Tensors, Jacobians, and Metrics

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Abstract

Class Notes 1993, et. seq., on Tensors and Diffeomoprhic maps

1. Some elementary features

Tensor analysis is concerned with the behavior of ordered sets of functions under special classes of differentiable mappings, ϕ, from an initial state of independent (base) variables to a final state of independent (base) variables.

Everything starts from the idea of a C1 map ϕ from an initial state $\{\xi^k\}$ to a final state $\{x^\mu\}$, and its differential, $d\phi$.

$$\phi \;:\; \xi^k \Rightarrow x^\mu = \phi^\mu(\xi^k) \qquad and \tag{1.1}$$

$$d\phi \;:\; d\xi^k \Rightarrow dx^\mu = \left[\partial\phi^\mu(\xi^n)/\partial\xi^k\right] d\xi^k. \tag{1.2}$$

The last expression defines the Jacobian mapping *linearly* connecting the differentials $d\xi^k$ and dx^μ, of the initial and final state.

$$\mathbb{J} = \left[\partial\phi^\mu(\xi^n)/\partial\xi^k\right]. \tag{1.3}$$

If the rank of the Jacobian is globally constant and equal to the dimension of the final state, the map is said to be onto (a submersion). If the rank of the Jacobian is constant and equal to the dimension of the initial state, the map is said to be 1-1 (a parametrization). Special situations occur when the map is both 1-1

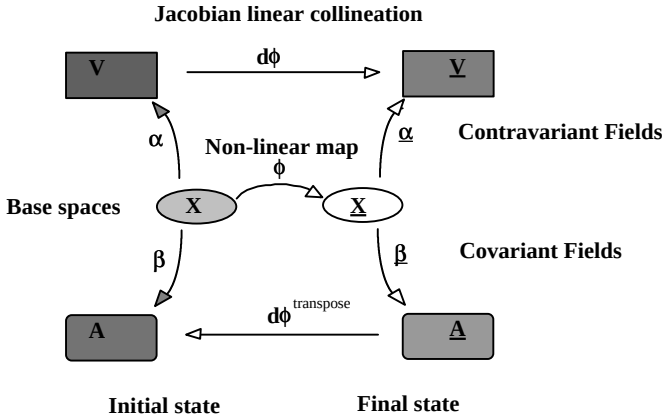

Figure 1.1:

and onto. These special maps are called diffeomorphisms, and they preserve the topology of the initial and final state.

Next consider ordered sets of differentiable functions with arguments in terms of the base variables. A single function will be defined as a tensor of rank 0. Over a base of N independent variables, an ordered set of N functions (components) will be defined as a tensor of rank 1, if the components of the ordered set on the final state can be **linearly** connected to the components of the ordered set on the initial state. What is remarkable is that there are two independent species of ordered sets of functions of the same rank: these species of tensors are epitomized by the contravariant vectors, $V^k(\xi^j)$, with upper indices (by convention) and the covariant vectors $A_\mu(x^\nu)$, with lower indices by convention). These two species of tensors of the same rank behave differently under the same mapping! The covariant tensors are more like "waves" and the contravariant tensors are more like "particles". Of course, the mapping can be considered a model of an evolutionary process.

The figure generalizes the situation for differentiable maps:

So, lets summarize. The map, ϕ, between initial $\{\xi^k\}$ and final states $\{x^\mu\}$ of base variables can be highly non-linear, but tensors are the restricted class of ordered sets of functions built over the base variables where the admissable arrays over the initial state must be linearly connected with the final state. The simplest ordered array is called a tensor of rank 1, or a vector. A tensor of rank

0 is a single scalar function. Tensors of higher rank will be described below). The linear mappings in the figure are often called collineations and correlations (in projective geometry). The key features of the linearity are governed by the components of the Jacobian matrix, its transpose, its adjoint, and if it exists, its inverse.

1.1. Contravariant Vectors

Physically, the covariant tensors are more like "waves" and the contravariant tensors are more like "particles". In many treatments of classical tensor analysis it is assumed that the Jacobian matrix has an inverse, and in addition, the Jacobian matrix is an element of the orthogonal group. Hence the inverse of the Jacobian is its transpose. For such a restriction, the equation for the linear behavior of the contravariant vector under the mapping is given by the matrix equation,

$$Contravariant: \qquad \left|V^k(\xi^n)\right\rangle \Rightarrow |V^\mu(\xi^n)\rangle = [\mathbb{J}^\mu_k(\xi^n)] \circ \left|V^k(\xi^n)\right\rangle \tag{1.4}$$

which maps the components of the contravariant vector on the initial state to the components of the contravariant vector on the final state. Note that the components $V^\mu(\xi^n)$ induced on the final state are functions whose arguments are on the initial state! Only if an inverse mapping function exists does the contravariant transformation rule produce an ordered array of functions on the final state whose arguments are the base variables of the initial state. Classical tensor analysis is restricted to such a constraint; the maps must be diffeomorphisms.

The transformation rule is modeled after the linear rule given for $d\phi$. The word contravariant is historical.

