```
> restart:with(linalg):
Warning, new definition for norm
Warning, new definition for trace
```

MINIMAL SURFACES

minimal.ms

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A theorem of Sophus Lie states that in 4D every complex holomorphic function generates a minimal surface.

This is a rather remarkable result that has not been utilized fully in understanding space-time evolutionary processes. Minimal surfaces are also associated with a position vector (in 3D) which is harmonic. The specification of a complex holomorphic function (two real functions) constrains 4D to yield a 2 D surface. This 2-D surface can be described by a parametric position vector in 3D, and that position vector is harmonic.

Consider an evolutionary system like a fluid in space time. Consider a complex holomorphic curve. This complex curve induces a two dimension subspace of space time. The subspace is a minimal surface. The "parametric" vector to the surface can be used to describe a vorticity field (or a velocity field). The construction utilizes the Weierstrass technique (see below). Such a vector is harmonic.

For Navier-Stokes like fluids, the viscous dissipation is a viscous coefficient times the vector Laplacian of the the velocity field. As the vector Laplacian vanishes for a harmonic vector field it follows that flows (that are position vectors to a minimal surface) do not experience viscous disspation due to shears.

Consider vector fields that are composed of a harmonic part and a non-harmonic part. Viscous dissipation will cause the non-harmonic part to decay. What is left is the Harmonic part, which generates a minimal surface as the (measureable) wake.

The Minimal surface generated by a complex curve, does not admit a single implicit real function in 3D for its description. Such minimal surfaces are artifacts of 4D space time.

However, there does exist a parametrization of such a minimal surface, and that is what the Weierstrass method is all about. A parametric version, 2D into 3D, but not implict version, 3D to a constant.

(Somehow this implies that the points off the surface are not open sets in 3D?? If it was implicit , the points off the surface are open sets of 3D.)

First, the holomorphic curve will be considered as a 4D vector field constructed over two parameters in the form R = [u,v,Real(Phi(z)),Imag(Phi(z))] where z = u+iv and Phi(z) is holomorphic (depends on z and not on z*)

The induced metric on the two surface will be used to compute the mean and Gauss curvatures.

Then the Weierstrass Method will be used to compute a 3D doubly parameterized Harmonic vector field (as fibers to the minimal surface). Then the Cartan frame matrix will be computed for this system and the various components of the Cartan matrix recomputed. The mean curvature will vanish.

```
> with(plots):with(liesymm): setup(u,v):
Warning, new definition for close
>
> Phi:=z^(2)-(a+I*b);
>
```

$$\Phi := z^2 - a - I\,b$$

The example is the generator of the Mandelbrot iterates. Hence every Mandelbrot iterate produces a minimal surface, as each iterate is holomorphic. Hence in the limit, the Mandelbrot bug is related to a minimal surface.

```
>
> F(z):=subs(z=u+I*v,Phi);
>
```

$$F(z) := (u + I\,v)^2 - a - I\,b$$

```
> phi(u,v):=evalc(Re(F(z)));
```

$$\phi(u, v) := u^2 - v^2 - a$$

```
> chi(u,v):=evalc(Im(F(z)));
```

$$\chi(u, v) := 2\,u\,v - b$$

Enter a complex position vector on 4Space

```
> R:=[u,v,phi(u,v),chi(u,v)];
```

$$R := [u, v, u^2 - v^2 - a, 2\,u\,v - b]$$

```
> Yu:=diff(R,u);
```

$$Yu := [1, 0, 2\,u, 2\,v]$$

```
> Yv:=diff(R,v);
```

$$Yv := [0, 1, -2\,v, 2\,u]$$

```
> EE:=simplify(dotprod(Yu,Yu));
```

$$EE := 1 + 4\,u^2 + 4\,v^2$$

```
> FF:=simplify(dotprod(Yu,Yv));
```

$$FF := 0$$

```
> GG:=simplify(dotprod(Yv,Yv));
```

$$GG := 1 + 4\,u^2 + 4\,v^2$$

```
> Yuu:=diff(Yu,u);
```

$$Yuu := [0, 0, 2, 0]$$

```
> Yuv:=diff(Yu,v);
```

$$Yuv := [0, 0, 0, 2]$$

```
> Yvv:=diff(Yv,v);
```

$$Yvv := [0, 0, -2, 0]$$

```
> H:=simplify((1+GG)*(Yuu[3]+Yuu[4])+(1+EE)*(Yvv[3]+Yvv[4])-2*FF*(Yuv[3]+Yuv[4]));
```

$$H := 0$$

Mean curvature of the two surface as computed above from the induced metric coefficients vanishes. Hence, the pull back surface from 4D to 2D is a minimal surface.

THE POSITION VECTOR to a MINIMAL SURFACE

Now construct the position vector to this surface (the double projection from 4D to a 2-surface leads to a parametrization of the 2 surface in 3D) using the Weierstrass formulas. The Weierstrass formulas generate a pair of conjugate minimal surfaces.

```
> Phi;
```

$$z^2 - a - I\,b$$

```
>
```

Define the Weierstrass integrals over the complex variable z

```
> RX:=((int((1-z*z)*Phi,z)));
```

$$RX := -\frac{1}{5}z^5 + \frac{1}{3}(1 + a + I\,b)\,z^3 - a\,z - I\,b\,z$$

```
> RY:=((int(I*(1+z*z)*Phi,z)));
```

$$RY := I\left(\frac{1}{5}z^5 + \frac{1}{3}(1 - a - I\,b)\,z^3 - a\,z - I\,b\,z\right)$$

```
> RZ:=((int(2*z*Phi,z)));
```

$$RZ := \frac{1}{2}z^4 + (-a - I\,b)\,z^2$$

```
>
>
```

Form the six vector (three real and thre imaginary components.

```
> RX:=subs(z=u+I*v,RX);RY:=subs(z=u+I*v,RY);RZ:=subs(z=u+I*v,RZ);
```

