Applications of Differential Forms

Maxwell Faraday and Maxwell Ampere Equations

R. M. Kiehn (in preparation - last update 10/31/97)
Physics Department, University of Houston, Houston, Texas

Abstract: The topological universality of the Maxwell Faraday and Maxwell Ampere equations is an artifact of C2 differential forms on a domain of dimension $n \geq 4$. Starting with a 1-form of (electromagnetic) Action, the Maxwell Faraday equations become a consequence of the Poincare lemma. Starting from an N-1 form density, the Maxwell Ampere equations become a consequence of the topological constraint that the N-1 form density is exact. The conservation of charge current is a consequence of the Poincare lemma. Geometrical structure constraining the deduced 2-form and the induced N-2 form establish equivalence classes of constitutive equations. Evolutionary processes acting on Maxwell-Faraday systems can be classified into reversible and irreversible categories, depending upon the Pfaff dimension of the Action 1-form. The perfect plasma equations are equivalent to the unique Hamiltonian dynamical systems on spaces of Pfaff dimension 3, and the Master equations describe reversible processes on the symplectic manifold of Pfaff dimension 4. Irreversible processes generate dynamical systems proportional to vector fields $(\mathbf{ExA} + \mathbf{B}\phi, \mathbf{A} \bullet \mathbf{B})$, on symplectic domains of Pfaff dimension 4.

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(Suggestions are appreciated)

INTRODUCTION

In this article, Classical Electromagnetism will be defined in terms of two topological statements or postulates: the existence of a non-exact global 1-form of potentials, *A*, and the existence of a global exact N-1 form of charge currents, *J*. Then, the ideas implied by these topological postulates will be expressed in terms of Cartan's theory of differential forms [1] along with complete details of the constructions on a four dimensional variety. The method will demonstrate that the laws of electromagnetism, as defined by a set of partial differential equations, or equivalently, an exterior differential system [2], are concepts independent from a choice of metric, or a choice of a group structure that is often used to define a connection [3]. Following this expose, the various topological components will be subjected first to geometrical constraints that will lead to a constitutive theory of signals as propagating discontinuities, and then to evolutionary processes by means of Cartan's Magic formula [4], demonstrating how equivalence classes of electromagnetic problems can be formulated in a topological manner.

The constitutive technique demonstrates that there exist characteristic wave solutions to Maxwell's equations for which the propagation velocity is not four-fold degenerate. Each of two states of polarization propagate in opposite directions with 4 (four) distinct speed. It is a characteristic of the Lorentz vacuum that these 4 speeds are degenerate. The method also indicates that there exists another type of vacuum, the chiral vacuum, that preserves the Lorentz symmetries. The only difference between the Lorentz vacuum and the chiral vacuum is in the conformal factor that represents the radiation impedance.

In the language of differential forms, the exterior derivative is related to the Kuratowski closure operator, which in simple terms implies that the exterior derivative is a limit set generator. This idea has explicit exhibition in terms of the elementary concept that the divergence of the **D** field implies that the field lines terminate on their limit points, the charges. However, the concept is much more general. For example, it will become apparent below that the **E** and **B** fields are the ”limit” points of the electromagnetic potentials. Of particular interest are those topological sub-systems whose limit sets are ”empty”. For then the system is closed, but not necessarily isolated. Such configurations can have harmonic components whose integral values are globally in relation to the integers - a topological raison d’etre for quantization.

In particular, it will be demonstrated that when the topology induced on the variety by the postulate of potentials is irreducibly of Pfaff dimension 3 or more, there can exist electromagnetic domains for which the magnetic field **B** is irreducibly 3 dimensional. Moreover, there can exist propagating modes for such fields such that magnetic field must have a longitudinal component [5]. This result settles the somewhat controversial arguments that have appeared in the literature in the last few years concerning the possible existence of longitudinal magnetic fields[6]. However, the existence of irreducibly three component magnetic field solutions to Maxwell’s equations can be accommodated without invoking a special gauge condition, or a constraint of a particular group, or claiming that electromagnetism is to be replaced by a Yang-Mills theory.

This article also injects into the modern literature of classical electromagnetism a discussion of the concept of the 3-form of Topological Torsion current, $A\hat{}F$. This concept has its foundations in the theory of Pfaff’s problem, with a recognizable 4 dimensional formulation appearing in the 1890 edition of Forsythe’s ”Theory of Differential Equations” [7]. The idea is at the foundation of what has been termed magnetic helicity density, a concept that apparently had its electromagnetic genesis with the study of plasmas in WWII. However, the concept of helicity density is but one component of the four dimensional Topological Torsion current. (The three dimensional form of helicity had its roots in the Frenet–Serret theory of space curves, as developed with Cartan’s Repere Mobile, and has also appeared in the theory of the Hopf Index. [8]) The concept of A^F has also appeared in differential geometry as the Chern-Simons term. Longitudinal (non-transverse) magnetic fields with three irreducible non-zero components do not exist if the 3-form of Topological Torsion current is identically zero.

The theory of 3-forms and their period integrals was investigated with respect to electromagnetism and other field theories by the present author [9-15], first with respect to the three form defined below as $A\hat{}G$ and then later with respect to its dual 3-form, defined below as $A\hat{}F$. The first application of $A\hat{}F$ was in the field of turbulence [16], where it was conjectured that the transition from streamline flow (uniquely integrable in the sense of Frobenius, such that $A\hat{}F = 0$) to a turbulent flow (not uniquely integrable in the sense of Frobenius, $A\hat{}F \neq 0$) must involve a topological change. Although the interest was focused on hydrodynamics, the electromagnetic format was always used to establish a credence level in the computations that were done by hand. In the modern world of symbolic calculators [17] on your desktop, this algebraic tedium has been alleviated.

A new result presented in this article is the determination of when the period integral of the 3-form of Topological Torsion is an evolutionary invariant. These results generalize the work based on the Hopf Index.

THE POSTULATES

On any set of elements it is possible to construct several different topologies. In this article, classical electromagnetism will be deduced from two topological postulates constraining sets on a domain of four independent variables. The topological postulates will lead to an exterior differential system, whose partial differential equations are precisely those of Maxwell, as given in the classical form by A. Sommerfeld [18] or J. C. Stratton [19]. One of the topological postulates, the Postulate of Potentials, when applied to a four dimensional space time variety restricts the sets of interest to manifolds which are not compact without boundary. The second postulate, the Postulate of Conserved Currents, will admit compact domains. The postulates lead to two types of fields: intensities involving six component functions that define **E** and **B** as components of a covariant tensor, and excitations involving six component functions that define **D** and **H** as components of a contravariant tensor density.

Maps that attempt to correlate localized regions of the two topologies produce geometric constraints that define constitutive relations between $\{\mathbf{D},\mathbf{H}\}$, and $\{\mathbf{E},\mathbf{B}\}$. The Maxwell equations themselves, however, are independent from these additional constitutive constraints. The presence of matter (charge and current distributions) on the domain is assumed to be represented by such constitutive constraints. This concept that presence of matter defines the constitutive properties is the analog of those gravitational theories where the presence of matter (mass) defines the metric coefficients.

Postulate of Potentials and Intensities.

On an ordered set of independent four variables (x, y, z, t), assume the existence an *exact* C1 2-form of electromagnetic intensities, $F = F_{\mu\nu}dx^{\mu}dx^{\nu}$. This exact 2-form is constructed from a single C2 differentiable 1-form, $A = A_{\mu}dx^{\mu}$, such that $F = dA$. The unique starting point for classical electromagnetism is the specification of a 1-form of potential functions, A. On the other hand, given F there are many 1-forms A that will produce the same F. This non-uniqueness is the basis of gauge theories, for the topology induced on the domain depends on both A and $F = dA$, and their intersections (the harmonic content of the potentials). If F vanishes on a domain, there is no electric or magnetic intensity in that region. If the induced Cartan topology is connected, $(A\hat{}F = 0)$, then the gauge theories involve flux quanta. If the induced Cartan topology is not a connected topology $(A\hat{}F \neq 0)$, then the gauge theories involve torsion and helicity as well as flux quanta.

Postulate of Currents and Excitations.

On the same ordered set of independent variables assume the existence of an *exact* N-1 form density or current, $J = J^{\mu}dx^{1}\hat{}...\widehat{dx^{\mu}}..dx^{N}$. This exact N-1 form or current is constructed from an N-2 form density of field excitations, $G^{\mu\nu}dx^{1}\hat{}...\widehat{dx^{\mu}}..\hat{}\widehat{dx^{\nu}}..dx^{N}$, such that $J = dG$. The existence of domains of compact support (where J is not zero) define regions with ionized matter content and/or free charges. Exterior to these domains are the regions $(J = 0)$ of "neutral fields". This domain $(J = 0)$ need not be simply connected in the topological sense. The harmonic content of the N-2 form G determines the topological features of charge distributions of composite neutral matter. While the 2-form F (if it exists) defines a non-compact symplectic domain, the N-2 = 4-2 = 2 form G can define a compact symplectic domain. The concept is summarized by: forces are not compact, sources can be compact.

The unique starting point would the specification of a N–2 form G. On the other hand, given a current distribution, J, there are many N-2 forms G that will produce the same J. The harmonic content of the N-2 form G determines the charge distributions of composite neutral matter and their topological features on the domain of interest.

The Field equations

From these two topological postulates, and the rules of the Cartan exterior calculus, the equations of Maxwell electrodynamics can be deduced without additional structure (such as a metric or a connection) being imposed on the domain.

