t^2), -2cz(y+ct)^2, \\ & -(y+ct)(x^2 + y^2 + z^2 + 2cty + c^2t^2)]4c/\lambda^{10}\end{aligned}$$

while the Torsion current is a zero vector

$$\mathbf{T}_4 = [0, 0, 0, 0].$$

In addition, for the superposed example, the spatial components of the Poynting vector are equal to the Spin current density vector multiplied by γ, such that

$$\mathbf{E} \times \mathbf{H} = \gamma\, \mathbf{S}, \quad \textit{with}\ \gamma = -(x^2 + y^2 + z^2 + 2cty + c^2t^2)/2c(y+ct)\lambda^2.$$

These results seem to give classical credence to the Planck assumption that vacuum state of Maxwell's electrodynamics supports quantized angular momentum, and that the energy flux must come in multiples of the spin quanta. In otherwords, these combined solutions have the appearance of the photon.

3. **The 3-form of Topological Torsion and the 3-form of Topological Spin**.

The formulation of Maxwell theory has an elegant (topological) representation in terms of exterior differential forms [4]. In this language topological features are readily apparent, with out the baggage of metric, coordinates or connections. The fundamental assumptions are that on a domain there exists a twice differentiable 1-form, A, of electromagnetic potentials, which induces a 2-form, F, of electromagnetic intensities. The 2-form is closed (and exact), such that the exterior differential system, $dF = 0$, generates the Maxwell-Faraday equations. The component functions of the 2-form transform as covariant tensor of rank 2. The fact that F is exact, implies that the domain of support for the field intensities cannot be compact without boundary, a fact that distinguishes classical electromagnetism from Yang-Mills field theories. Moreover, the fact that F is subsumed to be exact topologically excludes the concept of magnetic monopoles.

The second part of the exterior formalism assumes the existence of a twice differentiable N-2-form density, G. The form density G induces an N-1 form of charge-current densities, $dG = J$, which is closed. The exterior differential system $dG - J = 0$ generates the Maxwell-Ampere equations. By construction the N-1 form, J, is closed, and induces a topological invariant as $dJ = 0$.

In N=4 dimensions, the charge current density is a 3-form density, J. The invariance property is a global property in the sense that any integral of J over a closed 3 dimensional domain is a relative integral invariant of any infinitesimal group of motions. The formal statement is given by Cartan's magic formula[7], which describes continuous topological evolution in terms of the action of the Lie derivative, with respect to a vector field, acting on the exterior differential 3-form, J :

$$L_{\mathbf{V}}\Big(\iiint_{closed} J\Big) = \iiint_{closed} \{i(V)dJ + d(i(V)J)\} \;=\; \iiint_{closed} \{0 + d(i(V)J)\} \;=\; 0.$$

The Lie derivative is equal to zero for any 4-vector field V, when $dJ = 0$. The integral is a deformation invariant, for the result is valid even if the 4-vector field is distorted by an arbitrary function, $f\{x,y,z,t\}$ such that $\mathbf{V} \Rightarrow f(x,y,z,t)\mathbf{V}$. This idea is the foundation of the law of "charge conservation" in electromagnetic theory.

Of particular interest to this discussion is the existence of two other 3-forms, which may or may not have conservation properties. These 3-forms are associated with the property of transversality. If both three forms vanish, then the electromagnetic wave is said to be in a TEM mode. If only one of the two 3-forms vanishes, and the other does not, then the wave is said to be in either a TM or TE mode. The different transversality modes of electromagnetic waves is a well known concept experimentally, but their association to topological issues is novel herein. If the 2-form F was not exact, such topological concepts of transversality would be without meaning. It will be demonstrated that there can exist waves which are neither TE, TM, or TEM.

3.1 The Topological Torsion 3-form. A^F

The first 3-form of interest is the 3-form of topological torsion, $A\hat{}dA$.[6] This 3-form has 4 components that transform as a covariant tensor of rank 3. Using the classical notation, the components of the Torsion "4-vector" have the realization (relative to {x,y,z,t}):

$$A\hat{}dA = A\hat{}F = i(\mathbf{T}_4)dx\hat{}dy\hat{}dz\hat{}dt \Rightarrow \mathbf{T}_4 = [\mathbf{E}\times\mathbf{A}+\mathbf{B}\phi, \mathbf{A}\circ\mathbf{B}].$$

The physical dimension of the components of the Torsion current is that of angular momentum per unit volume times an impedance . Note that the classical helicity, $\mathbf{A}\circ\mathbf{B}$, forms only the fourth component of this third rank tensor. When the exterior derivative $d(A\hat{}dA) = 0$, then the Torsion "current" also satisfies a global conservation law. By direct computation,

$$d(A\hat{}dA) = F\hat{}F = -2\mathbf{E}\circ\mathbf{B}dx\hat{}dy\hat{}dz\hat{}dt.$$