$$RX := -\frac{1}{5}(u + I\,v)^5 + \frac{1}{3}(1 + a + I\,b)\,(u + I\,v)^3 - a\,(u + I\,v) - I\,b\,(u + I\,v)$$

$$RY := I\left(\frac{1}{5}(u + I\,v)^5 + \frac{1}{3}(1 - a - I\,b)\,(u + I\,v)^3 - a\,(u + I\,v) - I\,b\,(u + I\,v)\right)$$

$$RZ := \frac{1}{2}(u + I\,v)^4 + (-a - I\,b)\,(u + I\,v)^2$$

```
>
>
> xx:=simplify(factor(evalc(Re(RX))),trig);
```

$$xx := -\frac{1}{5}u^5 + 2\,u^3\,v^2 - u\,v^4 + \frac{1}{3}u^3 - u\,v^2 + \frac{1}{3}a\,u^3 - a\,u\,v^2 - b\,u^2\,v + \frac{1}{3}b\,v^3 - a\,u + b\,v$$

```
> xxi:=simplify(factor(evalc(Im(RX))),trig);
```

$$xxi := -u^4\,v + 2\,u^2\,v^3 - \frac{1}{5}v^5 + \frac{1}{3}b\,u^3 - b\,u\,v^2 + u^2\,v - \frac{1}{3}v^3 + a\,u^2\,v - \frac{1}{3}a\,v^3 - a\,v - b\,u$$

```
> yy:=simplify(factor(evalc(Re(RY))),trig);
```

$$yy := -u^4\,v + 2\,u^2\,v^3 - \frac{1}{5}v^5 + \frac{1}{3}b\,u^3 - b\,u\,v^2 - u^2\,v + \frac{1}{3}v^3 + a\,u^2\,v - \frac{1}{3}a\,v^3 + a\,v + b\,u$$

```
> yyi:=simplify(factor(evalc(Im(RY))),trig);
```

$$yyi := \frac{1}{5}u^5 - 2\,u^3\,v^2 + u\,v^4 + \frac{1}{3}u^3 - u\,v^2 - \frac{1}{3}a\,u^3 + a\,u\,v^2 + b\,u^2\,v - \frac{1}{3}b\,v^3 - a\,u + b\,v$$

```
> zz:=simplify(factor(evalc(Re(RZ))),trig);
```

$$zz := \frac{1}{2}u^4 - 3\,u^2\,v^2 + \frac{1}{2}v^4 - a\,u^2 + a\,v^2 + 2\,b\,u\,v$$

```
> zzi:=simplify(factor(evalc(Im(RZ))),trig);
```

$$zzi := 2\,u^3\,v - 2\,u\,v^3 - b\,u^2 + b\,v^2 - 2\,a\,u\,v$$

```
> RRR:=evalm([xx,yy,zz]);RRI:=[xxi,yyi,zzi];
```

$$RRR := \left[-\frac{1}{5}u^5 + 2\,u^3\,v^2 - u\,v^4 + \frac{1}{3}u^3 - u\,v^2 + \frac{1}{3}a\,u^3 - a\,u\,v^2 - b\,u^2\,v + \frac{1}{3}b\,v^3 - a\,u + b\,v,\right.$$
$$-u^4\,v + 2\,u^2\,v^3 - \frac{1}{5}v^5 + \frac{1}{3}b\,u^3 - b\,u\,v^2 - u^2\,v + \frac{1}{3}v^3 + a\,u^2\,v - \frac{1}{3}a\,v^3 + a\,v + b\,u,$$
$$\left.\frac{1}{2}u^4 - 3\,u^2\,v^2 + \frac{1}{2}v^4 - a\,u^2 + a\,v^2 + 2\,b\,u\,v\right]$$

$$RRI := \left[-u^4 v + 2 u^2 v^3 - \frac{1}{5} v^5 + \frac{1}{3} b u^3 - b u v^2 + u^2 v - \frac{1}{3} v^3 + a u^2 v - \frac{1}{3} a v^3 - a v - b u, \frac{1}{5} u^5 - 2 u^3 v^2 + u v^4 + \frac{1}{3} u^3 - u v^2 - \frac{1}{3} a u^3 + a u v^2 + b u^2 v - \frac{1}{3} b v^3 - a u + b v, 2 u^3 v - 2 u v^3 - b u^2 + b v^2 - 2 a u v\right]$$

```
> EI:=factor(diff(RRI,u));BI:=factor(diff(RRI,v));
> E:=diff(RRR,u);B:=diff(RRR,v);
```

$$EI := [-4 u^3 v + 4 u v^3 + b u^2 - b v^2 + 2 u v + 2 a u v - b, u^4 - 6 u^2 v^2 + v^4 + u^2 - v^2 - a u^2 + a v^2 + 2 b u v - a, 6 u^2 v - 2 v^3 - 2 b u - 2 a v]$$

$$BI := [-u^4 + 6 u^2 v^2 - v^4 - 2 b u v + u^2 - v^2 + a u^2 - a v^2 - a, -4 u^3 v + 4 u v^3 - 2 u v + 2 a u v + b u^2 - b v^2 + b, 2 u^3 - 6 u v^2 + 2 b v - 2 a u]$$

$$E := 0$$

$$B := 0$$

The two position vectors have real and imaginary components that are conjugates.
Eimag = -Breal, Ereal=Bimag.

```
>
>
> EE:=innerprod(E,E):BB:=innerprod(B,B):Poincare1:=EE-BB;Poincare2:=innerprod(E,B)
;
>
```