From Postulate 1, the first Maxwell pair of partial differential equations involve the six field intensities, (**E** and **B**) which, as the 2-form is exact, are well defined as the components of a covariant second rank tensor field. These six components have limited interpretations as two 3 dimensional vectors. These field intensities are constructed from the partial derivatives of the potentials that make up the . as an exact 1-form $F = dA$. The first Maxwell pair follows as the partial differential system, $ddA = dF = 0$. Magnetic monopoles do not exist in this formalism. In engineering format:

$$curl\ \mathbf{E} + \partial\mathbf{B}/\partial t = 0, \quad div\,\mathbf{B} = 0. \tag{1}$$

From Postulate 2, there exists an N-2 form G (of six field excitations **D** and **H**), such that $J = dG$. The second Maxwell pair is the partial differential system equivalent to these equations. From $ddG = dJ = 0$, the charge-conservation law is established as a consequence of the postulates. In engineering format:

$$curl\,\mathbf{H} - \partial\mathbf{D}/\partial t = \mathbf{J}, \quad div\,\mathbf{D} = \rho. \tag{2}$$

The two form G is homomorphic to the 2-form F (of field intensities, **E** and **B**) on even dimensional spaces (it has the same number of tensor components). A tensorial mapping between the 2nd rank covariant tensor F and the N-2 contravariant tensor density, G, imposes a perhaps non-linear geometrical constraint in the topological space.

$$G^{\mu\nu} = \chi^{\mu\nu\alpha\beta}F_{\alpha\beta.} \tag{3}$$

This geometrical constraint, and its assumed symmetries, can be put into correspondence with a set of constitutive relations, with singular sets that describe electromagnetic wave phenomena, in the vacuum, birefringent, Faraday, or optically active media.

It is important to remember that the designation {x,y,z,t} is not necessarily an orthogonal Cartesian set. The notation, and the components of the tensor coefficients, are only dependent upon the ordering of the independent variables. The set $\{\xi^1, \xi^2, \xi^3, \xi^4\}$would be as adequate as {x,y,z,t}.

It is assumed that the composite topological structure that defines a Maxwell system is determined by the sets $\{A, F, G, J\}$ and their intersections.

The First Maxwell Pair

Electromagnetism and The Pfaff Sequence

The 1-form of Potentials

One of the best ways to exemplify the techniques of Cartan's exterior differential forms is to apply them to an electromagnetic situation on a space of 4 dimensions. Most everyone has had some experience with electrodynamics on a four dimensional space-time, and therefore some of the simpler results of the Cartan Calculus will be of a familiar format. The familiarity will develop a level of credence in the more intricate results.

On the four dimensional space-time of independent variables, (x, y, z, t) the 1-form of Action (representing the postulate of potentials) can be written in the form

$$\mathrm{A} = \Sigma_{k=1}^{3} A_k(x,y,z,t)dx^k - \phi(x,y,z,t)dt. \qquad 4$$

For any 1-form, A, the first step in the Cartan method is to construct the Pfaff Sequence,

$$\{\mathrm{A}, d\mathrm{A}, \mathrm{A}\hat{\ } d\mathrm{A}, d\mathrm{A}\hat{\ } d\mathrm{A}\}. \qquad 5$$

This sequence can be constructed for an C1 differentiable 1-form. On a base variety of N dimensions (independent variables), the sequence will have 0 to N elements which are non-zero. The number of non-zero elements is defined as the Pfaff Dimension, $M \leq N$, of the 1-form, A. It represents the number of irreducible functions required to define the properties of the 1-form, A. The implication is that there exists a map from the N dimensional space to a space of M independent variables, upon which the 1-form, A, can be put into canonical form. All of the properties of the 1-form are well defined by functional substitution and pullback to the space N of the 1-form in irreducible minimal representation on space M.

The 2-form of Field Intensities (E and B)

Following the usual definitions of the exterior derivative, the components of the 2-form dA become

$$d\mathrm{A} = \{\partial A_k/\partial x^j - \partial A_j/\partial x^k\}dx^j\hat{\ }dx^k = F_{jk}dx^j\hat{\ }dx^k = \mathbf{B}_z dx\hat{\ }dy \ldots \mathbf{E}_x dx\hat{\ }dt \ldots \qquad 6$$

where in usual engineering notation,

$$\mathbf{E} = -\partial\mathbf{A}/\partial t - grad\phi, \qquad \mathbf{B} = curl\ \mathbf{A}. \qquad 7$$

Be aware that the engineering notation, where the six components of the second rank covariant tensor, F_{jk}, are grouped into two 3 component vectors, is deceptive, for the diffeomorphic transformational properties of the field intensities (**E** and **B**) are not that of Cartesian rank 1 three dimensional vectors, but are that of a second rank tensor field. Given the functional form of the components of the 1-form of Action, a Maple program (see faraday.pdf) may be used to compute all of the forms in this article.

Note that the Poincare lemma ($dd\omega = 0$) always leads to the first Maxwell pair of (Faraday Induction) equations: e.g.,

$$dd\mathrm{A} = \{curl\ \mathbf{E} + \partial\mathbf{B}/\partial t\}_x dy\hat{\ }dz\hat{\ }dt - .. + .. - div\,\mathbf{B}dx\hat{\ }dy\hat{\ }dz\} \Rightarrow 0, \qquad 8$$

or

$$\{curl\ \mathbf{E} + \partial\mathbf{B}/\partial t = 0, \quad div\,\mathbf{B} = 0\}. \qquad 9$$

This result is actually true for a variety of any dimension $\geq$ 4 and for any set of covariant symbols. If the dimension of the ordered domain exceeds 4, the concept of $ddA = dF = 0$, always yields the same set of partial differential equations for the first four variables. The first Maxwell set of equations is a nested set. The addition of new independent variables does not change the format of the first four Maxwell equations, but just adds to the set new equations involving field components defined over the new variables. The concept of Faraday Induction is universal, and should not restricted to the science of electromagnetism. It is valid for any physical system which can be described by a 1-form of Action with a Pfaff dimension 2 or larger, such as a fluid with vorticity.

The very existence of the **E** and **B** fields implies that the 2-form dA does not vanish. Hence, the 2-form defines a symplectic manifold of at least Pfaff dimension 2. As the 2-form is exact, the symplectic 2-manifold cannot be compact without boundary. (This result follows from Stokes theorem). The exact symplectic domain of the electromagnetic field intensities, **E** and **B**, must either be open or has a boundary. (This behavior is topological distinct from that of the field excitations, **D** and **H**. See piers97.pdf).

The 3-form of Topological Torsion

The topological torsion 3-form, A^*d*A, induces the electromagnetic torsion current. An engineering representation for the functional coefficients of the 3-form can be given in terms of the classic field intensities and potentials by the expression

$$\mathbf{T} = \{(\mathbf{E}\times\mathbf{A} + \mathbf{B}\phi);\mathbf{A}\bullet\mathbf{B}\} \equiv \{\mathbf{S},h\}. \qquad 10$$

In the language of differential forms another expression for the Topological Torsion current is given by the equivalent expression:

$$\mathrm{A}\hat{}d\mathrm{A} = \mathrm{A}\hat{}\mathrm{F} = i(\mathbf{T})dx\hat{}dy\hat{}dz\hat{}dt = \mathbf{S}^x dy\hat{}dz\hat{}dt..... - h dx\hat{}dy\hat{}dz \qquad 11$$

When this expression is non-zero, the Pfaff dimension of the 1-form of Action is not less than 3. Note that the fourth component of the Topological Torsion current is exactly that which historically has been defined as the helicity density of the field. However, by the construction above, $\mathbf{A}\bullet\mathbf{B}$ is to be recognized as the fourth component of a contravariant vector density, relative to transformations of space-time.

Existence of the Topological Torsion current can be proved by a constructive example. To this end, an interactive Maple formulation has been placed on the Internet [ranada2.pdf]. The user can manipulate the code to construct many different examples. For purposes of display in this written article consider the covariant set of potential functions:

$$A_\mu = [y/(1+x^2+y^2+z^2)^2,\ -x/(1+x^2+y^2+z^2)^2, k, \omega]\cos(-kz+\omega t) \qquad 12$$

Following the definitions, compute the components of the **E** and **B** fields to yield:

$$\mathbf{E} = [-\omega y, \omega x, 0]\sin(-kz+\omega t)/(1+x^2+y^2+z^2)^2 \qquad 13$$

$$\mathbf{B} = [-4xz, -4yz, -2(1+z^2-x^2-y^2)]\cos(kx-\omega t)/(1+x^2+y^2+z^2)^3 + [-kx,\ ky, 0]\sin(-kz+\omega t)/(1+x^2+y^2+z^2)^2 \qquad 14$$

It is apparent that the **B** field is irreducibly 3 dimensional almost everywhere, and that there exists a longitudinal component of B(3) perpendicular to the surface of constant phase.

The components of the Topological Torsion current become equal to :

$$\mathbf{T} \equiv [\mathbf{S},h] = [-4\omega xz, -4\omega yz, -2\omega(1+z^2-x^2-y^2), -2k(1+z^2-x^2-y^2)] \times \cos(kx-\omega t)^2/(1+x^2+y^2+z^2)^3 \qquad 15$$

The 4-form of Topological Parity

The 4-form of topological parity also has an expression in engineering format as,

$$d\mathrm{A}\hat{}d\mathrm{A} = -2(\mathbf{E}\bullet\mathbf{B})dx\hat{}dy\hat{}dz\hat{}dt = (div\,\mathbf{S} + \partial h/\partial t)dx\hat{}dy\hat{}dz\hat{}dt. \qquad 16$$

It is a straight forward procedure to compute the 4 divergence of the Topological Torsion current for the example of the preceding paragraph. The value turns out to be $(\mathbf{E}\bullet\mathbf{B}) = 0$. That is the example of the preceding paragraph is a demonstration of a 1-form of Action of

Pfaff dimension 3.