1.2. Covariant Vectors

Now consider another classic rule representing the total differential of a function on the final state,

$$d\Theta(x^\mu) = (\partial\Theta(x^\mu)/\partial x^\nu)dx^\nu = \underline{A}_\nu(x^\mu)dx^\nu = \langle \underline{A}_\nu(x^\mu)| \circ |dx^\nu\rangle \,. \tag{1.5}$$

Substitute the expression for $dx^\mu = [\mathbb{J}^\mu_k(\xi^n)] \circ \left|d\xi^k\right\rangle$, to obtain

$$\langle \underline{A}_\nu(x^\mu)| \circ |dx^\nu\rangle = \langle \underline{A}_\nu(x^\mu)| \circ [\mathbb{J}^\nu_k(\xi^n)] \circ \left|d\xi^k\right\rangle \,. \tag{1.6}$$

This leads to a linearly equivalent expression for the total differential on the initial state.

$$\langle A_k(x^\mu(\xi^n)| \circ \left|d\xi^k\right\rangle = \langle \underline{A}_\nu(x^\mu)| \circ [\mathbb{J}_k^\nu(\xi^n)] \circ \left|d\xi^k\right\rangle = \langle \underline{A}_\nu(x^\mu)| \circ |dx^\nu\rangle \tag{1.7}$$

The covariant vector on the final state $\underline{A}_\nu(x^\mu)$ is related to the covariant rule on the final state by means of the transpose of the Jacobian matrix.

$$\langle A_k(x^\mu(\xi^n)| \Leftarrow \langle \underline{A}_\nu(x^\mu)| \circ [\mathbb{J}_k^\nu(\xi^n)] \tag{1.8}$$

This operation is defined as a "pull-back" (from final to initial state), where the contravariant rule is defined as a "push-forward". Note the initial pull back is related to the Jacobian matrix transpose. For Jacobian matrices which are orthogonal, post multiplication by the inverse Jacobian matrix gives a "push forward" format for the behavior of covariant tensors. The maps considered by classical tensor analysis must be diffeomorphisms for the rules to have well defined meanings.

1.3. Differential forms

The concept of a differential form is epitomized by the expression for the total differential given on the final state,

$$A = \underline{A}_\nu(x^\mu)dx^\nu. \tag{1.9}$$

It has a functionally well defined pre-image on the initial state via the pull-back mechanism. The pull back exists if the map ϕ is differentiable, but not necessarily invertible. Hence a unique inverse map need not exist, and yet all differential forms on the final state are retrodictable to the initial state (but not predictable). Hence, the differential forms are well behaved even when the map from initial to final state does not preserve topology. In other words, differential forms can be used to study topological evolution, while ordinary tensor fields cannot. Tensor analysis is restricted to those maps where the dimension of the final state is the same as the dimension of the initial state (otherwise the Jacobian inverse does not exist). Differential forms can be used to study maps where the dimension of the initial state and the dimension of the final state are different.

2. Spaces that admit a Riemannian metric

Now consider the final state to be constrained such that the space has euclidean Cartesian properties. Then use the Pythagorean idea of length as a quadratic

form, $\ell^2 = x^2 + y^2 + z^2$, to produce the quadratic differential form,

$$(\delta s)^2 = (dx)^2 + (dy)^2 + (dz)^2 \tag{2.1}$$

Use functional substitution given by (1) to induce or pullback a value for $(ds)^2$ on the initial state:

$$(\delta s)^2 = \langle dx^\mu | \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} | dx^\nu \rangle = \langle d\xi^m | \, \mathbb{J}^{trans} \circ \mathbb{J} \, | d\xi^n \rangle = \langle d\xi^m | \, g_{mn} \, | d\xi^n \rangle \tag{2.2}$$

The quadratic differential form has the same value on initial and final state, but exists with different functional formats. However, the induced format on the final state is a **well defined function** on the final state. In particular the induced term, $g_{mn} = \mathbb{J}^{trans} \circ \mathbb{J}$, is a functionally well defined second rank tensor field defined on the initial state. It is the **metric** induced on the initial state by the mapping to a euclidean domain.

Note that the quadratic differential form is invariant by its very definition. The induced metric is a symmetric matrix by its very construction, which means it has a complete set of eigenvectors and can be diagonalized. However, the induced metric is diagonal only when the columns of the Jacobian matrix form an orthogonal set. Such is the case in many textbook applications. The diagonal elements are not constants, however, and their square roots are often called scale factors. (See Morse and Feshbach).

When the Jacobian matrix is such that $g_{mn} = \mathbb{J}^{trans} \circ \mathbb{J} = \lambda(\xi^k)\mathbb{I}$ then the metric is said to be conformal (all directions have the same scale factors). If $\lambda(\xi^k)$ is C1 then locally the domain is homogeneous and isotropic. If the $\lambda(\xi^k)$ is constant then the domain is globally homogeneous and isotropic. Note that the Lorentz metric $g_{mn} = \{1,\text{-}1,\text{-}1,\text{-}1\}$ is not strictly isotropic over space time, but it is spatially isotropic.