$$Poincare1 := 0$$

$$Poincare2 := 0$$

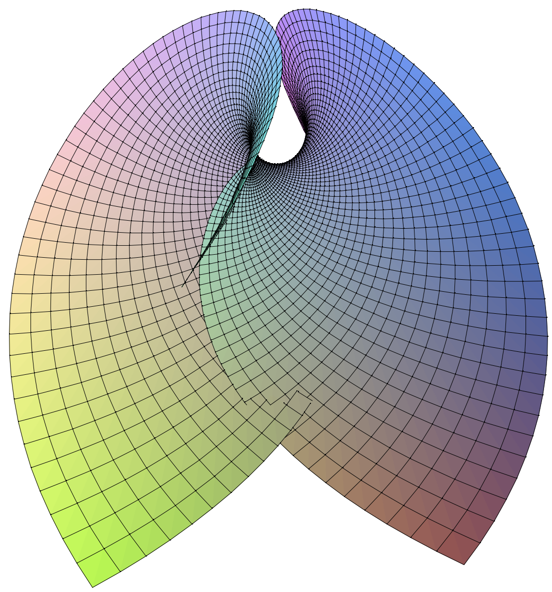
```
> a:=1;b:=1;
```

$$a := 1$$

$$b := 1$$

```
> plot3d(RRR,u=-1.3..1.3,v=-1.3..1.3,orientation=[134,86],numpoints=5000,style=PAT
CH,title=`Real part - Mandelbrot Minimal Surface`);
```

Real part - Mandelbrot Minimal Surface



Next, use the Frame method to compute the Mean curvature in terms of the position vector.

The position vector in R3

```
> RR:=[xx,yy,zz];
```

$$RR := \left[-\frac{1}{5}u^5 + 2u^3v^2 - uv^4 + \frac{2}{3}u^3 - 2uv^2 - u^2v + \frac{1}{3}v^3 - u + v,\ -u^4v + 2u^2v^3 - \frac{1}{5}v^5 + \frac{1}{3}u^3 - uv^2 + v + u,\ \frac{1}{2}u^4 - 3u^2v^2 + \frac{1}{2}v^4 - u^2 + v^2 + 2uv\right]$$

```
> Yu:=diff(RR,u);
```

$$Yu := [-u^4 + 6u^2v^2 - v^4 + 2u^2 - 2v^2 - 2uv - 1, -4u^3v + 4uv^3 + u^2 - v^2 + 1, 2u^3 - 6uv^2 - 2u + 2v]$$

```
> Yv:=diff(RR,v);
```

$$Yv := [4u^3v - 4uv^3 - 4uv - u^2 + v^2 + 1, -u^4 + 6u^2v^2 - v^4 - 2uv + 1, -6u^2v + 2v^3 + 2v + 2u]$$

```
> NNU:=crossprod(Yu,Yv);
```

$$\begin{aligned} NNU := [& (-4u^3v + 4uv^3 + u^2 - v^2 + 1)(-6u^2v + 2v^3 + 2v + 2u) - (2u^3 - 6uv^2 - 2u + 2v)(-u^4 + 6u^2v^2 - v^4 - 2uv + 1) \\ & , (2u^3 - 6uv^2 - 2u + 2v)(4u^3v - 4uv^3 - 4uv - u^2 + v^2 + 1) \\ & - (-u^4 + 6u^2v^2 - v^4 + 2u^2 - 2v^2 - 2uv - 1)(-6u^2v + 2v^3 + 2v + 2u), \\ & (-u^4 + 6u^2v^2 - v^4 + 2u^2 - 2v^2 - 2uv - 1)(-u^4 + 6u^2v^2 - v^4 - 2uv + 1) \\ & - (-4u^3v + 4uv^3 + u^2 - v^2 + 1)(4u^3v - 4uv^3 - 4uv - u^2 + v^2 + 1)] \end{aligned}$$

Scale the adjoint normal field here by rho

```
> rho:=innerprod(NNU,NNU)^(1/2);
```

$$\begin{aligned} \rho := (& 8v^{14} - 48v^{11}u - 128v^9u + 30v^{12} + 101v^8 - 168uv^5 - 192v^7u + u^{16} + 100v^6 + 68v^{10} + 4u^{10} - 4u^6 - 2u^{12} \\ & + 5u^8 - 120u^9v^5 + 8v^{14}u^2 - 120v^9u^5 + 56v^{10}u^6 + 28v^{12}u^4 - 48v^{11}u^3 + 70v^8u^8 + v^{16} + 24v^2 + 8u^{14}v^2 + 56u^{10}v^6 \\ & - 48u^{11}v^3 - 8u^{13}v - 160u^7v^7 + 64v^4 + 64u^2v^2 + 28u^{12}v^4 + 296u^6v^4 + 424u^4v^6 + 318u^4v^4 - 128u^7v^3 \end{aligned}$$

$$- 384 u^3 v^7 - 384 u^5 v^5 - 384 u^3 v^5 - 208 u^3 v^3 - 208 u^3 v^9 - 288 u^7 v^5 - 8 v^{13} u - 352 u^5 v^7 - 112 u^9 v^3 - 192 u^5 v^3 - 40 u^5 v - 16 u^{11} v + 84 u^8 v^2 + 116 u^6 v^2 + 130 u^8 v^4 + 20 u^{10} v^2 + 120 u^8 v^6 + 8 u^{12} v^2 + 48 u^{10} v^4 + 290 u^4 v^8 + 280 u^6 v^6 + 120 u^4 v^{10} + 160 u^6 v^8 + 148 u^2 v^{10} + 276 u^2 v^8 + 308 u^2 v^6 + 48 u^2 v^{12} + 92 u^4 v^2 + 196 u^2 v^4 + 4 - 16 u v + 8 u^2 - 48 u^3 v - 80 u v^3)^{1/2}$$