When the Action 1-forms has domains of {x,y,z,t} where $(\mathbf{E} \bullet \mathbf{B}) \neq 0,$ the Pfaff dimension is 4, and the space supports a symplectic non-compact irreducibly 4 dimensional manifold. The singular points of this space are where $2(\mathbf{E} \bullet \mathbf{B}) \Rightarrow 0,$ and these sets form a contact submanifold of Pfaff dimension = 3. On the 4D symplectic manifold, the 4-divergence of the Torsion current does not vanish. On the contact 3 dimensional manifold, the Torsion current obeys a ”conservation law”

$$(div\,\mathbf{S} + \partial h/\partial t) = 0 \qquad 17$$

(or an ”equation of continuity”).

On domains where the Pfaff dimension is 3 (and not 4) there exists a 3 dimensional period integral related to the Hopf index, $\iiint_{3\_cycle} A\hat{}F$. It will be demonstrated below that if the domain is of Pfaff dimension 3, then evolutionary processes in the direction of the electromagnetic charge current 4 vector, J, leave the integral of the Topological Torsion current over a 3 dimensional boundary as an evolutionary invariant. Even more remarkable is the fact that such a statement is valid in domains where the Pfaff dimension is 4, not 3, if the current flow is on the surface defined by $(\mathbf{E} \bullet \mathbf{B}) = 0$. See the section below on the Hopf Index.

The Second Maxwell Pair

Field excitations and sources

The second postulate assumes the existence of a N-2 form density given by the expression,

$$G = G^{34}dx\hat{}dy... + G^{12}dx\hat{}dt... = \mathbf{D}^z dx\hat{}dy...\mathbf{H}^z dz\hat{}dt... \qquad 18$$

Exterior differentiation produces an N-1 form, $J = \mathbf{J}^z dx\hat{}dy\hat{}dt... - \rho dx\hat{}dy\hat{}dz$. Matching the coefficients of the exterior expression $dG = J$ leads to the second Maxwell pair,

$$curl\,\mathbf{H} - \partial\mathbf{D}/\partial t = \mathbf{J} \qquad 19$$

and

$$div\,\mathbf{D} = \rho. \qquad 20$$

The fact that J is exact leads to the charge conservation law, $dJ = ddG = 0,$ or

$$\partial\mathbf{J}^x/\partial x + \partial\mathbf{J}^y/\partial y + \partial\mathbf{J}^z/\partial z + \partial\rho/\partial t = 0. \qquad 21$$

Recall that it should not be presumed that the set of symbols {x,y,z,t} is on some Cartesian type of space. The arrangement and the exterior differential equations are dependent on the ordering of the set {x,y,z,t} only, and not the explicit form of the symbols.

3-forms and 4-forms

The N-2 form density, $G,$ need not be closed. Its limit set, $J,$ defined by the exterior derivative of $G,$ is the distribution of (free) charge-current densities on the domain. On a four dimensional domain the N-2 form of charge current is a 3 form density. Although it is a classic assumption to impose constitutive constraints relating the components of F to the components of G, it is important to recognize that even without such geometrical constraints there exist several composite forms and form densities that are distinct.

For example, the 4-form

$$F\hat{}F = -2(\mathbf{E} \bullet \mathbf{B})dx\hat{}dy\hat{}dz\hat{}dt \qquad 22$$

is not the same as the 4-form density

$$F\hat{}G = (\mathbf{D} \bullet \mathbf{E} - \mathbf{B} \bullet \mathbf{H})dx\hat{}dy\hat{}dz\hat{}dt \qquad 23$$

nor the 4-form density

$$A\hat{}J = (\mathbf{A} \bullet \mathbf{J} - \rho\phi)dx\hat{}dy\hat{}dz\hat{}dt. \qquad 24$$

There are three distinct and important three forms,

$$A\hat{}F \quad (The\ Topological\ \ Torsion\ 3 - form) \qquad 25$$

$$A\hat{}G \quad (The\ Spin\ 3 - form) \qquad 26$$

$$J\ (the\ \ Maxwell\ current) \qquad 27$$

Note that there is a closure constraint that leads to the fundamental identity

$$d(A\hat{}G) = F\hat{}G - A\hat{}J. \qquad 28$$

The right hand side is an exact 4-form that yields an invariant measure (a relative integral invariant) on the domain. If the RHS is not zero, then the 4 dimensional (spin) domain cannot be compact without boundary. Only if the RHS is zero can the spin domain be compact, and then there can exist 3 dimensional cycles but they must come in pairs.

The relationship of the field excitations, **D** and **H**, to the field intensities, **E** and **B**, belongs to the theory of constitutive constraints of structure on the fundamental topological features of the fields. This topic is taken up in the next section. The idea is that the presence of matter (a neutral charge-current distribution) forms equivalence classes of constraints between the field excitations and the preceding field intensities. The vacuum will be considered to be a case where the limit sets, J, of the N-2 form, G, vanish. Note that the limit sets, J, represent non-harmonic components of G.

Electromagnetic Signals

Signals are Propagating Discontinuities

In a remarkable piece of work published in 1932, V. Fock extended Bateman’s concepts and deduced the characteristic system for the solutions of Maxwell’s equations which are not unique in a neighborhood. He clearly formulated the idea that electromagnetic signals were propagating discontinuities. (The concept of a signal and its precise relationship to electromagnetism was not clearly defined by Einstein) To paraphrase Victor Fock:

> ”The laws of propagation of light in empty space are thoroughly understood. They find their expression in the well-known equations of Maxwell. However, we are not interested in the general case of light propagation, but only in the propagation of a signal advancing with maximum speed; i.e., the propagation of a wave front. Ahead of the front of the wave all components of the field vanish.

> Behind it some of them are different from zero. Therefore, some of the field components must be discontinuous at the front. On the other hand, given the field (a solution to the equations) on some surface moving in space, the derivatives of the field are, in general, determined by Maxwell's equations. Hence the value of the field at an infinitely near surface is also (uniquely) determined (by analytical continuation) and discontinuities are impossible. The only case when this is not so is when the form and the motion of the surface satisfies certain special conditions subject to which the values of the derivatives is not determined by the values of the field components themselves. Such a surface is called a characteristic surface, or briefly, a characteristic. Thus discontinuities of the field can occur only on a characteristic, but since there must certainly be discontinuities at a wave front (the signal), such a front is clearly a characteristic."

These ideas were championed by Luneberg (and others) at the end of WWII. The key idea in Fock's work is the result that the characteristic solutions remain characteristic solutions (discontinuities remain discontinuities - signals remain signals) only under a limited set of coordinate transformations. The laws of Maxwell as tensor equations are well behaved with respect to ALL diffeomorphisms, but the characteristic solutions to Maxwell's equations retain their topological properties only with respect to a restricted class of transformations. Fock proves that the linear class of transformations that preserve signals is the Lorentz group, for which a finite propagation speed is an invariant concept. However, the part of Fock's development that has been ignored is the idea that there also exists a non-linear transformation group (the Moebius fractional projective transformations) which also preserve discontinuities, signals, and other properties of characteristics. Such signals are not restricted to a finite propagation speed. These results will not be pursued further herein, but that which will be developed in the language of differential forms is the idea that the characteristics are determined from a prolonged exterior differential system.

The characteristic system

For the work presented herein, the source free system of forms $\Sigma_0 = \{F, G\}$, $\{J = 0\}$ will be treated as a closed ideal, $d\Sigma_0 = \{dF, dG\} = 0$. The Cartan technique will be used to search for two systems of dual vector fields that define point sets which are complimentary in a union and intersection sense. This problem is equivalent to finding those point sets upon which the field amplitudes admit discontinuities, but still satisfy Maxwell's equations. In other words, the solutions to Maxwell's equations are not unique upon the singular point set. There are two ways to go about this problem, the ray method and the wave method. In this article, with its emphasis on three forms, only the wave method will be pursued in detail. However see [Phys Rev 1991] for both the ray and wave analysis.

Consider the set of covariant fields k which annihilate the system of forms of the closed ideal in an intersection sense. These point sets are given by the covariant vectors

$$k = \{\mathbf{k}, \omega\} \tag{29}$$

such that the l-form $k = \mathbf{k} \bullet d\mathbf{r} - \omega\, dt$ will have null intersections with the closed ideal of forms $\{\Sigma_0\}$; i.e., search for k such that

$$k\hat{}\{\Sigma_0\} = 0, \tag{30}$$

or

$$k\hat{}F = 0, \quad k\hat{}G = 0. \tag{31}$$

These equations of characteristics have an engineering format as the set

$$\mathbf{k} \times \mathbf{E} - \omega\mathbf{B} = 0, \quad \mathbf{k} \bullet \mathbf{B} = 0, \tag{32}$$

$$\mathbf{k} \times \mathbf{H} + \omega \mathbf{D} = 0, \quad \mathbf{k} \bullet \mathbf{D} = 0, \qquad 33$$

from which it is apparent that **k** has the direction of the momentum flux $\mathbf{D} \times \mathbf{B}$. It is to be noted that these six independent equations for the covariant wave field **k** are the usual sets of equations which define the point sets upon which field discontinuities may exist.[12] The 1-form can be multiplied by any non-zero function without changing the format of the equations. In fact it is more convenient to use the covariant field in the format $k = \{\mathbf{k}, \omega\} = \omega\{\mathbf{n}, 1\}$. (The equivalent analysis in terms of rays, implies that the ray vector is in the direction of the energy flux, $\mathbf{E} \times \mathbf{H}$. The energy flux and the momentum flux are not necessarily in the same direction).