It is now possible to ask for further transformations of the coordinates such that the metric coefficients remain invariant. The foundations of special relativity are embedded in a search for all transformations that leave the Lorentz metric invariant. V. Fock has proved that there are two possibilities: The only linear set of maps are the Lorentz transformations. However, there is another NON-LINEAR map, the fractional Moebius (Projective) mappings that also preserve the quadratic form given by the Lorentz metric. These non-linear features of Relativity and Electromagnetism have been little explored, but they admit the

concept of electromagnetic signal propagation occurring faster than the speed of light.

Having the knowledge that an induced metric is computable, it is possible to start from another tack, and subsume the existence of some form of metric on the initial state, and then ask how does this metric evolve as the point of interest changes. It is also possible to ask questions about metrics induced on subspaces and determine how such subspace metrics are deformed due to some evolution of the embedding space. Evolutionary processes that preserve the line element are called isometries. Certainly the simple concepts of rotation and translation are isometries, for distances along paths between connectable points (perhaps in subspaces) are preserved. Bending of a thin flat piece of soft copper is approximately an isometry that is also a deformation, and not a rotation or translation. *A* distance between a pair of points on a connected path is preserved under the deformation. Note that the distance between a pair of points along a non-connectable path is not preserved. *Only* the simplest of evolutionary processes conserve distance, yet physics is inundated with the constraints of the Lorentz transformations. Pity.

Problem 1.
Consider the map $x = r\sin(\varphi)\cos(\theta),\ y = r\sin(\varphi)\sin(\theta),\ z = r\cos(\varphi)$.
Compute the Jacobian of the map and the induced metric.

From the idea that a tensor invariant is a product of a contravariant and a covariant tensor, and from the concept that $d\xi^m$ is the role model for contravariant vectors, the length of a vector can be defined in terms of its measure coefficients, g_{mn}, or the metric of the domain.

$$v^2 = V^m g_{mn} V^n \tag{2.3}$$

From this formula, the implication is that $A_m = g_{mn}V^n$ is a covariant vector deduced from the contravariant vector, V^n. The metric tensor acts as a lowering operator converting a Contravariant-vector into a Covariant-vector by "lowering" the index. The reciprocal metric can be considered as a dual operator, or raising (the indices) tensor. Suppose that $d\xi^m - V^m dt = 0$. Then

$$v^2(dt)^2 = (V^m dt) g_{mn} (V^n dt) = \langle d\xi^m |\, g_{mn}\, | d\xi^n \rangle = (\delta s)^2, \tag{2.4}$$

an expression relating distance to speed and time interval. The important thing to remember is that:

$$dt \text{ is a perfect differential, } but\ \delta s\ is\ \textbf{not}. \tag{2.5}$$

$$(\delta s)^2\ is\ an\ invariant, but\ (dt)^2\ is\ \textbf{not}. \tag{2.6}$$

3. Basis Frames and Connections

For tensors of Rank 1 and above, the idea is to find a method of differentiation that will yield the same values on either the initial or final state. The ideas stem from the desire to formulate a differential form on both the initial and the final state of a map which is well defined functionally. Consider the scalar differential expression on the final state $A_\mu(x)dx^\mu$. Next substitute from the map for the differentials $dx^\mu = J^\mu_m d\xi^m$, and then use the map itself to convert functions of x into functions of ξ. These are the procedures of functional substitution, or the "Pullback". The end result is to convert a covariant vector $A_\mu(x)$ on the final state into a covariant vector $A^{'}_m(\xi)$ on the initial state. The covariant transformation law is via the *transpose* of the Jacobian matrix.

$$A^{'}_m(\xi)d\xi^m \Leftarrow A_\mu(x(\xi))J^\mu_m(\xi)d\xi^m \Leftarrow A_\mu(x)dx^\mu \tag{3.1}$$

$$A^{'}_m(\xi) = A_\mu(x(\xi))J^\mu_m(\xi) \tag{3.2}$$

As a scalar invariant in tensor analysis is a inner product of a contravariant and a covariant, and as $d\xi^m$ is a set of contravariants on the initial state, it follows that $A^{'}_m(\xi)$ is a covariant vector on the initial state. This transposition-pullback rule is the proper definition of a covariant transformation, and is based on the concept of a scalar invariant as an inner product, or contraction, between a covariant and a contravariant.

On the right hand side, the product, $A_\mu(x(\xi))J^\mu_m(\xi)$, looks like a product of a set of expansion coefficients on the final state (the $A\mu(x)$) and a set of m basis vectors (with arguments on the initial state). These basis vectors have μ components which is the proper dimension of the final state (the Jacobian matrix $[J^\mu_m(\xi)]$). The differential on the left should be the same as the differential on the right. Using the chain rule, the total differential of the term on the RHS becomes:

$$d\{A_\mu(x(\xi))\ J^\mu_m(\xi)\} = [\partial A_\mu(x(\xi)/\partial x^\sigma)(J^\sigma_n(\xi))\ J^\mu_m(\xi) + A_\mu(x(\xi))\ \partial J^\mu_m(\xi)/\partial\xi^n]d\xi^n \tag{3.3}$$

By the Leibniz rule the differential on the right consists of two parts: first, the differential of the expansion factors multiplying a matrix of basis vectors, and second, the differential of the basis vectors multiplied by the expansion factors. In the Cartesian space of engineering, the basis set consists of constant vectors; hence their differentials are zero. In general spaces, this is not true, and it is a topological problem to develop expressions for the differentials of the basis vectors.