This result is equivalent to the 4-divergence of the Torsion 4-vector:

$$div_3(\mathbf{E}\times\mathbf{A}+\mathbf{B}\phi) + \partial(\mathbf{A}\circ\mathbf{B})/\partial t = -2\mathbf{E}\circ\mathbf{B}.$$

Define the quantity of "Torsion" or "Helicity" as the integral

$$Torsion\ (Helicity) \doteq \iiint_{closed} A\hat{}dA = \iiint_{closed} A\hat{}F$$

If $A\hat{}dA$ is closed then the Torsion is a deRham period integral with values whose ratios are rational. The closure condition implies that the evolution of the Torsion is an invariant of an infinitesimal group of motions. A global conservation law for the torsion current ("helicity conservation") requires that the second Poincare invariant must vanish: $\mathbf{E}\circ\mathbf{B} = 0$. For the example non-transverse wave solutions presented in section 2.1 and 2.2, $\mathbf{E}\circ\mathbf{B} \neq 0$, and the Torsion (or helicity) is not conserved. That is

$$L_{\mathbf{V}}(\iiint_{closed} A\hat{}dA) = \iiint_{closed} -2\mathbf{E}\circ\mathbf{B}i(V)dx\hat{}dy\hat{}dz\hat{}dt \neq 0.$$

It is remarkable that if the example solutions of section 2.1 and 2.2 are combined, the two torsion currents cancel identically, and not only is $F\hat{}F = 0$, but also $A\hat{}dA = 0$. Hence the superposed solution generates a TM solution.

3.2 The Topological Spin density A^G

In addition to the 3-form of Torsion current and the 3-form of charge current density, there exists a third 3-form, the 3-form of Spin density, $A\hat{}G$. [5] Using the classical notation, the 4 components of the Spin density "4-vector" have the realization (relative to {x,y,z,t}):

$$A\hat{}G = i(\mathbf{S}_4)dx\hat{}dy\hat{}dz\hat{}dt \Rightarrow \mathbf{S}_4 = [\mathbf{A}\times\mathbf{H} + \mathbf{D}\phi, \mathbf{A}\circ\mathbf{D}].$$

The physical dimensions of the components of this three form density is that of angular momentum per unit volume, or a spin density. The divergence of this 3-form density defines the first Poincare invariant,

$$d(A\hat{}G) = F\hat{}G - A\hat{}J = \{(\mathbf{B}\circ\mathbf{H} - \mathbf{D}\circ\mathbf{E}) - (\mathbf{A}\circ\mathbf{J} - \rho\phi)\}dx\hat{}dy\hat{}dz\hat{}dt.$$

For both the example non-transverse wave solutions given in sections 2.1 and 2.2, the Spin density 3-form is not zero, but the first Poincare invariant vanishes. Therefore it is possible to demonstrate that for these wave solutions, the Spin integral

$$Spin \doteq \iiint_{closed} A\hat{}G,$$

as a deRham period integral, is a topological quantity whose possible values form rational ratios. The Spin is also a deformation or relative integral invariant of any infinitesimal group of motions, for when the first Poincare invariant vanishes,

$$L_{\mathbf{V}}(\iiint_{closed} A\hat{}G) = 0.$$

For the example solutions of section 2.1 and 2.2, Spin is an evolutionary invariant while Torsion-Helicity is not. Such waves are neither TE or TM. The example given in section 2.3 above is the first known (TM) solution to the vacuum Maxwell equations where the Torsion vanishes, but the conserved Spin does not. (The concept of the topological Spin 3-form, A^G, was created in 1977.[5])

4. The Hopf Map

The example solutions given in section 2 above were inspired by the work of Ranada [?] who investigated the applications of the Hopf map to the problem of finding solutions to the Maxwell equations. Recall that the Hopf map can be written as the common constraint on the map Φ from $R4(x,y,z,s)$ to $R3(X,Y,Z)$ given by the expressions:

$$X = 2(ys - xz)$$
$$Y = -2(yz + xs)$$
$$Z = -(z^2 + s^2) + (x^2 + y^2)$$

such that

$$R^2_{(3)} = X^2 + Y^2 + Z^2 = (x^2 + y^2 + z^2 + s^2)^2 = (R^2_{(4)})^2$$

Fixing the value of $R^2_{(4)} = \pm A$ determines a sphere in R4 and also a sphere, $R^2_{(3)} = A^2$, in R3. Permuting symbols and changing signs of the components in R4 give other similar expressions relating the quadratic form in R3 to the Quartic form in R4.