The choice of the scaling factor rho will determine the torsion coefficients of the system. As the position vector is paramtrically defined, the 1-form, small omega, vanishes identicall, and therefore there is no affine (translational dislocation shear) torsion for the minimal surfaces. However, the rotational shear disclination torsion coefficients exist, except for the scaling factor rho chosen above. The Mean curvature of the Holomorphic curves vanishes for all scaling factors.

```
> rho:=1;
>
```

$$\rho := 1$$

This vector (surface normal) NNU can be computed from the Adjoint Matrix operation on the two tangent vectors Yu and Yv. The basis frame utilizes this surface normal with arbitrary scaling

```
> NN:=([factor(NNU[1]),factor(NNU[2]),simplify(factor(NNU[3]))]);
```

$$NN := [2 u (v^2 + 1 + u^2) \%1, 2 v (v^2 + 1 + u^2) \%1, (v^2 - 1 + u^2)(v^2 + 1 + u^2) \%1]$$

$$\%1 := v^4 + 2 v^2 + 2 u^2 v^2 - 4 u v + u^4 - 2 u^2 + 2$$

```
> FF:=array([[Yu[1],Yv[1],NN[1]/rho],[Yu[2],Yv[2],NN[2]/rho],[Yu[3],Yv[3],NN[3]/rh
  o]   ] );
```

$$FF := \begin{bmatrix} -u^4 + 6 u^2 v^2 - v^4 + 2 u^2 - 2 v^2 - 2 u v - 1 & 4 u^3 v - 4 u v^3 - 4 u v - u^2 + v^2 + 1 & 2 u (v^2 + 1 + u^2) \%1 \\ -4 u^3 v + 4 u v^3 + u^2 - v^2 + 1 & -u^4 + 6 u^2 v^2 - v^4 - 2 u v + 1 & 2 v (v^2 + 1 + u^2) \%1 \\ 2 u^3 - 6 u v^2 - 2 u + 2 v & -6 u^2 v + 2 v^3 + 2 v + 2 u & (v^2 - 1 + u^2)(v^2 + 1 + u^2) \%1 \end{bmatrix}$$

$$\%1 := v^4 + 2 v^2 + 2 u^2 v^2 - 4 u v + u^4 - 2 u^2 + 2$$

The Repere Mobile or FRAME MATRIX, FF. note that the frame matrix is not orthonormal.!!

```
> detFF:=simplify((det(FF)));
```

$$detFF := 8 v^{14} - 48 v^{11} u - 128 v^9 u + 30 v^{12} + 101 v^8 - 168 u v^5 - 192 v^7 u + u^{16} + 100 v^6 + 68 v^{10} + 4 u^{10} - 4 u^6 - 2 u^{12} + 5 u^8 - 120 u^9 v^5 + 8 v^{14} u^2 - 120 v^9 u^5 + 56 v^{10} u^6 + 28 v^{12} u^4 - 48 v^{11} u^3 + 70 v^8 u^8 + v^{16} + 24 v^2 + 8 u^{14} v^2 + 56 u^{10} v^6 - 48 u^{11} v^3 - 8 u^{13} v - 160 u^7 v^7 + 64 v^4 + 64 u^2 v^2 + 28 u^{12} v^4 + 296 u^6 v^4 + 424 u^4 v^6 + 318 u^4 v^4 - 128 u^7 v^3 - 384 u^3 v^7 - 384 u^5 v^5 - 384 u^3 v^5 - 208 u^3 v^3 - 208 u^3 v^9 - 288 u^7 v^5 - 8 v^{13} u - 352 u^5 v^7 - 112 u^9 v^3 - 192 u^5 v^3 - 40 u^5 v - 16 u^{11} v + 84 u^8 v^2 + 116 u^6 v^2 + 130 u^8 v^4 + 20 u^{10} v^2 + 120 u^8 v^6 + 8 u^{12} v^2 + 48 u^{10} v^4 + 290 u^4 v^8 + 280 u^6 v^6 + 120 u^4 v^{10} + 160 u^6 v^8 + 148 u^2 v^{10} + 276 u^2 v^8 + 308 u^2 v^6 + 48 u^2 v^{12} + 92 u^4 v^2 + 196 u^2 v^4 + 4 - 16 u v + 8 u^2 - 48 u^3 v - 80 u v^3$$

For the case at hand, the deteminant is non-zero globally, hence an inverse always exists.

```
> FFINVD:=evalm(FF^(-1));
```

$$FFINVD := \begin{bmatrix} -\frac{v^4 + 2 v^2 - 6 u^2 v^2 + 2 u v + u^4 + 1 - 2 u^2}{\%1} & \frac{-4 u^3 v + 4 u v^3 + u^2 - v^2 + 1}{\%1} & -2 \frac{3 u v^2 - v + u - u^3}{\%1} \\ -\frac{4 u v^3 - v^2 - 4 u^3 v + 4 u v - 1 + u^2}{\%1} & -\frac{v^4 - 6 u^2 v^2 + 2 u v + u^4 - 1}{\%1} & 2 \frac{v^3 - 3 u^2 v + v + u}{\%1} \\ 2 \frac{u}{\%2} & 2 \frac{v}{\%2} & \frac{v^2 - 1 + u^2}{\%2} \end{bmatrix}$$

$$\%1 := 2 + 6 v^2 + 7 v^4 + 4 v^6 + v^8 - 4 u v - 8 u v^3 - 4 u v^5 + 2 u^2 + 6 u^2 v^2 + 8 u^2 v^4 + 4 u^2 v^6 - 8 u^3 v - 8 u^3 v^3 - u^4 + 4 u^4 v^2 + 6 u^4 v^4 - 4 u^5 v + 4 u^6 v^2 + u^8$$

$$\%2 := 5 v^8 - 12 u v^5 - 4 v^7 u + 11 v^6 + v^{10} + u^{10} - u^6 + u^8 - 4 u^7 v + 8 v^2 + 13 v^4 + 14 u^2 v^2 + 10 u^6 v^4 + 10 u^4 v^6 + 18 u^4 v^4 - 12 u^3 v^5 - 24 u^3 v^3 - 12 u^5 v^3 - 12 u^5 v + 5 u^8 v^2 + 8 u^6 v^2 + 5 u^2 v^8 + 16 u^2 v^6 + 9 u^4 v^2 + 21 u^2 v^4 + 2 + u^4$$