If the characteristic system is viewed as a set of six equations in 12 unknowns representing the components of the fields, then the problem does not admit a solution without further constraints. To alleviate this problem, it is convenient to impose geometrical constraints in the form of constitutive relations on the topological system. The format of the constitutive relations can be guided by the the tensor format, where for a linear system it is presumed that

$$G^{\mu\nu} = \chi^{\mu\nu\alpha\beta} F_{\alpha\beta} \qquad 33$$

Due to the antisymmetries of the forms, the 256 components of the tensor density, $\chi^{\mu\nu\alpha\beta}$ in four dimensions may be reduced to 36 distinct values, [14] a reduction which leads to the 6 vector formalism of Sommerfeld; namely, in matrix format,

$$\begin{aligned} \mathbf{D} &= [\epsilon]\circ\mathbf{E} + [\gamma]\circ\mathbf{B} \qquad 34 \\ \mathbf{H} &= [\gamma^{\dagger}]\circ\mathbf{E} + [\mu^{-1}]\circ\mathbf{B}. \end{aligned}$$

The 6x6 constitutive matrix is assumed to be Hermitian for the nondissipative cases considered herein, and in that case can be partitioned into real and imaginary parts. However a slightly more general system, which can incorporate certain discontinuous non-linear effects, is obtained if it is presumed that the matrix $[\gamma]$ can be hermitian or anti-hermitian matrix, $[\gamma^{\dagger}] = \pm[\gamma]$. The various optical effects that have been measured in classical electromagnetism then can be into correspondence with symmetries induced by the diagonal and off diagonal matrix elements of the partitioned representation (See Post).

$$\begin{bmatrix} \mathbf{D} \\ \mathbf{H} \end{bmatrix} = \begin{bmatrix} dielectric\_birefr. & Fresnel\_Fizeau \\ Fresnel\_Fizeau & magnetic\_birefr. \end{bmatrix} \begin{matrix} \mathbf{E} \\ \mathbf{B} \end{matrix} + \qquad 35$$
$$\sqrt{-1} \begin{bmatrix} dielectric\_Faraday & Opt\_Act \\ Opt\_Act & magnetic\_Faraday \end{bmatrix} \begin{bmatrix} \mathbf{E} \\ \mathbf{B} \end{bmatrix}$$

The chiral vacuum

In the Lorentz vacuum case, the classic assumption reduces the Hermitian constitutive matrix to the format

$$\begin{bmatrix} \mathbf{D} \\ \mathbf{H} \end{bmatrix} = \begin{bmatrix} \varepsilon_0 \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} & \gamma \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \\ \gamma^{\dagger} \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} & 1/\mu_0 \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \end{bmatrix} \begin{matrix} \mathbf{E} \\ \mathbf{B} \end{matrix} \qquad 36$$

where ε_0 and $1/\mu_0$ are presumed to be constants, and the chirality function γ is presumed to be zero. Note that it is a simple exercise for the constitutive constraint given above, including the chirality factor, to derive the wave equation for the field components:

$$curl\,curl\,\mathbf{E} - \varepsilon_0\mu_0\partial^2\mathbf{E}/\partial t^2 = (\mp\gamma^{\dagger} + \gamma)(curl\,\mathbf{E} + \partial\mathbf{B}/\partial t). \qquad 37$$

When the chirality factor is imaginary, the RHS = zero, giving the classic wave equation, for any value of γ. This result is also true if the chirality factor is a function of space, independent from time, but with a gradient orthogonal to the **B** field. The concept of a chiral vacuum has not received attention from a classical point of view. Note that the determinant of the constitutive matrix $(\varepsilon_0/\mu_0 - \gamma^{\dagger}\gamma)$ is the square of the reciprocal of an impedance. What is the value of γ? Could it be proportional to the reciprocal of the Hall impedance, $e^2/\hbar$?

Fresnel -Kummer surfaces

The six constitutive equations given above may be used to eliminate half of the unknowns in the six equations for the characteristic system. For example, the elimination of **E** and **H** in the characteristic equations leaves a homogeneous system of six equations in six variables. The Cramers determinantal condition on this homogeneous set must be satisfied if a solution exists. The Cramers determinant condition, $\mathrm{H}(n_x, n_y, n_z) = 0$ leads to a quartic Kummer surface in the components of the three vector of reciprocal phase velocity, $\mathbf{n} = \mathbf{k}/\omega$. By using the matrix representation,

$$[\mathbf{n}^{\times}] = \begin{matrix} 0 & n_z & -n_y \\ -n_z & 0 & n_x \\ n_y & -n_x & 0 \end{matrix} \qquad 38$$

the Kummer surface can be defined by the equation:

$$\mathrm{H}(n_x, n_y, n_z) = det([\epsilon] + [\gamma][\mathbf{n}^{\times}] + [\mathbf{n}^{\times}][\pm\gamma^{\dagger}] + [\mathbf{n}^{\times}][\mu^{-1}][\mathbf{n}^{\times}]) = 0. \qquad 39$$

In the three-dimensional space of variables (n_x, n_y, n_z) the implicit hypersurface $\mathrm{H} = 0$ is of fourth degree and creates an extension of the usual Fresnel wave surface to include not only anisotropic birefringence, but also electric and magnetic Faraday rotation, optical activity, and Fresnel-Fizeau phenomena in combination.

In the case of birefringence symmetries alone, the quartic function $\mathrm{H} = 0$ splits into two doubly degenerate quadratic factors which are the usual representations of the Fresnel ellipsoids, one ellipsoid for each state of polarization. In the case of the Lorentz symmetries, the Kummer surface is fourth order degenerate with the phase velocity of propagation equal for both states of polarization and for both directions of propagation.

The general Fresnel wave surfaces are distorted ellipsoids which do not have a center of symmetry. Hence an arbitrary line through the origin in reciprocal phase velocity space will intercept the Fresnel wave surface(s) in four distinct points representing four distinct phase velocities: two magnitudes in the outbound direction for each state of polarization, and two different velocities in the inbound direction, one for each state of polarization. A Maple based program (Fresnel.pdf) is available for computing the Fresnel wave surfaces.

The little appreciated result is that the propagation velocity of a signal can be sensitive to the Fresnel direction of propagation, which of course implies that the constitutive equations do not have the symmetries of the Lorentz vacuum. These effects were experimentally verified in dual polarized ring laser experiments conducted by Sanders.[6] When the solutions to the Kummer surface equation are not degenerate (four distinct roots to the fourth order polynomial), then the solutions to the wave equation are best described in terms of quaternions, and not real or complex vectors.

Evolutionary Processes and Electromagnetism

Consider an arbitrary process defined by the contravariant 4 vector field $\mathbf{W} = \rho\{\mathbf{V}; \mathbf{1}\}$ which will cause the evolution of a physical system defined by a 1-form of Action, A. Subsume that the evolution obeys Cartan's Magic formula [??]

$$\mathrm{L}_{(\mathbf{V})}A = i(\mathbf{V})dA + d(i(\mathbf{V})A) \qquad 40$$

1. The term $W = i(\mathbf{V})dA$ is defined as the 1-form of "virtual work".
2. The function $U = i(V)A$ is defined as the "internal energy" of interaction.
3. The sum of these two terms, $Q = W + U$, defines the 1-form of "heat", the result of an evolutionary process, $\mathbf{V}$, acting on the physical system, A.

From these definitions, it is apparent that Cartan's magic formula not only represents an evolutionary process, but also is formally equivalent to the cohomological description of the First Law of Thermodynamics.

$$\begin{aligned} \mathrm{L}_{(\mathbf{V})}A &= i(\mathbf{V})dA + d(i(\mathbf{V})A) = Q \\ &= W + dU = Q \end{aligned} \qquad 41$$

For an electromagnetic 1-form of Action, the 1-form of virtual Work, $W = (i(\mathbf{W})d\mathrm{A})$, generates an expression that defines the components of the Lorentz force law:

$$W = (i(\mathbf{W})d\mathrm{A}) = (\rho\{\mathbf{E} + \mathbf{V} \times \mathbf{B}\}_k dx^k - \{\rho\mathbf{V} \bullet \mathbf{E}\}dt). \qquad 42$$

The question now arises as to the Pfaff dimension of the 1-form of virtual work, and its physical implications. The simplest case is when the Pfaff dimension of the virtual work 1-form is zero (globally). Then each of the four components of the virtual work 1-form must vanish. Such a constraint is impossible if the Pfaff dimension of the 1-form of Action is 4, for then there do not exist and eigenvectors of the matrix of coefficients, F_{mn}, with zero eigen values.

The Hamiltonian Extremal Class

When the Pfaff dimension of the Action 1-form is 3, the induced 2-form F is of rank 3 on a four dimensional domain. That is, the coefficients of the 2-form form a 4x4 anti-symmetric matrix, with two null eigenvalues. There exists, then, two distinct eigenvectors with a null eigen value. These two eigenvectors define two evolutionary fields which are extremal vector fields in the sense of the calculus of variations. That is, both vector fields produce a zero value for the Virtual Work 1-form. Such extremal evolutionary processes leave the integral of the 1-form of Action stationary, and are called ”extremal” fields. (If the integration path is not closed, then stationary variational principle imposes the additional requirement that $\rho\{\mathbf{A} \bullet \mathbf{V} - \phi\} \Rightarrow 0$ on the boundary. The integration path must start and terminate on the zero set of this (Lagrange) function.)

One of the two vector fields with null eigen values is not only extremal, it is characteristic, in the sense that not only does the virtual work vanish locally, but so does the internal energy of interaction. Such vectors are both extremal (virtual work is zero) and associated (interaction internal energy is zero). The added boundary condition in the calculus of variations is not required for the characteristic field. This *unique characteristic* evolutionary vector field, is proportional to the Torsion vector for situations where the Pfaff dimension is 3. Evolution along the direction of the characteristic vector field is always adiabatic.