If the Jacobian matrix is 1-1 and onto, then the Jacobian itself can be used as a global basis set for all vector spaces on the domain. Such a global basis set has a non-zero determinant everywhere. It follows that

$$d(\mathbb{J}^{-1}\circ\mathbb{J}) = d\mathbb{J}^{-1}\circ\mathbb{J} + \mathbb{J}^{-1}\circ d\mathbb{J} = d\mathbb{I} = 0. \tag{3.4}$$

Hence, the derivatives of the basis vectors are linear combinations of the original basis vectors. The coefficients of these linear connections are called the (Cartan) connection coefficients:

$$d\mathbb{J} = \mathbb{J} \circ (-d\mathbb{J}^{-1} \circ \mathbb{J}) = \mathbb{J} \circ \mathbb{C} \tag{3.5}$$

In index notation the idea is that the partial derivatives of the Jacobian can be written as linear combinations of the elements of the Jacobian itself:

$$\partial J^{\mu}_{m}(\xi)/\partial\xi^{n} = J^{\mu}_{k}(\xi)C^{k}_{mn} \tag{3.6}$$

The topological idea is that the differentials of the basis vectors are closed in a linear sense; i.e., they can be linearly constructed from the original basis. The dimensionality of the vector space does not change in nearby differential neighborhoods.

It may not be possible to use the Jacobian matrix as a basis set on the final state, but suppose that such a global basis set, $\mathbb{B}$, can be found by other means. Then the Cartan connection formula becomes

$$d\mathbb{B} = \mathbb{B} \circ (-d\mathbb{B}^{-1} \circ \mathbb{B}) = \mathbb{B} \circ \mathbb{C}. \tag{3.7}$$

4. The Repere Mobile

Cartan constructed such a basis $\mathbb{B}$ by considering a position vector, $\mathbf{R}(s) = [x(s), y(s), z(s)]$, to an arbitrary space curve in some euclidean space. The curve may be considered as a map from an initial state $\{s\}$ to a final state $\{x, y, z\}$. In practice, the process usually involves a two stage map from $\{s\} \Rightarrow \{\xi\} \Rightarrow \{x, y, z\}$, as the position

vector $\mathbf{R}(\xi)$ is usually given as a function of some other parameter (called time in many physical systems). A constraint is placed on the map $\{s\} \Rightarrow \{\xi\}$ such that s is the parameter of "arc" length along the curve.

Differentiating this position vector with respect to s leads to the "unit tangent" vector, $\mathbf{t} = d\mathbf{R}/ds = (d\mathbf{R}(\xi)/d\xi)(d\xi/ds)$, tangent to the curve. Actually, the differential process, and the "normalization or inner product", $\mathbf{t} \circ \mathbf{t} = 1$, defines what is meant by arc length, s. Note that this is a metric concept, for $\mathbf{t} = d\mathbf{R}/ds = (\partial\mathbf{R}/\partial\xi)d\xi/ds$ and

$$\mathbf{t} \circ \mathbf{t} = \langle d\xi/ds | \left[(\partial\mathbf{R}/\partial\xi)\right]^{trans} \left[(\partial\mathbf{R}/\partial\xi)\right] | d\xi/ds \rangle = \langle d\xi/ds | \left[g\right] | d\xi/ds \rangle = g(d\xi/ds)^2 \tag{4.1}$$

It is apparent that $[g]$ is the induced metric on the space $\{s\}$ with the single entry or scale factor of value equal to $(ds/d\xi)^2$ (in order to make $\mathbf{t} \circ \mathbf{t} = 1$).

Next, differentiate the unit tangent vector $\mathbf{t}$ with respect to the arc length s, defining the unit normal vector, $\mathbf{n}$

$$\square\mathbf{n} = d\mathbf{t}/ds \tag{4.2}$$

The inner product of the euclidean space can be used to demonstrate that $\mathbf{n}$ and $\mathbf{t}$ are orthogonal, $\mathbf{n} \circ \mathbf{t} = 2d\mathbf{t}/ds \circ \mathbf{t} = 0$ when $\square$, the curvature is not zero. From the completeness of euclidean space, use either the cross product, or the Gram Schmidt process, to construct the unit bi-normal, $\mathbf{b} = \mathbf{n} \times \mathbf{t}$. The basis frame is then constructed as

$$\mathbb{B} = \left[\begin{array}{ccc} \mathbf{t} & \mathbf{n} & \mathbf{b} \end{array} \right] \tag{4.3}$$

which Cartan called the *Repere Mobile, or "Moving Frame"*.

This process of basis set construction is of course the well known Frenet-Serret process, but Cartan extended the idea (later) to projective, not euclidean, spaces where the concept of length and inner product are not needed. The Frenet-Serret-Cartan basis is an element of the orthogonal group, for $\mathbb{B}^{trans} \circ \mathbb{B} = \mathbb{I}$. It follows that the Cartan connection, $\mathbb{C}$, is **anti-symmetric** for the Frenet Basis Frame.

$$d\mathbb{B} = \mathbb{B} \circ \mathbb{C}, \quad with \quad -\mathbb{C}^{trans} = \mathbb{C} = \left[\begin{array}{ccc} 0 & \square & 0 \\ -\square & 0 & \tau \\ 0 & -\tau & 0 \end{array} \right] ds. \tag{4.4}$$

The connection coefficients, $\square$ and τ are defined as the Frenet curvature and the Frenet torsion of the space curve.