$$-4\,u\,v+4\,u^2-12\,u^3\,v-12\,u\,v^3$$

The 1-form components of the differential position vector with respect to the Basis Frame, F.

```
> dR:=innerprod(FFINVD,[d(xx),d(yy),d(zz)]);
```

$$dR := [\,\mathrm{d}(u),\,\mathrm{d}(v),\,0\,]$$

```
> sigma1:=(wcollect(dR[1]));
```

$$\sigma 1 := \mathrm{d}(u)$$

```
> sigma2:=wcollect(dR[2]);
```

$$\sigma 2 := \mathrm{d}(v)$$

Note that sigma1 is du and sigma2 is dv for a parametric Monge surfaces!!

```
> omega:=(wcollect(dR[3]));
```

$$\omega := 0$$

Note that this term vanishes for a parametric Monge surface, hence parametric Monge surfaces exhibit no TORSION!! of the Affine type (that is there is no translational shear defects!)

```
>
```

Compute the Cartan Matrix of connection forms from C=[F(inverse)] times d[F]

```
> dFF:=array([[d(FF[1,1]),d(FF[1,2]),d(FF[1,3])],[d(FF[2,1]),d(FF[2,2]),d(FF[2,3])
  ],[d(FF[3,1]),d(FF[3,2]),d(FF[3,3])]]);
```

$dFF :=$

$$[-4\,u^3\,\mathrm{d}(u)+12\,u\,\mathrm{d}(u)\,v^2+12\,u^2\,v\,\mathrm{d}(v)-4\,v^3\,\mathrm{d}(v)+4\,u\,\mathrm{d}(u)-4\,v\,\mathrm{d}(v)-2\,\mathrm{d}(u)\,v-2\,u\,\mathrm{d}(v)\,,$$
$$12\,u^2\,\mathrm{d}(u)\,v+4\,u^3\,\mathrm{d}(v)-4\,\mathrm{d}(u)\,v^3-12\,u\,v^2\,\mathrm{d}(v)-4\,\mathrm{d}(u)\,v-4\,u\,\mathrm{d}(v)-2\,u\,\mathrm{d}(u)+2\,v\,\mathrm{d}(v)\,,$$
$$2\,\mathrm{d}(u)\,(v^2+1+u^2)\,\%2+2\,u\,\%3\,\%2+2\,u\,(v^2+1+u^2)\,\%1\,]$$
$$[-12\,u^2\,\mathrm{d}(u)\,v-4\,u^3\,\mathrm{d}(v)+4\,\mathrm{d}(u)\,v^3+12\,u\,v^2\,\mathrm{d}(v)+2\,u\,\mathrm{d}(u)-2\,v\,\mathrm{d}(v)\,,$$
$$-4\,u^3\,\mathrm{d}(u)+12\,u\,\mathrm{d}(u)\,v^2+12\,u^2\,v\,\mathrm{d}(v)-4\,v^3\,\mathrm{d}(v)-2\,\mathrm{d}(u)\,v-2\,u\,\mathrm{d}(v)\,,$$
$$2\,\mathrm{d}(v)\,(v^2+1+u^2)\,\%2+2\,v\,\%3\,\%2+2\,v\,(v^2+1+u^2)\,\%1\,]$$
$$[\,6\,u^2\,\mathrm{d}(u)-6\,\mathrm{d}(u)\,v^2-12\,u\,v\,\mathrm{d}(v)-2\,\mathrm{d}(u)+2\,\mathrm{d}(v)\,,\,-12\,u\,\mathrm{d}(u)\,v-6\,u^2\,\mathrm{d}(v)+6\,v^2\,\mathrm{d}(v)+2\,\mathrm{d}(v)+2\,\mathrm{d}(u)\,,$$
$$\%3\,(v^2+1+u^2)\,\%2+(v^2-1+u^2)\,\%3\,\%2+(v^2-1+u^2)\,(v^2+1+u^2)\,\%1\,]$$
$$\%1 := 4\,v^3\,\mathrm{d}(v)+4\,v\,\mathrm{d}(v)+4\,u\,\mathrm{d}(u)\,v^2+4\,u^2\,v\,\mathrm{d}(v)-4\,\mathrm{d}(u)\,v-4\,u\,\mathrm{d}(v)+4\,u^3\,\mathrm{d}(u)-4\,u\,\mathrm{d}(u)$$
$$\%2 := v^4+2\,v^2+2\,u^2\,v^2-4\,u\,v+u^4-2\,u^2+2$$
$$\%3 := 2\,v\,\mathrm{d}(v)+2\,u\,\mathrm{d}(u)$$

```
> cartan:=evalm(FFINVD&*dFF):
```

The interior connection coefficients (can be Christoffel symbols on the parameter space

```
> Gamma11:=(wcollect(cartan[1,1]));
```

$$\Gamma 11 := \Bigg(-\frac{(v^4+2\,v^2-6\,u^2\,v^2+2\,u\,v+u^4+1-2\,u^2)\,(-4\,u^3+4\,u+12\,u\,v^2-2\,v)}{\%1}$$
$$+\frac{(-4\,u^3\,v+4\,u\,v^3+u^2-v^2+1)\,(-12\,u^2\,v+4\,v^3+2\,u)}{\%1}-2\,\frac{(3\,u\,v^2-v+u-u^3)\,(-6\,v^2-2+6\,u^2)}{\%1}\Bigg)\mathrm{d}(u)+\Bigg($$
$$-\frac{(v^4+2\,v^2-6\,u^2\,v^2+2\,u\,v+u^4+1-2\,u^2)\,(-4\,v+12\,u^2\,v-4\,v^3-2\,u)}{\%1}$$
$$+\frac{(-4\,u^3\,v+4\,u\,v^3+u^2-v^2+1)\,(-4\,u^3+12\,u\,v^2-2\,v)}{\%1}-2\,\frac{(3\,u\,v^2-v+u-u^3)\,(2-12\,u\,v)}{\%1}\Bigg)\mathrm{d}(v)$$
$$\%1 := 2+6\,v^2+7\,v^4+4\,v^6+v^8-4\,u\,v-8\,u\,v^3-4\,u\,v^5+2\,u^2+6\,u^2\,v^2+8\,u^2\,v^4+4\,u^2\,v^6-8\,u^3\,v-8\,u^3\,v^3-u^4$$
$$+4\,u^4\,v^2+6\,u^4\,v^4-4\,u^5\,v+4\,u^6\,v^2+u^8$$

```
> Gamma12:=(wcollect(cartan[1,2]));
```