Ranada suggested the 4-potential

$$\mathbf{A} = [y, -x, -s](2/\pi)/\lambda^4 , \quad \phi = 0/\lambda^4, \quad where\ \lambda^2 = s^2 + x^2 + y^2 + z^2.$$

This 4 potential will generate the field intensities

$$\mathbf{E} = [0, 0, 0]$$

and

$$\mathbf{B} = [-2(sy + zx), +2(sx - yz), +(-s^2 + x^2 + y^2 - z^2)](4/\pi)/\lambda^6.$$

When the right hand side of the Hopf map is mapped projectively (by choosing $s = 1$), then the format used by Ranada becomes apparent. Ranada discusses the knottedness of the magnetic field lines of such a solution to the Maxwell-Faraday equations, by computing the Torsion current. Only the fourth component of the Torsion vector survives, yielding the helicity value

$$h = \mathbf{A} \circ \mathbf{B} = -8s/\pi^2\lambda^8$$

Unfortunately, as it stands, the Ranada 4-potential does not satisfy the Maxwell-Ampere equation for the vacuum. Substitution of the field intensities into the Maxwell-Ampere equation, using the Lorentz vacuum constraints, generates a finite current (although the charge density is zero). Although the Ranada suggestion does generate a conserved finite Torsion-helicity integral, it is not an acceptable vacuum solution to the Maxwell equations, as the charge current 4-vector is not zero. Moreover, the solution is static. (It is to be noted that the Ranada suggestion also generates a finite spin current, but the Spin current is not a conserved quantity in the Ranada example). Be that as it may be, the Ranada suggestion inspired the search for a vacuum solution, and led to the results of section 2 above.

Consider the modification of the Hopf map obtained by the substitution, $\{x, y, z, ct\} \to \{x, y, z, ict\}$, such that the complex map relates the position vector to a Hyperbolic surface of one sheet in a Minkowski space to the position vector of a sphere in R3; i.e., consider

$$X = 2(y \bullet ict - xz)$$
$$Y = -2(yz + x \bullet ict)$$
$$Z = -(z^2 - c^2t^2) + (x^2 + y^2)$$

such that

$$R^2_{(3)} = X^2 + Y^2 + Z^2 = (x^2 + y^2 + z^2 - c^2t^2)^2 = (R^2_{(m)})^2$$

A similar result occurs when the original Hopf map undergoes the complex linear transformation, $\{x, y, z, ct\} \to \{ix, iy, iz, ct\}$. However, in this case the map is from a hyperbolic surface of three sheets to the sphere in R3. The light cone in R4 maps to the origin in R3.

5. Globally Transverse Waves and Wave Solutions that are related to Minimal surfaces.

It is pertinent to discuss other vacuum-solution equivalence classes for which both the divergence of the Torsion vector and the divergence of the Spin vector vanish identically. The transverse equivalence class of solutions was developed by Bateman (and Whittaker) based upon the concept of self dual fields. In particular Bateman demonstrated that if two complex functions $\alpha(x, y, z, t)$ and $\beta(x, y, z, t)$ could be found that satisfied the eikonal expression, then the complex potential 1-form

$$A = \alpha d\beta - \beta d\alpha$$

would generate a solution to the vacuum Maxwell equations. Moreover, from the theory of the eikonal, any function of primitive eikonal solutions would satisfy the eikonal and the wave equation. As an example, Bateman suggested the two primitive functions,

$$\alpha = (x + iy)/(z + r), \quad \beta = r - ct$$

Indeed, such functions produce a vacuum solution to the Maxwell equations and also satisfy the eikonal expression. These solutions create TEM modes and are ”globally transverse”, for not only are both the first and second Poincare invariants zero, but also the Torsion and Spin currents vanish identically. In addition, the **E** and the **B** fields so generated from the Bateman equivalence class are complex and can have zero squares: $\mathbf{E} \circ \mathbf{E} = 0$, $\mathbf{B} \circ \mathbf{B} = 0$, as well as $\mathbf{E} \circ \mathbf{B} = 0$.

These globally transverse TEM fields, generated above, satisfy Bateman’s general formula for self conjugate fields,

$$\mathbf{M} \circ \mathbf{M} = (\mathbf{E} \pm ic\mathbf{B}) \circ (\mathbf{E} \pm ic\mathbf{B}) = 0,$$

a formula which Osserman has shown is related to the existence of two 2-dimensional conjugate minimal surfaces in the four dimensional domain. Reversing the argument, if a complex function can be found that will generate a minimal surface in 4 dimensions, then it may be used to construct vacuum solutions to the Maxwell equations, for which both the first and the second Poincare invariants are zero.
In particular, the superposed solution of section 2.3 above generates the field intensities:

References

Although the ideas are straightforward, the algebraic complexity of evaluating examples can be almost overwhelming. To this end, a symbolic mathematics program based on Maple had been used to generate the objects defined above. Given a set of 4 potentials, the program will compute the field intensities, the field excitations, the charge current densities if they exist, the Torsion 4-vector and the Spin 4-vector, if they exist, and the first and second Poincare invariants. The Maple program may be found at http://www.uh.edu/~rkiehn/pdf/maxwell.zip. A printout of an example worksheet can be found as a pdf file at http://www.uh.edu/~rkiehn/pdf/maxhopf.pdf

1. The notation is that to be found in Sommerfeld , Stratton. Sommerfeld carefully distingushes between intensities and excitations on thermodynamic grounds.
2. Liebermann
3. Lahtakia
4. Kiehn and Pierce
5. Kiehn
6. Kiehn
7. Bateman
8. Ranada