Problem 2
Prove that $\mathbb{C}^{trans} = -\mathbb{C}$ for any basis frame which is orthogonal, $\mathbb{B}^{trans} \circ \mathbb{B} = \mathbb{I}$.

As an example consider the map

$$\phi : s \Rightarrow \xi \Rightarrow \{x, y, z\} = \mathbf{R}(\xi) = \left[\xi, \xi^2, \xi^3\right] \tag{4.5}$$

Then

$$d\mathbf{R}/ds = \mathbf{t} = (\partial\mathbf{R}/\partial\xi)d\xi/ds = \{1, 2\xi, 3\xi^2\}(1/\lambda) \tag{4.6}$$

with

$$1/\lambda \ = (d\xi/ds) = \pm/\sqrt{1 + 4\xi^2 + 9\xi^4}. \tag{4.7}$$

The unit normal field becomes

$$\square\mathbf{n} = d\mathbf{t}/ds = \{0, 2, 6\xi\}/\lambda + \{1, 2\xi, 3\xi^2\}d(1/\lambda)/ds \tag{4.8}$$

But $d(1/\lambda)/ds = -(1/\lambda^2)(d\lambda/d\xi)(d\xi/ds) = -(1/\lambda^3)(d\lambda/d\xi)$ such that

$$\square\mathbf{n} = d\mathbf{t}/ds = [-\{0, 2, 6\xi\}(1 + 4\xi^2 + 9\xi^4) + \{1, 2\xi, 3\xi^2\}]d(\lambda)/d\xi \tag{4.9}$$

Problem 3
Use Maple to construct the Frenet Frame for an arbitrary space curve.
Plot for arbitrary space curve.

If and when the basis set is constructed from the induced metric, $g_{mn} = J^{\mu}_{m} J^{\mu}_{n} = g_{nm}$, the connection coefficients enjoy certain symmetries and are called Christoffel symbols, $\Gamma^{k}_{mn} = \Gamma^{k}_{nm}$.

There are three important cases to consider: the case when the differential map is 1-1, onto, or 1-1 and onto. First consider the 1-1 and onto case in which the dimension of the initial state is the same as the dimension of the final state. The the Jacobian matrix is square, and with zero determinant everywhere. The basis set on the final state can be considered to be complete in the sense that there are enough Jacobian columns to equal the dimension of the space.

5. Classical Field Theory of a Single Parameter – LaGrange Euler Equations

Consider a map from the 1 dimensional domain, t, to the 2N+1 dimensional domain of $\{\mathbf{q}, \mathbf{v}, t\}$ Assume the existence of a function $L(\mathbf{q}, \mathbf{v}, t)$ on the target space that somehow describes a physical system. If the map is differentiable, then

$$dq^k = (\partial\phi^k/\partial t)\, dt \; and \; dv^k = (\partial\phi^{N+k}/\partial t)\, dt. \tag{5.1}$$

Further assume that the 2N+1 space is locally constrained such that

$$dq^k - v^k dt = e^k dt \Rightarrow 0. \tag{5.2}$$

The e^k can be interpreted as error, or fluctuation, components that tend to zero under the assumption of kinematic perfection. By functional substitution, pull back the function $L(\mathbf{q}, \mathbf{v}, t)$ such that it becomes a function of $L^{'}(t)$ alone.

The fundamental assumption is the curvature differentials should agree on both domains:

$$dL^{'} = dL(\mathbf{q}, \mathbf{v}, t) \tag{5.3}$$

Apply the chain rule on the RHS:

$$dL = (\partial L/\partial t)dt + (\partial L/\partial q^k)dq^k + (\partial L/\partial v^k)dv^k. \tag{5.4}$$

From the Leibniz rule:

$$(\partial L/\partial v^k)dv^k = d(\partial L/\partial v^k v^k) - \{d(\partial L/\partial v^k)\}v^k \tag{5.5}$$

This trick is equivalent to the "integration by parts" in the calculus of variations. Substitution of this result into the expression for the differentials on the two domains 5.3 yields

$$d(\partial L/\partial v^k v^k - L^{'}) = \{d(\partial L/\partial v^k)/dt - \partial L/\partial q^k\}v^k dt + \{\partial L/\partial q^k e^k - \partial L/\partial t\}dt \tag{5.6}$$

It is conventional to define

$$p_{k=}\partial L/\partial v^k \; and \; f_k^{cons} = \partial L/\partial q^k \tag{5.7}$$

which are the covariant momentum and force. Substitution yields

$$d(p_k v^k - L) = (dp_k/dt - f_k^{cons})v^k dt - (f_k^{cons} e^k - \partial L/\partial t)dt \tag{5.8}$$

The term $(p_k v^k - L)$ is defined classically as the Hamiltonian, H. The term $(dp_k/dt - f_k^{cons}) = f_k^{diss}$ is classically the Lagrangian dissipative force. The term $\{\partial L/\partial q^k e^k - \partial L/\partial t\}$ is a combination of the fluctuation and time dependent losses.