$$\Gamma 12 := \Bigg(-\frac{(v^4+2\,v^2-6\,u^2\,v^2+2\,u\,v+u^4+1-2\,u^2)\,(-4\,v+12\,u^2\,v-4\,v^3-2\,u)}{\%1}$$

$$+\frac{(-4u^3v+4uv^3+u^2-v^2+1)(-4u^3+12uv^2-2v)}{\%1}-2\frac{(3uv^2-v+u-u^3)(2-12uv)}{\%1}\Bigg)d(u)+\Bigg($$

$$-\frac{(v^4+2v^2-6u^2v^2+2uv+u^4+1-2u^2)(4u^3-12uv^2-4u+2v)}{\%1}$$

$$+\frac{(-4u^3v+4uv^3+u^2-v^2+1)(-2u+12u^2v-4v^3)}{\%1}-2\frac{(3uv^2-v+u-u^3)(-6u^2+6v^2+2)}{\%1}\Bigg)d(v)$$

$$\%1 := 2+6v^2+7v^4+4v^6+v^8-4uv-8uv^3-4uv^5+2u^2+6u^2v^2+8u^2v^4+4u^2v^6-8u^3v-8u^3v^3-u^4+4u^4v^2+6u^4v^4-4u^5v+4u^6v^2+u^8$$

```
> Gamma21:=(wcollect(cartan[2,1]));
```

$$\Gamma 21 := \Bigg(-\frac{(4uv^3-v^2-4u^3v+4uv-1+u^2)(-4u^3+4u+12uv^2-2v)}{\%1}$$

$$-\frac{(v^4-6u^2v^2+2uv+u^4-1)(-12u^2v+4v^3+2u)}{\%1}+2\frac{(v^3-3u^2v+v+u)(-6v^2-2+6u^2)}{\%1}\Bigg)d(u)+\Bigg($$

$$-\frac{(4uv^3-v^2-4u^3v+4uv-1+u^2)(-4v+12u^2v-4v^3-2u)}{\%1}$$

$$-\frac{(v^4-6u^2v^2+2uv+u^4-1)(-4u^3+12uv^2-2v)}{\%1}+2\frac{(v^3-3u^2v+v+u)(2-12uv)}{\%1}\Bigg)d(v)$$

$$\%1 := 2+6v^2+7v^4+4v^6+v^8-4uv-8uv^3-4uv^5+2u^2+6u^2v^2+8u^2v^4+4u^2v^6-8u^3v-8u^3v^3-u^4+4u^4v^2+6u^4v^4-4u^5v+4u^6v^2+u^8$$

```
> Gamma22:=(wcollect(cartan[2,2]));
```

$$\Gamma 22 := \Bigg(-\frac{(4uv^3-v^2-4u^3v+4uv-1+u^2)(-4v+12u^2v-4v^3-2u)}{\%1}$$

$$-\frac{(v^4-6u^2v^2+2uv+u^4-1)(-4u^3+12uv^2-2v)}{\%1}+2\frac{(v^3-3u^2v+v+u)(2-12uv)}{\%1}\Bigg)d(u)+\Bigg($$

$$-\frac{(4uv^3-v^2-4u^3v+4uv-1+u^2)(4u^3-12uv^2-4u+2v)}{\%1}$$

$$-\frac{(v^4-6u^2v^2+2uv+u^4-1)(-2u+12u^2v-4v^3)}{\%1}+2\frac{(v^3-3u^2v+v+u)(-6u^2+6v^2+2)}{\%1}\Bigg)d(v)$$

$$\%1 := 2+6v^2+7v^4+4v^6+v^8-4uv-8uv^3-4uv^5+2u^2+6u^2v^2+8u^2v^4+4u^2v^6-8u^3v-8u^3v^3-u^4+4u^4v^2+6u^4v^4-4u^5v+4u^6v^2+u^8$$

The second fundamental form or shape matrix comes from the third row of the Cartan matrix

```
> h1:=wcollect(cartan[3,1]);
```

$$h1 := \Bigg(2\frac{u(-4u^3+4u+12uv^2-2v)}{\%1}+2\frac{v(-12u^2v+4v^3+2u)}{\%1}+\frac{(v^2-1+u^2)(-6v^2-2+6u^2)}{\%1}\Bigg)d(u)$$

$$+\Bigg(2\frac{u(-4v+12u^2v-4v^3-2u)}{\%1}+2\frac{v(-4u^3+12uv^2-2v)}{\%1}+\frac{(v^2-1+u^2)(2-12uv)}{\%1}\Bigg)d(v)$$

$$\%1 := 5v^8-12uv^5-4v^7u+11v^6+v^{10}+u^{10}-u^6+u^8-4u^7v+8v^2+13v^4+14u^2v^2+10u^6v^4+10u^4v^6+18u^4v^4-12u^3v^5-24u^3v^3-12u^5v^3-12u^5v+5u^8v^2+8u^6v^2+5u^2v^8+16u^2v^6+9u^4v^2+21u^2v^4+2+u^4-4uv+4u^2-12u^3v-12uv^3$$

```
> gamma1:=wcollect(cartan[1,3]);
```