For evolution in the direction of any extremal vector field (which may or may not be adiabatic), the components of the virtual work 1-form can vanish; e.g., the Pfaff dimension of the Virtual work is zero. Such an evolutionary process defines what is call the ”perfect” plasma in classical electromagnetic theory, for then there is no ”ohmic” dissipation, $(\{\rho\mathbf{V} \bullet \mathbf{E}\} = 0)$ and the Lorentz force vanishes, $\rho\{\mathbf{E} + \mathbf{V} \times \mathbf{B}\} \Rightarrow 0$. This condition is subsumed as the classic constraint for the sophomore problem of a charge particle moving in crossed magnetic and electric fields.

Extemal and Characteristic vector fields always admit a Hamiltonian representation for the Action 1-form.

$$A \Rightarrow pdq - h(p,q,t)dt. \tag{43}$$

The Helmholtz Symplectic Class (the Master Equation)

A somewhat more general class of evolutionary possibilities exists on those domains where the Pfaff dimension of the 1-form of Action is 3, but the Pfaff dimension of the virtual work 1-form is 1, not zero. In this case the virtual work 1-form must be closed, $d(i(\mathbf{W})dA) \Rightarrow 0$. Such a constraint defines the Helmholtz class of evolutionary processes, and leads to the ”conservation of vorticity” in the hydrodynamic case, and to the concept of ”frozen in magnetic flux” in the electromagnetic case. The closure condition implies that the Lorentz force need not be zero, but it should have zero curl: $d(\rho\{\mathbf{E} + \mathbf{V} \times \mathbf{B}\}_k dx^k - \{\rho\mathbf{V} \bullet \mathbf{E}\}dt) = 0$. First consider that case where ρ is a constant. Then, the necessary condition to satisfy the closure condition, for arbitrary displacements of the independent variables, is that

$$curl\,\{\mathbf{E} + \mathbf{V} \times \mathbf{B}\} = 0 \tag{44}$$

and similarly

$$\partial\{\mathbf{E} + \mathbf{V} \times \mathbf{B}\}/\partial t + grad\{\mathbf{V} \bullet \mathbf{E}\} = 0. \tag{45}$$

Substituting the Maxwell result, $curl\,\mathbf{E} = -\partial\mathbf{B}/\partial t,$ leads to the Master equation of the Imperfect Plasma:

$$-\partial\mathbf{B}/\partial t + curl\{\mathbf{V} \times \mathbf{B}\} = 0. \tag{46}$$

Such evolutionary processes are defined as symplectomorphisms in the modern literature.

For non-constant ”charge density distributions” ρ the Helmholtz closure constraint requires that

$$\rho(\mathbf{E} + \mathbf{V} \times \mathbf{B}) = -grad(\Theta). \qquad 47$$

In this case, the Master equation is modified slightly to account for a non-constant distributions, ρ :

$$-\partial\mathbf{B}/\partial t + curl\{\mathbf{V} \times \mathbf{B}\} = grad \ln\rho \times (\mathbf{E} + \mathbf{V} \times \mathbf{B}). \qquad 48$$

In elementary physics the scaling function ρ is to recognized as the charge distribution. In elementary mathematics, the scaling function ρ is to be recognized as the integrating factor.

There are two important cases that must be examined for the Helmholtz class of evolutionary processes. Either the Pfaff dimension is even and of dimension 4, or it is odd and of Pfaff dimension 3. If the Pfaff dimension of the Action is 4, then the symplectic *manifold* condition of maximal rank over the 4 dimensional domain requires that the second Poincare invariant is not zero:

$$dA\hat{}dA = -2(\mathbf{E} \bullet \mathbf{B})dx\hat{}dy\hat{}dz\hat{}dt \neq 0. \qquad 49$$

The Helmholtz evolutionary process condition requires that

$$\rho(\mathbf{E} \bullet \mathbf{B}) = \mathbf{B} \bullet grad\,\Theta \neq 0 \qquad 50$$

Therefore, there must exist a gradient (of pressure or temperature, Θ) in the direction of the **B** field lines. Similarly, there is a dissipation if the motion is in the direction of the gradient, for then

$$\rho(\mathbf{E} \bullet \mathbf{V}) = \mathbf{V} \bullet grad\,\Theta. \qquad 51$$

There is no "ohmic" dissipation in the direction orthogonal to *grad* Θ .

Physically, then, in a symplectic system exhibiting symplectic evolution, there must exist a component of electric force **E** that accelerates charged particles along the magnetic **B** field lines, and that component of force, as an artifact of the symplectic constraints, must be the ultimate source of the magnetic dynamo. A similar situation holds in hydrodynamics where fluid mass can be accelerated along the lines of vorticity. For the extremal, non-symplectic case, the Lorentz force must vanish, and there is no magnetic dynamo action.

From the argument developed above for symplectic systems, the Bernoulli-Casimir energy function Θ is either of the type *TS* and/or of the type *PV*. For a solid, assume the former representation dominates. Then the "Lorentz force" must have the form of a spatial gradient of the temperature, $\rho(\mathbf{E} + \mathbf{V} \times \mathbf{B}) = grad(kT)$. For motion that is along the magnetic field lines, the term $\mathbf{V} \times \mathbf{B} \Rightarrow 0$. Then, incorporating the empirical Ohmic relation, $\mathbf{j} = \sigma\mathbf{E}$, it is apparent that the symplectic case leads to a derivation of flux equations in the Thompson format for thermal power:

$$\mathbf{j} = (1/\rho\sigma)grad(kT) \qquad 52$$

The suggestion is that the source of magnetic dynamo forces in plasmas is to be associated with a temperature gradient and the existence of differential velocity fluctuations in a symplectic system.

The Torsion Vector and Irreversible Evolutionary Processes (Pfaff dimension 4)

Assume that the Pfaff dimension of the domain of interest is 4, hence the space is symplectic. However, consider evolutionary fields that are not constrained to be symplectic such that $dW \neq 0$. Direct evaluation of the virtual work 1-form, $W = i(\mathbf{W})d\mathrm{A}$ yields (the Lorentz force)

$$W = i(\mathbf{W})d\mathrm{A} = (\{\rho\mathbf{E} + \mathbf{J}x\mathbf{B}\}_k dx^k - \{\mathbf{J} \bullet \mathbf{E}\}dt) \qquad 53$$

The obvious first choice for the evolutionary vector field has been based on the classic assumption that $\mathbf{W} = \{\mathbf{J}; \rho\} \Rightarrow \rho\{\mathbf{V}; 1\}$ The expression for virtual work becomes

$$W = \rho(\{\mathbf{E} + \mathbf{V}x\mathbf{B}\}_k dx^k - \{\mathbf{V} \bullet \mathbf{E}\}dt). \qquad 54$$

However, another perhaps not so obvious a candidate for a solution vector field is the expression for the Torsion current. That is, examine the evolution along the unique four dimensional vector field,

$$\mathbf{T} = \{(\mathbf{E} \times \mathbf{A} + \mathbf{B}\phi); \mathbf{A} \bullet \mathbf{B}\}. \qquad 55$$

The expression for virtual work becomes

$$W = i(\sigma\mathbf{T})d\mathrm{A} = \sigma(\{(\mathbf{A} \bullet \mathbf{B})\mathbf{E} + (\mathbf{E} \times \mathbf{A}) \times \mathbf{B}\}_k dx^k - \{\mathbf{E} \bullet \mathbf{B}\phi\}dt) = \sigma(\mathbf{E} \bullet \mathbf{B})\mathrm{A}. \qquad 56$$

The torsion current is an associated field relative to the 1-form of Action, in the sense that

$$i(\sigma\mathbf{T})\mathrm{A} \Rightarrow 0. \qquad 57$$

Evolution in the direction of the Torsion vector does not produce any internal energy of interaction, even though the process is not extremal. In Pfaff dimension 4, the Torsion vector is not extremal, but amazingly enough it decays to a process which is characteristic.

It follows that the Lie derivative of the Action along the direction of the Torsion current is an isovector process in the sense that

$$\mathrm{L}_{(\sigma\mathbf{T})}\mathrm{A} = \Gamma\mathrm{A} = \sigma(\mathbf{E} \bullet \mathbf{B})\mathrm{A} = Q. \qquad 58$$

By direct computation,

$$\mathrm{L}_{(\sigma\mathbf{T})}d\mathrm{A} = d\Gamma\hat{}\mathrm{A} + \Gamma d\mathrm{A} = dQ \qquad 59$$

from which it follows that

$$Q\hat{}dQ = \Gamma^2\mathrm{A}\hat{}d\mathrm{A}. \qquad 60$$

If the topological parity $\Gamma = \sigma(\mathbf{E} \bullet \mathbf{B})$ does not vanish, then the Torsion current $\sigma\mathbf{T}$ represents an irreversible non-conservative process. For such processes the heat 1-form, Q, does not admit an integrating factor.

The formula $\mathrm{L}_{(\sigma\mathbf{T})}\mathrm{A} = \Gamma\mathrm{A}$ was the fundamental principle used by the present author in1974 to describe ”An Extension of Hamilton’s Principle to Include Dissipative Systems”. It was not known at that time the such processes implied the existence of a symplectic structure, nor the fact that these processes were not symplecto-morphisms.

More on Characteristic Vectors

On the 4D manifold, those point sets upon which $(\mathbf{E} \bullet \mathbf{B}) = 0$ are special, for then the symplectic manifold becomes a contact manifold, and admits a unique direction field on the 3D submanifold, a Hamiltonian vector field. From the 4D point of view, the rank of the matrix of 2-form coefficients, dA, is 2 (not 4), which implies that there are two null eigen vectors (on 4 D). One of these eigen vectors is the Hamiltonian field such that $i(\mathbf{V})dA = 0$, but $i(\mathbf{V})A = U$ is unspecified. In otherwords, one of the null eigenvectors on the 4D space is extremal, but not associated.