Suppose the system is without fluctuation losses such that $dq^k - v^k dt = 0$. Then the differential equivalence constraint is recognized as a **Power Theorem**:

$$dH/dt = f_k^{diss} v^k, \tag{5.9}$$

or the time rate of change of the energy, H, is the product of the dissipative force times the velocity.

If the energy (or Hamiltonian) is a constant of the "motion" then for arbitrary displacements, dq^k, it follows that

$$\{d(\partial L/\partial v^k)/dt - \partial L/\partial q^k\} = 0 \tag{5.10}$$

which are the LaGrange Euler equations of motion. A constrained possibility for zero energy change is given by motions on an "equipotential" surface. This surface is defined as the zero set of the function, $\varphi(\mathbf{q}, \mathbf{v}, t) = 0$.

So classical mechanics is replicated above without any assumption of metric, or a specified set of coordinates, or a connection, and without the explicit use of a variational principle!!! the technique demonstrates when a flow v^k and a kinematic particle velocity, dq^k/dt are the same, and when they differ due to possible fluctuations.(Recall that Newton started his calculus with the term of *fluxions*. Are the fluctuations expressed above what he had in mind??)

Consider once again the term involving the fluctuations, $e^k dt$. In what sense can these fluctuations be ignored? The simplest case is when the e^k are zero, the case of kinematic perfection. On the otherhand it may be true that a sufficient "null" situation would be that the integral of the fluctuations over a closed curve vanishes. In that case - by Stokes theorem - the e^k must have zero "curl". As the curl operation is a coordinate independent differential process (the exterior derivative), the idea of null fluctuations in the large is related not to the fact that the fluctuations vanish identically, but instead is related to the idea that the fluctuations "average" to zero in some cyclic sense. More on this point later.

5.1. Multiple parameter fields

Consider a map from the n dimensional domain, $\{x, y, z, t, ...\}$ to the 2N+n dimensional domain of $\{\mathbf{\Psi}^\mu, \mathbf{\Psi}_k^\mu\}$. Assume the existence of a function $L\{\mathbf{\Psi}^\mu, \mathbf{\Psi}_k^\mu\}$

on the target space that somehow describes a physical system. If the map is differentiable, then not only is it possible to pull back L but also it is possible to pull back dL. The objective as before is to rewrite $dL\prime = dL$.

Further, constrain the differentials of the field amplitudes such that

$$d\mathbf{\Psi}^\mu - \mathbf{\Psi}^\mu_k dx^k = \omega^\mu \Rightarrow 0, \text{ in some sense.} \tag{5.11}$$

In the single parameter mapping, the map itself insures that there could be unique solutions to the equation, $\omega^\mu = 0$. In the multiple parameter case, the existence of unique solutions to the equations, $\omega^\mu = 0$ is generally not possible. This problem is the problem of Pfaff and a solution set is called a foliation. In that which follows it will be assumed that the fluctuation forms are not zero, but they are closed. That is, the fundamental assumption (or constraint) in that which follows is that $d\omega^\mu = 0$. This constraint implies that the integral of the fluctuations over a boundary of the domain vanishes. Fluctuations can take place in the interior, but are not detected outside the boundary. The closure constraint implies that

$$\partial\mathbf{\Psi}^\mu_k/\partial x^j - \partial\mathbf{\Psi}^\mu_j/\partial x^k = 0. \tag{5.12}$$

Following the procedures of the previous section for a single parameterized mapping, construct the differentials dL and dL', with

$$dL = (\partial L/\partial\mathbf{\Psi}^\mu)d\mathbf{\Psi}^\mu + (\partial L/\partial\mathbf{\Psi}^\mu_k)d\mathbf{\Psi}^\mu_k. \tag{5.13}$$

Expand the factor

$$d\mathbf{\Psi}^\mu_k = (\partial\mathbf{\Psi}^\mu_k/\partial x^j)dx^j = (\partial\mathbf{\Psi}^\mu_j/\partial x^k)dx^j \tag{5.14}$$

using the closure constraint.

Then recognize that the Leibniz rule permits the substitution

$$(\partial L/\partial\mathbf{\Psi}^\mu_k)\partial\mathbf{\Psi}^\mu_j/\partial x^k = \partial\{\partial L/\partial\mathbf{\Psi}^\mu_k)\mathbf{\Psi}^\mu_j\}/\partial x^k - \{\partial(\partial L/\partial\mathbf{\Psi}^\mu_k)/\partial x^k\}\mathbf{\Psi}^\mu_j. \tag{5.15}$$

Substituting the last 5 expressions into the desired equation $dL' = dL$ leads to a power theorem for fields:

$$[\partial\{(\partial L/\partial\mathbf{\Psi}^\mu_k)\mathbf{\Psi}^\mu_j - (L')\delta^k_j\}/\partial x^k]dx^j =$$

$$[\{\partial(\partial L/\partial\mathbf{\Psi}^\mu_k)/\partial x^k - \partial L/\partial\mathbf{\Psi}^\mu\}\mathbf{\Psi}^\mu_j]dx^j - (\partial L/\partial\mathbf{\Psi}^\mu)\omega^\mu \tag{5.16}$$

The term $(\partial L/\partial \mathbf{\Psi}^{\mu})\omega^{\mu}$ represents the "fluctuations" in the fields.