$$\gamma 1 := \Bigg(-(v^4+2v^2-6u^2v^2+2uv+u^4+1-2u^2)$$

$$(2(v^2+1+u^2)\%1+4u^2\%1+2u(v^2+1+u^2)(4uv^2-4v+4u^3-4u))/(\%2)$$

$$+\frac{(-4\,u^3\,v+4\,u\,v^3+u^2-v^2+1)\,(4\,u\,v\,\%1+2\,v\,(v^2+1+u^2)\,(4\,u\,v^2-4\,v+4\,u^3-4\,u))}{\%2}-2\,(3\,u\,v^2-v+u-u^3)$$

$$\left.(2\,u\,(v^2+1+u^2)\,\%1+2\,(v^2-1+u^2)\,u\,\%1+(v^2-1+u^2)\,(v^2+1+u^2)\,(4\,u\,v^2-4\,v+4\,u^3-4\,u))\,/\,(\%2)\right)\mathrm{d}(u)+$$

$$\left(-\frac{(v^4+2\,v^2-6\,u^2\,v^2+2\,u\,v+u^4+1-2\,u^2)\,(4\,u\,v\,\%1+2\,u\,(v^2+1+u^2)\,(4\,v^3+4\,v+4\,u^2\,v-4\,u))}{\%2}\right.$$

$$+\frac{(-4\,u^3\,v+4\,u\,v^3+u^2-v^2+1)\,(2\,(v^2+1+u^2)\,\%1+4\,v^2\,\%1+2\,v\,(v^2+1+u^2)\,(4\,v^3+4\,v+4\,u^2\,v-4\,u))}{\%2}-2$$

$$(3\,u\,v^2-v+u-u^3)$$

$$\left.(2\,v\,(v^2+1+u^2)\,\%1+2\,(v^2-1+u^2)\,v\,\%1+(v^2-1+u^2)\,(v^2+1+u^2)\,(4\,v^3+4\,v+4\,u^2\,v-4\,u))\,/\,(\%2)\right)\mathrm{d}(v)$$

$$\%1:=v^4+2\,v^2+2\,u^2\,v^2-4\,u\,v+u^4-2\,u^2+2$$

$$\%2:=2+6\,v^2+7\,v^4+4\,v^6+v^8-4\,u\,v-8\,u\,v^3-4\,u\,v^5+2\,u^2+6\,u^2\,v^2+8\,u^2\,v^4+4\,u^2\,v^6-8\,u^3\,v-8\,u^3\,v^3-u^4$$
$$+4\,u^4\,v^2+6\,u^4\,v^4-4\,u^5\,v+4\,u^6\,v^2+u^8$$

```
> h2:=(wcollect(cartan[3,2]));
```

$$h2:=\left(2\,\frac{u\,(-4\,v+12\,u^2\,v-4\,v^3-2\,u)}{\%1}+2\,\frac{v\,(-4\,u^3+12\,u\,v^2-2\,v)}{\%1}+\frac{(v^2-1+u^2)\,(2-12\,u\,v)}{\%1}\right)\mathrm{d}(u)$$
$$+\left(2\,\frac{u\,(4\,u^3-12\,u\,v^2-4\,u+2\,v)}{\%1}+2\,\frac{v\,(-2\,u+12\,u^2\,v-4\,v^3)}{\%1}+\frac{(v^2-1+u^2)\,(-6\,u^2+6\,v^2+2)}{\%1}\right)\mathrm{d}(v)$$

$$\%1:=5\,v^8-12\,u\,v^5-4\,v^7\,u+11\,v^6+v^{10}+u^{10}-u^6+u^8-4\,u^7\,v+8\,v^2+13\,v^4+14\,u^2\,v^2+10\,u^6\,v^4+10\,u^4\,v^6$$
$$+18\,u^4\,v^4-12\,u^3\,v^5-24\,u^3\,v^3-12\,u^5\,v^3-12\,u^5\,v+5\,u^8\,v^2+8\,u^6\,v^2+5\,u^2\,v^8+16\,u^2\,v^6+9\,u^4\,v^2+21\,u^2\,v^4+2+u^4$$
$$-4\,u\,v+4\,u^2-12\,u^3\,v-12\,u\,v^3$$

```
> gamma2:=(wcollect(cartan[2,3]));
```

$$\gamma 2:=\left(\right.$$

$$-\frac{(4\,u\,v^3-v^2-4\,u^3\,v+4\,u\,v-1+u^2)\,(2\,(v^2+1+u^2)\,\%1+4\,u^2\,\%1+2\,u\,(v^2+1+u^2)\,(4\,u\,v^2-4\,v+4\,u^3-4\,u))}{\%2}$$

$$-\frac{(v^4-6\,u^2\,v^2+2\,u\,v+u^4-1)\,(4\,u\,v\,\%1+2\,v\,(v^2+1+u^2)\,(4\,u\,v^2-4\,v+4\,u^3-4\,u))}{\%2}+2\,(v^3-3\,u^2\,v+v+u)$$

$$\left.(2\,u\,(v^2+1+u^2)\,\%1+2\,(v^2-1+u^2)\,u\,\%1+(v^2-1+u^2)\,(v^2+1+u^2)\,(4\,u\,v^2-4\,v+4\,u^3-4\,u))\,/\,(\%2)\right)\mathrm{d}(u)+$$

$$\left(-\frac{(4\,u\,v^3-v^2-4\,u^3\,v+4\,u\,v-1+u^2)\,(4\,u\,v\,\%1+2\,u\,(v^2+1+u^2)\,(4\,v^3+4\,v+4\,u^2\,v-4\,u))}{\%2}\right.$$