The other null eigenvector is both and extremal and associated; that is, it is a characteristic vector field. That vector field (to within a factor) is given by the Torsion Vector field, $\mathbf{T} = \{(\mathbf{E} \times \mathbf{A} + \mathbf{B}\phi); \mathbf{A} \bullet \mathbf{B}\}$ subject to the closure condition that $4Div\mathbf{T} = 2(\mathbf{E} \bullet \mathbf{B}) = 0.$ In other words, the Torsion Vector defines the adiabatic direction field in space-time, which preserves every element of the Pfaff sequence locally.

$$\mathrm{L}_{(\sigma\mathbf{T})}\mathrm{A} = 0, \qquad 61$$

and

$$\mathrm{L}_{(\sigma\mathbf{T})}d\mathrm{A} = 0. \qquad 62$$

The characteristic field is usually associated with propagating discontinuities, or signals. (Wave phenomena)

Conjugate and Minimal Surface Wave Functions

Bateman's Solutions

In 1914, in a small monograph entitled Electrical and Optical Wave Motion, H. Bateman [1] introduced a number of interesting solutions to Maxwell's equations that emulate propagating singular strings (not plane waves). Bateman is perhaps more famous for his work on the equations that describe the decay chains of radioactive species. However, as pointed out by Whittaker [2], it was Bateman who determined in 1910 that the Maxwell equations were invariant with respect to the conformal group, a much wider group than the Lorentz transformations. Bateman in 1910 also recognized the relationship of his work to the tensor calculus of Ricci and Levi-Civita, several years before the Einstein development of general relativity [3].

Maxwell's equations are defined as the Maxwell Faraday equations for the field intensities,

$$curl\,\mathbf{E} + \partial\mathbf{B}/\partial t = 0, \qquad div\,\mathbf{B} = 0 \qquad 1$$

and the Maxwell Ampere equations for the field excitations, with source.

$$curl\,\mathbf{H} - \partial\mathbf{D}/\partial t = \mathbf{J} \qquad div\mathbf{D} = \rho \qquad 2$$

The vacuum is defined by the constraints of a source free domain, such that the current density vanishes, $\mathbf{J} = 0,$ and the charge density vanishes, $\rho = 0$. In addition, the vacuum is defined by a set of generalized constitutive constraints between the field intensities and the field excitations, which take the form of the 6x6 matrix equation,

$$\left| \begin{matrix} \mathbf{D} \\ \mathbf{H} \end{matrix} \right\rangle = \begin{bmatrix} \varepsilon_0 & 0 & 0 & \gamma & 0 & 0 \\ 0 & \varepsilon_0 & 0 & 0 & \gamma & 0 \\ 0 & 0 & \varepsilon_0 & 0 & 0 & \gamma \\ \gamma^\dagger & 0 & 0 & 1/\mu_0 & 0 & 0 \\ 0 & \gamma^\dagger & 0 & 0 & 1/\mu_0 & 0 \\ 0 & 0 & \gamma^\dagger & 0 & 0 & 1/\mu_0 \end{bmatrix} \circ \left| \begin{matrix} \mathbf{E} \\ \mathbf{B} \end{matrix} \right\rangle . \qquad 3$$

Bateman presumed, as have most authors, that the vacuum state requires that $\gamma = 0$. Indeed, the Lorentz vacuum will be defined as the case where $\gamma = 0,$ and the Chiral vacuum will be defined as the case when $\gamma \neq 0$.

Assuming

$$\mathbf{D} = \varepsilon_0 \mathbf{E} \qquad \mathbf{H} = \mathbf{B}/\mu_0 \qquad 8$$

the Maxwell-Ampere equation yields

$$\mathbf{J} = curl\,\mathbf{H} - \partial\mathbf{D}/\partial t = \{curl\mathbf{B} - \varepsilon\mu\partial\mathbf{E}/\partial t\}/\mu \Rightarrow 0 \qquad 2$$
$$\rho = div\mathbf{D} = div\mathbf{E}/\varepsilon \Rightarrow 0$$

Each term must vanish for the ”vacuum” condition of no charge density and no current density. Suppose the conditions are true. Then differentiating the first expression with respect to time, and taking the curl of the Maxwell-Faraday expression, combine to yield

$$grad\,div\mathbf{E} - curl\,curl\,\mathbf{E} - \varepsilon\mu\partial^2\mathbf{E}/\partial t^2$$

In other words a necessary condition for the Lorentz vacuum is that the fields satisfy the Vector Wave Equation (with $div\mathbf{E} = 0$).

Following Bateman, form the inner 3D product of the Maxwell Faraday equation with $\mathbf{H} = \mathbf{B}/\mu,$ and the inner product of the source free Maxwell Ampere equation with **E**. Use the constitutive definitions for the Lorentz vacuum where $\mathbf{H} = \mathbf{B}/\mu$ and $\mathbf{D} = \varepsilon\mathbf{E}$. Subtract the second resultant from the first, to create the equation (assuming $\gamma = 0$), to produce the famous Poynting equation,

$$div(\mathbf{E} \times \mathbf{H}) + \mathbf{H} \circ \partial\mathbf{B}/\partial t + \mathbf{E} \circ \partial\mathbf{D}/\partial t \Rightarrow \qquad 4$$
$$div(\mathbf{E} \times \mathbf{H}) + \partial(1/2\mathbf{B}^2/\mu + 1/2\varepsilon\mathbf{E}^2)/\partial t = 0.$$

The result is an equation of continuity in terms of the field variables. By comparison to a ”fluid”, this ”equation of continuity” yields a field energy density, $\rho_e,$ and an energy current density, $\rho_e\mathbf{v},$ given by the expressions:

$$\rho_e c^2\mathbf{v} = (\mathbf{E} \times \mathbf{H}) = (\mathbf{D} \times \mathbf{B})c^2 \quad and \quad \rho_e c^2 = (1/2\mathbf{B}^2/\mu + 1/2\varepsilon\mathbf{E}^2).$$

It should be remembered that these equations can be complex. The energy current density and the energy density can be formed from complex numbers. Bateman finds the extraordinary result, equivalent to the expression,

$$\rho_e^2(1/\mu\varepsilon - \mathbf{v}\circ\mathbf{v}) = \rho_e^2(c^2 - \mathbf{v}\circ\mathbf{v}) \qquad 6$$
$$\equiv (1/c^2)\{[(1/2)(\mathbf{D}\circ\mathbf{E}) - (1/2)(\mathbf{B}\circ\mathbf{H})]^2 + (\mathbf{E}\circ\mathbf{B}/Z_{freespace})^2\}.$$

under the assumption that $\varepsilon\mu c^2 = 1$. The factor (μ/ε) is the square of the radiation impedance of free space, $Z_{freespace} = \sqrt{\mu/\varepsilon}$. It is apparent that the first term on the right is the first Poincare (conformal) invariant equivalent to the Lagrange energy density of the field (the difference between the deformation and the kinetic energy densities). The second term is the second Poincare invariant of the field, and is to be associated with topological parity and thermodynamic irreversibility [4]. Bateman remarks that ”the rate at which energy flows through the field is less that the velocity of light”, unless the two Poincare invariants on the RHS vanish. The importance of the null Poincare invariants becomes obvious, as they furnish the requirement that the field energy propagates with the speed of light. It is important to remember that these equations can involve complex vector fields.

In general, for the Lorentz vacuum, if the energy density of the field is defined as

$$Ham = (1/2)(\mathbf{D}\circ\mathbf{E}) + (1/2)(\mathbf{B}\circ\mathbf{H}) = 1/2\mathbf{B}^2/\mu + 1/2\varepsilon\mathbf{E}^2$$

while the field Lagrangian is defined classically as

$$Lag = (1/2)(\mathbf{D}\circ\mathbf{E}) - (1/2)(\mathbf{B}\circ\mathbf{H}) = 1/2\varepsilon\mathbf{E}^2 - 1/2\mathbf{B}^2/\mu.$$

These are all classic results. The objective is to see what happens to these expressions for the Chiral Vacuum. The Chiral adjective is appropriate, for a pure imaginary γ replicates certain features of media which are optically active. The classic example of an optically active media is a solution of right handed helical molecules, such as sugar, in water. The phemomena has practical use in the wine industry and has been used to permit the grower to determine the sugar content of his grapes. (This is the basis of the words $\circ$*brix* often found on French wine labels)

An objective of this article is to study those cases where γ is a domain constant, but not zero, and then to determine what are the consequences of such an assumption. Such an assumption, which if applicable to the vacuum, would imply that the Chiral vacuum, and therefor the universe itself, does not have a center of symmetry.

It is also of some historical perspective that, in 1914, Bateman noticed that a complex 3-dimensional vector, $\mathbf{M} = \mathbf{B} \pm i\sqrt{\varepsilon\mu}\,\mathbf{E}$ could be used to express both the Maxwell Faraday and the Maxwell Ampere equations for the Lorentz vacuum as one combined set of complex vector equations. Sometimes it is possible to find a conjugate pair of solutions $\mathbf{M}$ and $\mathbf{M}'$ that satisfy the complex equation

$$\mathbf{M}\circ\mathbf{M}'=0. \qquad 7$$

Each solution satisfies the equation

$$\mathbf{M}\circ\mathbf{M} =(\mathbf{B}^2 - \varepsilon\mu\mathbf{E}^2) \pm 2i(\sqrt{\varepsilon\mu}\,\mathbf{E}\circ\mathbf{B}) = \mathbf{I}_1 \pm 2i\,\mathbf{I}_2, \qquad 7$$

where $\mathbf{I}_1$ and $\mathbf{I}_2$ are the Poincare conformal invariants of the field, $\mathbf{M}$.

If the complex solution vector satisfies the complex equation of constraint,

$$\mathbf{M}\circ\mathbf{M} =(\mathbf{B}^2 - \varepsilon\mu\mathbf{E}^2) + 2i(\sqrt{\varepsilon\mu}\,\mathbf{E}\circ\mathbf{B}) = 0, \qquad 7$$

then such a vector not only satisfies both the Maxwell Faraday and the Maxwell Ampere (source free) equations for a Lorentz vacuum, but also - according to the derived result in equation 6 - propagates the field energy with the speed of light. Such solutions were defined by Bateman as self conjugate solutions. (Translate to self dual solutions in modern day language of Yang Mills theory.)