Now define the **Stress Energy Tensor** W_k^j (a mixed second rank non-symmetric tensor) as

$$W_j^k = [(\partial L/\partial \mathbf{\Psi}_k^{\mu})\mathbf{\Psi}_j^{\mu} - L'\delta_j^k]. \tag{5.17}$$

The Eulerian (covariant) dissipative **force** field f_{μ}^{diss} on the target domain

$$f_{\mu}^{diss} = \{\partial(\partial L/\partial \mathbf{\Psi}_k^{\mu})/\partial x^k - \partial L/\partial \mathbf{\Psi}^{\mu}\} \tag{5.18}$$

has a covariant pre-image as a Lagrangian force F_j^{diss} on the initial space of parameters as

$$F_j^{diss} = f_{\mu}^{diss}\mathbf{\Psi}_j^{\mu} \tag{5.19}$$

The field momentum tensor is given by the expression

$$\Pi_{\mu}^k = (\partial L/\partial \mathbf{\Psi}_k^{\mu}). \tag{5.20}$$

The fundamental identity (without the fluctuation term) becomes the Power Theorem,

$$\{\partial[W_j^k]/\partial x^k\}dx^j - F_j^{diss}dx^j + (\partial L/\partial \mathbf{\Psi}^{\mu})\omega^{\mu} = 0. \tag{5.21}$$

This expression is actually a differential 1-form related to the first law of thermodynamics (see below). When the differentials dx^j are presumed to be arbitrary, and when the fluctuation term is ignored, then the Power Theorem becomes the classical stress-energy theorem,

$$\partial[W_j^k]/\partial x^k = F_j^{diss} = f_{\mu}^{diss}\mathbf{\Psi}_j^{\mu}, \tag{5.22}$$

which reads "The divergence of the Stress Energy Tensor = the Lagrangian Force". This fundamental formalism was obtained with out metric, without a connection and is in accurate tensor format. The stress energy tensor is a mixed 2nd rank tensor field, and the Lagrangian force is a covariant vector. Note that $F_j^{diss}dx^j$ is a 1-form whose closed loop integral represents the dissipative work of thermodynamics. On the other hand, the curl of F_j^{diss} is zero, then the system is not irreversible.

In the conservative case, where the dissipative force vanishes, the "equations of motion" or "Field Equations" are the LaGrange-Euler equations,

$$\{\partial(\partial L/\partial \mathbf{\Psi}_k^{\mu})/\partial x^k - \partial L/\partial \mathbf{\Psi}^{\mu}\} = 0. \tag{5.23}$$

5.2. Applications of the Field Theory Result.

5.2.1. The single parameter case, or "particle theory"

Before giving a multiple parameter example, it is useful to show that the multiple parameter arguments lead to the same results as in the single parameter case. Let the map ϕ be defined as:

$$\phi : \{t\} \Rightarrow \{\mathbf{x}, \mathbf{v}\} \; with \; \mathbf{\Psi}^\mu \Leftrightarrow \mathbf{x} \; and \; \mathbf{\Psi}^\mu_k \simeq \mathbf{\Psi}^\mu_t \Leftrightarrow \mathbf{v}. \tag{5.24}$$

Note that $\omega^\mu \Leftrightarrow d\mathbf{x} - \mathbf{v}dt$ and $d\omega^\mu = 0$ implies that $d\mathbf{v}\hat{}dt = 0$, or $\mathbf{v} = \mathbf{v}(t)$. The constraint is satisfied automatically by the original single parameter mapping assumption. The constraint $\omega^\mu = 0$ implies that $\mathbf{v} = \partial\mathbf{x}/\partial t$, which is equivalent to the kinematic topology hypothesis of elementary texts.

Suppose that

$$L(\mathbf{x}, \mathbf{v}) = m\mathbf{v}^2/2 - U(\mathbf{x}) = KE - PE \tag{5.25}$$

such that the momentum tensor is a single covector

$$\Pi^k_\mu \Rightarrow p_\mu = \partial L/\partial v^\mu = m\mathbf{v}, \tag{5.26}$$

and the dissipative force

$$f^{diss}_\mu = dp_\mu/dt - \partial L/\partial x^\mu = d\mathbf{p}/dt - gradU(x) \tag{5.27}$$

in agreement with Hamilton's principle. The conservative "equations of motion" are

$$d\mathbf{p}/dt - gradU(x) = 0. \tag{5.28}$$

The stress energy tensor has a single component

$$W^k_j = [\partial L/\partial \mathbf{\Psi}^\mu_k)\mathbf{\Psi}^\mu_j\} - L'\delta^k_j] = p_\mu v^\mu - L = \mathbf{p} \circ \mathbf{v} - \{m\mathbf{v}^2/2 - U(\mathbf{x})\} =$$

$$m\mathbf{v}^2/2 + U(\mathbf{x}) = KE + PE = Total \; Energy = the \; Hamiltonian. \tag{5.29}$$

When L and therefor H are not explicit functions of time, then the "divergence" of the stress-energy tensor vanishes (the divergence is equal to $\partial W/\partial t = 0$ for a 1 parameter system) yielding the classic result which is called the conservation

of energy. In this case the LaGrange-Newtonian dissipative force vanishes, such that $d\mathbf{p}/dt - gradU(x) = 0$.