$$-\frac{(v^4-6\,u^2\,v^2+2\,u\,v+u^4-1)\,(2\,(v^2+1+u^2)\,\%1+4\,v^2\,\%1+2\,v\,(v^2+1+u^2)\,(4\,v^3+4\,v+4\,u^2\,v-4\,u))}{\%2}+2$$

$$(v^3-3\,u^2\,v+v+u)$$

$$\left.(2\,v\,(v^2+1+u^2)\,\%1+2\,(v^2-1+u^2)\,v\,\%1+(v^2-1+u^2)\,(v^2+1+u^2)\,(4\,v^3+4\,v+4\,u^2\,v-4\,u))\,/\,(\%2)\right)\mathrm{d}(v)$$

$$\%1:=v^4+2\,v^2+2\,u^2\,v^2-4\,u\,v+u^4-2\,u^2+2$$

$$\%2:=2+6\,v^2+7\,v^4+4\,v^6+v^8-4\,u\,v-8\,u\,v^3-4\,u\,v^5+2\,u^2+6\,u^2\,v^2+8\,u^2\,v^4+4\,u^2\,v^6-8\,u^3\,v-8\,u^3\,v^3-u^4$$
$$+4\,u^4\,v^2+6\,u^4\,v^4-4\,u^5\,v+4\,u^6\,v^2+u^8$$

Next compute the expansion_twist:

The abnormality for the parametric surface will show up as a non-zero entry in the [3,3] slot of the Cartan Matrix. The abnormaility is always an exact differental for parametric and Monge surfaces. The expansion_twist (Big Omega) is not zero unless the euclidean scaling is subsumed.

Therefore implicit Monge surfaces will admit disclination defects (Torsion of the second kind due to rotations)

```
> Omega:=(wcollect(factor((cartan[3,3]))));
>
```

$$\Omega := 4\,\frac{(2\,u\,v^4 - v^3 + 2\,u\,v^2 + 4\,u^3\,v^2 - v - 5\,u^2\,v - 2\,u^3 + 2\,u^5 + u)\,\mathrm{d}(u)}{(v^2+1+u^2)\,(v^4+2\,v^2+2\,u^2\,v^2 - 4\,u\,v + u^4 - 2\,u^2 + 2)} + 4\,\frac{(2\,v^5 + 4\,u^2\,v^3 + 4\,v^3 - 5\,u\,v^2 + 3\,v + 2\,u^4\,v - u^3 - u)\,\mathrm{d}(v)}{(v^2+1+u^2)\,(v^4+2\,v^2+2\,u^2\,v^2 - 4\,u\,v + u^4 - 2\,u^2 + 2)}$$

(Omega vanishes for the euclidean type of normalization.

```
> shape11:=-factor(gamma1&^d(v)/d(u)&^d(v));
```

$$shape11 := 2\,v^2 + 2 - 2\,u^2$$

```
> shape12:=-factor(gamma1&^d(u)/d(v)&^d(u));
```

$$shape12 := 4\,u\,v - 2$$

```
> shape21:=-factor(gamma2&^d(v)/d(u)&^d(v));
```

$$shape21 := 4\,u\,v - 2$$

```
> shape22:=-factor(gamma2&^d(u)/d(v)&^d(u));
```

$$shape22 := -2\,v^2 - 2 + 2\,u^2$$

```
>
> SHAPE:=array([[shape11,shape12],[shape21,shape22]]):
> HH:=simplify(trace(SHAPE)/2):
> print(`Mean Curvature is `,factor(HH));
```

$$\textit{Mean Curvature is}\ , 0$$

```
> KK:=simplify(det(SHAPE)):
> print(`Gauss Curvature is `,factor(KK));
```

$$\textit{Gauss Curvature is}\ , -4\,v^4 - 8\,v^2 - 8\,u^2\,v^2 - 8 + 8\,u^2 - 4\,u^4 + 16\,u\,v$$

```
>
>
>
> xxi;yyi;zzi;
```

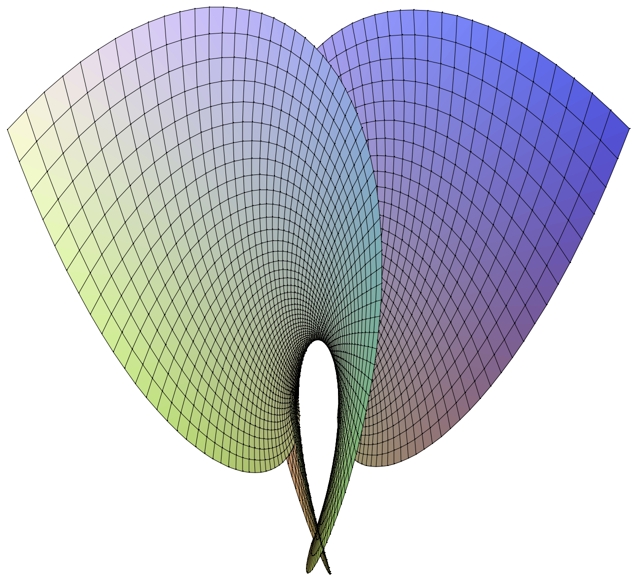
$$-u^4\,v + 2\,u^2\,v^3 - \frac{1}{5}v^5 + \frac{1}{3}u^3 - u\,v^2 + 2\,u^2\,v - \frac{2}{3}v^3 - v - u$$

$$\frac{1}{5}u^5 - 2\,u^3\,v^2 + u\,v^4 + u^2\,v - \frac{1}{3}v^3 - u + v$$

$$2\,u^3\,v - 2\,u\,v^3 - u^2 + v^2 - 2\,u\,v$$

```
> plot3d([xxi,yyi,zzi],u=-1.3..1.3,v=-1.3..1.3,orientation=[134,86],numpoints=5000
  ,style=PATCH,title=`Imag part - Mandelbrot Minimal Surface`);
```

Imag part - Mandelbrot Minimal Surface



>

Mandelbrodt minimal surface.