The Bateman self conjugate condition requires that the (complex) magnetic energy density be the same as the (complex) electric energy density, and the (complex) Electric field be orthogonal to the (complex) Magnetic field. Both of these Poincare conformal invariants must be zero to satisfy the Bateman self duality condition. It is the self dual solutions, these self conjugate solutions, that satisfy the Eikonal expression, and therefore, as Bateman points out [5], can represent propagating electromagnetic discontinuities [6]. The Poincare invariants are additive, such that it is conceivable to construct a self-conjugate solution from two or more non-self conjugate solutions, each of which has different Poincare invariants, but which are equal to zero under addition.

Bateman apparently did not notice that the complex constraint equation of self duality on **M** is precisely the conditions that the complex position vector generated by **M** defines a minimal surface [7]. Moreover, Bateman did not notice that most of his results are to be obtained for a Chiral vacuum, when the chiral components of the constitutive equation satisfy the constraint, $[\gamma]^{\dagger} + [\gamma] = 0$. The details of the Chiral Vacuum condition are explored in the the next section. [Note that the Bateman ideas can be transcribed to more modern form by treating the the source free vacuum Maxwell equations as an exterior differential ideal, $dF = 0$ and $dG = 0$. See below.]

THE CHIRAL VACUUM

Use the (complex) Chiral Vacuum constitutive equations

$$\mathbf{D} = \varepsilon_0\mathbf{E} + [\gamma]\circ\mathbf{B} \qquad \mathbf{H} = [\gamma^{\dagger}]\circ\mathbf{E} + \mathbf{B}/\mu_0 \tag{8}$$

along with the Maxwell Faraday equations and the Maxwell Ampere equations, and replicate the steps of the preceeding section. For simplicity, assume that the matrix

$$[\gamma] = (g + \sqrt{-1}\,\gamma)[\mathbf{I}] \text{ and } [\gamma \dagger] = \alpha(g - \sqrt{-1}\,\gamma)[\mathbf{I}] \quad with \quad \alpha = \pm 1$$

Substitution into the Maxwell Ampere equation yields

$$\begin{aligned} \mathbf{J} &= curl\,\mathbf{H} - \partial\mathbf{D}/\partial t = \{curl\mathbf{B} - \varepsilon\mu\partial\mathbf{E}/\partial t\}/\mu \\ &\quad + g(\alpha\, curl\, E - \partial B/\partial t) - \sqrt{-1}\,\gamma(\alpha\, curl\, E + \partial B/\partial t) \\ \rho &= div\mathbf{D} = \varepsilon div\mathbf{E} + (g + \sqrt{-1}\,\gamma)(div\,\mathbf{B}) \end{aligned} \tag{2}$$

The point of this exercise is to note that the Chiral Vacuum produces no charge currents or charge densities, subject to certain conditions, if the field intensities satisfy the vector wave equation, (with $div\mathbf{E} = 0$, and no magnetic monopoles, $div\mathbf{B} = 0$ - a result which is valid if the field intensities are derived from a set of potentials).

The three cases to be considered are:

$$1.\ g = 0, \gamma = 0, The\ Lorentz\ Vacuum\ case.$$

$$2.\ g = 0, any\ \gamma, alpha = 1.$$

$$3.\ any\ g,\ \gamma = 0, alpha = -1.$$

Similar substitutions of the Chiral constiutive equations lead to the Poynting equation in the form:

$$div(\mathbf{E}\times\mathbf{H}) + \mathbf{H}\circ\partial\mathbf{B}/\partial t + \mathbf{E}\circ\partial\mathbf{D}/\partial t = 0 \Rightarrow \qquad 9$$
$$div(\mathbf{E}\times\mathbf{H}) + \partial(1/2\mathbf{B}^2/\mu + 1/2\varepsilon\mathbf{E}^2)/\partial t = -\{(\alpha+1)g - \sqrt{-1}\,(\alpha-1)\gamma\}\mathbf{E}\circ\partial\mathbf{B}/\partial t.$$

If the RHS of the equation above vanishes, then the Poynting theorem of equation (6) is retrieved without change in form. The three constraints for which the equivalence between the Chiral vacuum and the Lorentz vacuum is true, are the same as above.

The next step is to evaluate the expressions for the total field Hamiltonian energy density and the Lagrange density of the Chiral Vacuum. The expression for the Hamiltonian energy density becomes

$$Ham = (1/2)(\mathbf{D}\circ\mathbf{E}) + (1/2)(\mathbf{B}\circ\mathbf{H}) =$$
$$1/2\mathbf{B}^2/\mu + 1/2\varepsilon\mathbf{E}^2 + \{(\alpha+1)g - \sqrt{-1}\,(\alpha-1)\gamma\}\mathbf{E}\circ\mathbf{B}/2$$

while the field Lagrangian is becomes:

$$Lag = (1/2)(\mathbf{D}\circ\mathbf{E}) - (1/2)(\mathbf{B}\circ\mathbf{H}) =$$
$$1/2\varepsilon\mathbf{E}^2 - 1/2\mathbf{B}^2/\mu - \{(\alpha-1)g - \sqrt{-1}\,(\alpha+1)\gamma\}\mathbf{E}\circ\mathbf{B}/2$$

These results indicate that there are slight modifications to the formulas, modifications that are dependent upon the second Poincare invariant. However, for systems where the field intensities are deducible from a 1-form of potentials, and the 1-form is of Pfaff dimension 3 or less, then $\mathbf{E}\circ\mathbf{B}$ vanishes, and all computations of Hamiltonian or Lagrangian energy densities are identical for the Lorentz Vacuum, or for the Chiral vacuum. It is only for cases where the 1-form of potentials is of Pfaff dimension 4, such that $\mathbf{E}\circ\mathbf{B}\neq 0,$ that the Chiral factors can make a difference in the expressions for energy density. Futhermore the chiral effect does not influence the Hamiltonian energy density, it only effects the Lagrangian energy density of the field, in that it no longer is a difference of two quadratic forms. The case $\alpha = 1, g = 0$ implies that the Lagrangian has a pure imaginary contribution proportional to $\mathbf{E}\circ\mathbf{B}\neq 0,$ while the case $\alpha = -1, \gamma = 0,$ implies that the Lagrangian has a pure imaginary contribution proportional to $\mathbf{E}\circ\mathbf{B}\neq 0.$

These are a rather startling results for they demonstrate that the Lorentz vacuum and the Chiral vacuum can be formally indistinguishable, except for the impedance of free space (which is related to the determinant of the constitutive tensor

$$1/Z_{freespace} = \sqrt{\varepsilon/\mu - \alpha(g^2+\gamma^2)}\,.$$

There is a lot of open questions: If the vacuum is chiral, is there an equal distribution of right handed vs. left handed photons coming from distant objects.? Is the redshift sensitive to photon chirality?

Is $\quad g^2 + \gamma^2 \approx (e^2/\hbar)^2$

Whittaker Bateman Cartan Pfaff dimension 2 stuff

In modern language of differential forms, Bateman determined that if two functions $\alpha(x,y,z,t)$ and $\beta(x,y,z,t)$ used to define the 2-form

$$F = d\alpha\hat{}d\beta = d(1/2(\alpha d\beta - \beta d\alpha)),$$

such that the field components

$$\mathbf{E} = (\partial\alpha/\partial t)\nabla\beta - (\partial\beta/\partial t)\nabla\alpha \;\; and \;\;\; \mathbf{B} = \nabla\alpha \times \nabla\beta,$$

satisfied the Maxwell Faraday equations, $curl\,\mathbf{E} + \partial\mathbf{B}/\partial t = 0,$ then the N-2 form defined by construction as

$$G = i(d\alpha)\hat{}i(d\beta)\Omega = i(d\alpha)\hat{}i(d\beta)dx\hat{}dy\hat{}dz\hat{}dt$$

would satisfy the Maxwell Ampere equations $curl\,\mathbf{H} - \partial\mathbf{D}/\partial t = 0$ with

$$\mathbf{H} = (\partial\alpha/\partial t)\nabla\beta - (\partial\beta/\partial t)\nabla\alpha \;\; and \;\;\; \mathbf{D} = \nabla\alpha \times \nabla\beta.$$

In terms of the general constitutive relation, $\mathbf{D} = [\varepsilon] \circ \mathbf{E} + [\gamma] \circ \mathbf{B}$ and $\mathbf{H} = [1/\mu] \circ \mathbf{B} + [\gamma^{\dagger}] \circ \mathbf{E}$, it would appear that the two function construction considered above would imply an unusual constitutive map of the form $\mathbf{D} = [\gamma] \circ \mathbf{B}$ and $\mathbf{H} = -[\gamma] \circ \mathbf{E}$. Again the ant-hermitian constraint appears in the matrix form $[\gamma] + [\gamma^{\dagger}] = 0$. Such excitation fields have chiral parts only. It is not necessary for the fields to satisfy a wave equation under the assumptions stated. However, the presumption has to be made that $[\varepsilon] \circ \mathbf{E}$ and $[1/\mu] \circ \mathbf{B}$ are zero. If $[\varepsilon]$ and $[1/\mu]$ are non-zero constants, then the $\mathbf{E}$ and $\mathbf{B}$ fields must satisfy the wave equation, with a propagation speed equal furnished by $1/\sqrt{\varepsilon\mu}$. Any pair of functions can be added to a solution of the wave equation, and Maxwell's equations for a chiral vacuum are satisfied.