When kinematic perfection is not assumed, then the fluctuation terms are of the form

$$(\partial L/\partial \mathbf{\Psi}^{\mu})\omega^{\mu} = (\partial L/\partial \mathbf{x}^{\mu})(d\mathbf{x}^{\mu} - \mathbf{v}^{\mu}dt) = f_{\mu}^{cons}(d\mathbf{x}^{\mu} - \mathbf{v}^{\mu}dt) \tag{5.30}$$

Note that this fluctuation term in power can vanish when the fluctuations - if they exist - are orthogonal to the gradient of the potential. In other words, if the fluctuations are tangential to the equipotential surfaces there is no dissipative power loss (or gain). Only fluctuations transverse to the equipotential surfaces will contribute to the power theorem. This idea would imply that the transverse component of the fluctuations must vanish on the boundary of a conservative system, and as in electrostatics, the boundary (of a conductor) is an equipotential surface

5.2.2. The two parameter case, or "string theory"

Let the map ϕ be defined as:

$$\phi : \{s, ct\} \Rightarrow \{\mathbf{\Psi}^{\mu}, \mathbf{\Psi}_{s}^{\mu}, \mathbf{\Psi}_{ct}^{\mu}\} = \{\mathbf{\Psi}, \mathbf{\Psi}_{s}, \mathbf{\Psi}_{ct}\} \, . \tag{5.31}$$

The target space consists of a single field amplitude (or wave function, $\mathbf{\Psi}$) that corresponds to a "coordinate", but has several (two) components that correspond to a "velocity".

Note that $\omega^{\mu} \Leftrightarrow d\mathbf{\Psi} - \mathbf{\Psi}_{ct}d(ct) - \mathbf{\Psi}_{s}ds$. Now suppose that the LaGrange function for this system is given in the form

$$L(\mathbf{\Psi}, \mathbf{\Psi}_{s}, \mathbf{\Psi}_{ct}) = \{\mathbf{\Psi}_{s}^{2} - \mathbf{\Psi}_{ct}^{2}\}/2 \tag{5.32}$$

Then the field momentum is the two component contravector

$$\Pi_{\mu}^{k} = \partial L/\partial \mathbf{\Psi}_{k}^{\mu} \Rightarrow \pi^{k} = \begin{bmatrix} \mathbf{\Psi}_{s} \\ -\mathbf{\Psi}_{ct} \end{bmatrix} . \tag{5.33}$$

The stress energy tensor becomes a two by two matrix

$$W_{j}^{k} = [\partial L/\partial \mathbf{\Psi}_{k}^{\mu})\mathbf{\Psi}_{j}^{\mu}\} - L'\delta_{j}^{k}] = \begin{bmatrix} \{\mathbf{\Psi}_{s}^{2} + \mathbf{\Psi}_{ct}^{2}\}/2 & \{\mathbf{\Psi}_{s}\mathbf{\Psi}_{ct}\} \\ -\{\mathbf{\Psi}_{s}\mathbf{\Psi}_{ct}\} & \{\mathbf{\Psi}_{s}^{2} + \mathbf{\Psi}_{ct}^{2}\}/2 \end{bmatrix} \tag{5.34}$$

Note that the stress energy tensor is NOT symmetric (therefore the system will have a spin component of angular momentum as well as a classical component of angular momentum – see below).

The LaGrange force becomes a "scalar" 1 component object

$$f_\mu^{diss} = \{\partial(\partial L/\partial \mathbf{\Psi}_k^\mu)/\partial x^k - \partial L/\partial \mathbf{\Psi}^\mu\} = \partial(\mathbf{\Psi}_s)/\partial s - \partial(\mathbf{\Psi}_{ct})/\partial(ct) - 0 \quad (5.35)$$

Suppose that the fluctuations are zero, and that $\mathbf{\Psi}_s \Rightarrow \partial\mathbf{\Psi}/\partial s$ and $\mathbf{\Psi}_{ct} \Rightarrow \partial\mathbf{\Psi}/\partial(ct)$. Then if this "dissipative" force is to vanish, the amplitude function, $\mathbf{\Psi}$, satisfies the wave equation in the two variables, s and ct. That is, the Field Equations are:

$$\partial^2\mathbf{\Psi}/\partial s^2 - \partial^2\mathbf{\Psi}/\partial(ct)^2 = 0. \quad (5.36)$$

Problem 4.

Work out the divergence of the stress energy tensor for the string above. Also do the same problem over except assume that the initial parameters are the complex pair, $\{s, ict\}$. Show that the "equation of motion" is Laplace's equation. Is the stress-energy tensor symmetric?

Problem 5.

Set up the basic formulas in Maple, such that for a given L you can get Maple to deduce the Stress-Energy Tensor and the Field Equations.