Now consider complex vector fields such that

$$\mathbf{B}^2 - \varepsilon\mu\mathbf{E}^2 = (\mathbf{B} - i\sqrt{\varepsilon\mu}\,\mathbf{E}) \circ (\mathbf{B} + i\sqrt{\varepsilon\mu}\,\mathbf{E}) = 0.$$

which implies that

$$(\nabla\alpha \times \nabla\beta)^2 - \varepsilon\mu\{(\partial\alpha/\partial t)\nabla\beta - (\partial\beta/\partial t)\nabla\alpha\}^2$$

This equation can be satisfied if both the square of the complex vector $\mathbf{B}$ and the complex vector $\mathbf{E}$ vanish, separately. The two complex functions would have to satisfy

An Example Chiral Solution

A maple program that will compute and test configurations for a chiral vacuum are provided at http://www22.uh.edu/csdc/maple/maxcompl.mws.

Consider the two functions

$$\alpha = (x + iy)/(z + r) \qquad \beta = r - ct \quad where \quad r = \sqrt{x^2 + y^2 + z^2}$$

Construct

$$\mathbf{E} = (\partial\alpha/\partial t)\nabla\beta - (\partial\beta/\partial t)\nabla\alpha \;\; and \;\;\; \mathbf{B} = \nabla\alpha \times \nabla\beta,$$

for which $\mathbf{E} \circ \mathbf{E} = 0$, $\mathbf{B} \circ \mathbf{B} = 0$, and $\mathbf{E} \circ \mathbf{B} = 0$. The functions α and β satisfy the eikonal equation.

$$(\nabla\beta)^2 - (1/c\partial\beta/\partial t)^2 = 0$$

and **E** and **B** satisfy the vector wave equation:

$$grad\ div\,\mathbf{E}\ -\ curl\ curl\ \mathbf{E}\ -\ \varepsilon\mu\partial^2\mathbf{E}/\partial t^2 = 0$$

Minimal Surfaces

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Now of even more extraodinary interest is the fact that Bateman's self conjugate solutions, based on a complex vector with zero square, define a minimal surface. Before going on note that the norm of the self conjugate vector solution is given by the expression

$$\mathbf{M}^*\circ\mathbf{M} = [c^2\mathbf{B}^*\circ\mathbf{B} + \mathbf{E}^*\circ\mathbf{E}]$$

which is proportional to the energy density of the field. If this term is non-zero, it will turn out that the minimal surface will be regular.

From Osserman, a generalized minimal surface in E^N is a non-constant map from a 2-manifold, M, with a conformal structure (over regular regions) such that the coordinates of E^N are harmonic on M. Let the map be defined by

$$\Psi : \{\alpha,\beta\} \Rightarrow \mathbf{X}^k(\alpha + i\beta)$$

Define the complex vector

$$\mathbf{M} = \partial(\mathbf{X}^k(\alpha+i\beta))/\partial\alpha - i\partial(\mathbf{X}^k(\alpha+i\beta))/\partial\beta \Rightarrow \mathbf{E} + ic\mathbf{B}$$

Then if $\mathbf{M}\circ\mathbf{M} = 0$ the two functions α and β form a set of isothermal coordinates on a minimal surface. The induced metric $g_{\alpha\beta} = \sum_k \partial\mathbf{X}^k/\partial\alpha \circ \partial\mathbf{X}^k/\partial\beta$ generates a conformal structure, and if $\mathbf{M}^*\circ\mathbf{M} \neq 0$, the minimal surface is regular (without self intersections or pinch points). For real **E** and **B** the minimal surfaces are always regular except when both components vanish identically (No field intensities). For complex **E** and **B** the minimal surfaces can have singularities, which are of particular interest from a topological point of view.

The self conjugate (minimal surface) condition requires that the **E** field is orthogonal to the **B** field, and the electric and magnetic energy densities are equal. The conclusion is reached that propagating electromagnetic waves are to be associated with minimal surfaces. The associated minimal surface is always regular and without singularities for real, non-zero **E** and **B**. When the **E** and **B** fields are complex, which in a physical sense implies the existence of elliptical polarization, another interpretation is possible.

This association of electromagnetic wave propagation with minimal surface theory was apparently unknown to Bateman, and not appreciated by the present author until only very recently, following a nth re-reading of Osserman's book on A Survey of Minimal Surfaces. According to Osserman, the complex 3-vector representations of minimal surfaces were known to Enneper and Weierstrass. A study of the minimal surfaces generated in E4 is given by Kommerell. The minimal surfaces so generated in E4 by this class of vector fields will have 3-dimensional images that are not always regular. In general, two dimensional non-regular surfaces may have "singularities" consisting of "curves of double points" created by intersections of two local surface patches, or of ”triple” points consisting of intersections of three local surface patches, or of curves of double points which terminate on ”Pinch” points within the interior of the surface. These three types of self intersection singularities are the only three ”stable” singularities in the sense of Whitney. Recall that Whitney proved that any N manifold can be embedded in 2N+1 euclidean space, and immersed in a 2N euclidean space. The induced surfaces may be orientable or non-orientable. The non-orientable examples are characterized by the Klein-Bottle, or the Projective Plane, and the orientable surfaces by the Sphere. Each surface may have tubular handles, holes and distortions. Of interest to this work are not just any surface, but those surfaces which in particular are minimal surfaces. If the surface has no singularities, then the surface is said to be regular or embedded. The constraint of regularity implies that the surface normal vector never goes to zero over the surface, or the induced metric on the surface is always invertible. This implies that are always two linearly independent directions on a regular domain of the surface. If the lines of self intersection are divergence free on the domain (meaning that they stop or start only on boundary points, or are closed upon themselves, then the surface is said to be immersed in 3-Dimensions. The points where the divergence of the lines of intersection is not zero are defined as Pinch points. Such surfaces cannot be immersed in 3-D. The Pinch points are signatures of the fact the surface resides in 4-Dimensions (as an immersion), and cannot be immersed in 3-Dimensions. A flow vector field may have domains where it is irrotational or solenoidal, and these domains may be separated by a surface. If the surface of separation is a minimal surface, then the flow on this surface is harmonic. The minimal surface need not be regular, and may have lines of self-intersection. These lines of surface self-intersections (lines of singular double points) are not necessarily solenoidal. In fact, the Pinch points are points where the lines of self-intersection terminate not on themselves and not on a boundary, but in the surface interior. The Pinch points may be viewed as the ”sources” of the divergence of the lines of self-intersection. The classic example is given by Whitney’s

The Bohm-Aharanov Index and the Hopf Index

There are two topological concepts that can be constructed from the 1-form of Action, or Maxwell potentials, A. The first involves domains where the 2-form $F = dA \Rightarrow 0$ (the electromagnetic intensities vanish). The second involves domains where the 4-form $F\hat{}F = dA\hat{}dA \Rightarrow 0$. These constraints produce the conditions of closure on the 1-form A and the 3-form $A\hat{}dA$ but neither differential form need be exact. For example if the domain $dA \Rightarrow 0$ defines holes in a piece of a connected surface, then for closed cycles that bound the hole and are in the domain $dA = 0$, the value of the line integral, $\oint_{1-cycle} A \circ dl \simeq \text{integer}\, 2\pi$. This integer is the Bohm-Aharanov index and is to be associated with the flux-quantum h/e in quantum mechanics. It is an index that counts the number of ”holes” in the compliment to the domain of the field intensities.

Consider the evolution of the integral of the Action 1-form over a cycle relative to the 4-vector V

$$L_{(V)} \oint_{1-cycle} A = \oint_{1-cycle} i(V)(dA) + \oint_{1-cycle} d[i(V)(A)] \quad = \oint_{1-cycle} W + 0$$

If the Action in a relative integral invariant relative to the evolution generated by the field V, then the RHS must be zero. The statement is equivalent to the concept that the virtual work 1-form, W, be exact. Such is the Cartan constraint that the evolutionary field has a

Hamiltonian representation.

The three dimensional integral $\iiint_{closed\,3-cycle} A\hat{}F$ also has values that "count" the number of obstructions in the domain where $F\hat{}F = 0$. This integer is defined as the Hopf Index. Both of these topological objects are constructed from harmonic forms, and are properties of the potentials that generate the intensive variables.

Consider the evolution of the 3-form $A\hat{}F$ in the direction of the charge current 4 vector, $J = dG$.

$$\begin{aligned} L_{(J)} \iiint_{closed\,3-cycle} (A\hat{}F) &= \iiint_{closed\,3-cycle} i(J)(dA\hat{}dA) + \iiint_{closed\,3-cycle} d[i(J)(A\hat{}F)] \\ &= \iiint_{closed\,3-cycle} 2(\mathbf{E} \circ \mathbf{B})\, J + 0 \end{aligned}$$

On domains of Pfaff dimension 3, the RHS vanishes, as $(\mathbf{E} \circ \mathbf{B}) = 0$. Hence the integral of the 3-form $A\hat{}F$ over a closed cycle is an evolutionary invariant with respect to evolution in the direction of the charge-current 4 vector. (The Hopf index is a relative integral invariant). Note that if the integration domain is a boundary, and if $d(\mathbf{E} \circ \mathbf{B})\hat{}J = 0,$ then by Stokes theorem, the integral over a boundary of $A\hat{}F$ is an evolutionary invariant. The requirement $d(\mathbf{E} \circ \mathbf{B})\hat{}J = 0$ is equivalent to the idea that the current 4-vector is confined to the hyper-surface, $\mathbf{E} \circ \mathbf{B} = 0$.

Reprise

Note that although all of the symbols used above are familiar in the realm of electromagnetism, the topological results and formulas obtained apply to any set of symbols, for example a fluid. The Faraday Maxwell equations are universal ideas on systems of C2 functions.

For a computational example in Maple, see http://www.uh.edu/~rkiehn/pdf/ranada.pdf

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