```
>
> restart:
> with(linalg):with(liesymm):with(difforms):defform(u=0,v=0,x=0,y=0,z=0,t=0,lambda=0,a=const,b=const,c=const,p=const,n=const,k=const,omega=const,e=const,Gamma=const);

>
Warning, new definition for norm
Warning, new definition for trace
Warning, new definition for close
Warning, new definition for `&^`
Warning, new definition for d
Warning, new definition for mixpar
Warning, new definition for wdegree
```

# HOLDER NORMS 4D examples

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## A flat surface in 4D of Pfaff dimension

The 1-form is modeled such that the Topological Parity and the Torsion current are not zero.

```
> A1:=0;A2:=0;A3:=u(x,y);A4:=v(x,y);A:=evalm([A1,A2,A3]);phi:=-A4;
```

$$A1 := 0$$

$$A2 := 0$$

$$A3 := u(x, y)$$

$$A4 := v(x, y)$$

$$A := [0, 0, u(x, y)]$$

$$\phi := -v(x, y)$$

```
> A1 := 0;
```

$$A1 := 0$$

```
> lambda:=factor(subs(A[1]^2+A[2]^2+A[3]^2+phi^2))^(1/2);
```

$$\lambda := \sqrt{u(x, y)^2 + v(x, y)^2}$$

```
> Action:=A1*d(x)+A2*d(y)+A3*d(z)+A4*d(t);Fields:=d(Action);TTORSION:=Action&^d(Action);TPARITY:=d(Action)&^d(Action);
```

$$Action := u(x, y)\, d(z) + v(x, y)\, d(t)$$

$$Fields := \left(\frac{\partial}{\partial x} u(x, y)\right)(d(x) \mathbin{\&\wedge} d(z)) + \left(\frac{\partial}{\partial y} u(x, y)\right)(d(y) \mathbin{\&\wedge} d(z)) + \left(\frac{\partial}{\partial x} v(x, y)\right)(d(x) \mathbin{\&\wedge} d(t)) + \left(\frac{\partial}{\partial y} v(x, y)\right)(d(y) \mathbin{\&\wedge} d(t))$$

$$TTORSION := \left(u(x, y)\left(\frac{\partial}{\partial x} v(x, y)\right) - v(x, y)\left(\frac{\partial}{\partial x} u(x, y)\right)\right) \&\wedge(d(z), d(x), d(t)) + \left(u(x, y)\left(\frac{\partial}{\partial y} v(x, y)\right) - v(x, y)\left(\frac{\partial}{\partial y} u(x, y)\right)\right) \&\wedge(d(z), d(y), d(t))$$

$$TPARITY := \left(2\left(\frac{\partial}{\partial x}\mathrm{u}(x,y)\right)\left(\frac{\partial}{\partial y}\mathrm{v}(x,y)\right)-2\left(\frac{\partial}{\partial y}\mathrm{u}(x,y)\right)\left(\frac{\partial}{\partial x}\mathrm{v}(x,y)\right)\right)\&\wedge(\mathrm{d}(x),\mathrm{d}(z),\mathrm{d}(y),\mathrm{d}(t))$$

```
> B:=curl([A[1],A[2],A[3]],[x,y,z]);EP:=evalm(-grad(phi,[x,y,z]));EV:=-[diff(A[1],t),diff(A[2],t),diff(A[3],t)];E:=evalm(EV+EP);Parity:=innerprod(E,B);Helicity:=innerprod(A,B);
```

These are E and B fields on for the potentials NOT made homogeneous of degree 1

$$B := \left[\frac{\partial}{\partial y}\mathrm{u}(x,y), -\left(\frac{\partial}{\partial x}\mathrm{u}(x,y)\right), 0\right]$$

$$EP := \left[\frac{\partial}{\partial x}\mathrm{v}(x,y), \frac{\partial}{\partial y}\mathrm{v}(x,y), 0\right]$$

$$EV := [0, 0, 0]$$

$$E := \left[\frac{\partial}{\partial x}\mathrm{v}(x,y), \frac{\partial}{\partial y}\mathrm{v}(x,y), 0\right]$$

$$Parity := \left(\frac{\partial}{\partial y}\mathrm{u}(x,y)\right)\left(\frac{\partial}{\partial x}\mathrm{v}(x,y)\right)-\left(\frac{\partial}{\partial x}\mathrm{u}(x,y)\right)\left(\frac{\partial}{\partial y}\mathrm{v}(x,y)\right)$$

$$Helicity := 0$$

The potentials produce a non-zero, but static E and B fields. It is noteworthy that the Helicity is zero and the Parity is not zero. Hence there must be a spatial term to the torsion flow. All of this is for the inhomogeneous 1-form.

```
> lambda:=factor(subs(A[1]^2+A[2]^2+A[3]^2+phi^2))^(1/2);
```

$$\lambda := \sqrt{\mathrm{u}(x,y)^2+\mathrm{v}(x,y)^2}$$

```
> NA:=evalm([A1,A2,A3,-phi]/lambda);
```

$$NA := \left[0, 0, \frac{\mathrm{u}(x,y)}{\sqrt{\mathrm{u}(x,y)^2+\mathrm{v}(x,y)^2}}, \frac{\mathrm{v}(x,y)}{\sqrt{\mathrm{u}(x,y)^2+\mathrm{v}(x,y)^2}}\right]$$

```
>
> JAC:=jacobian(NA,[x,y,z,t]);DETJAC:=det(JAC);MINPPOLYNOMIAL:=minpoly(JAC,alpha);
`Eigenvalues`:=eigenvalues(JAC);
>
```

$JAC :=$

$$[0, 0, 0, 0]$$

$$[0, 0, 0, 0]$$

$$\left[-\frac{1}{2}\frac{\mathrm{u}(x,y)\left(2\,\mathrm{u}(x,y)\left(\frac{\partial}{\partial x}\mathrm{u}(x,y)\right)+2\,\mathrm{v}(x,y)\left(\frac{\partial}{\partial x}\mathrm{v}(x,y)\right)\right)}{\%1^{(3/2)}}+\frac{\frac{\partial}{\partial x}\mathrm{u}(x,y)}{\sqrt{\%1}},\right.$$

$$\left.-\frac{1}{2}\frac{\mathrm{u}(x,y)\left(2\,\mathrm{u}(x,y)\left(\frac{\partial}{\partial y}\mathrm{u}(x,y)\right)+2\,\mathrm{v}(x,y)\left(\frac{\partial}{\partial y}\mathrm{v}(x,y)\right)\right)}{\%1^{(3/2)}}+\frac{\frac{\partial}{\partial y}\mathrm{u}(x,y)}{\sqrt{\%1}}, 0, 0\right]$$

$$\left[-\frac{1}{2}\frac{\mathrm{v}(x,y)\left(2\,\mathrm{u}(x,y)\left(\frac{\partial}{\partial x}\mathrm{u}(x,y)\right)+2\,\mathrm{v}(x,y)\left(\frac{\partial}{\partial x}\mathrm{v}(x,y)\right)\right)}{\%1^{(3/2)}}+\frac{\frac{\partial}{\partial x}\mathrm{v}(x,y)}{\sqrt{\%1}},\right.$$

$$-\frac{1}{2}\frac{v(x,y)\left(2\,u(x,y)\left(\frac{\partial}{\partial y}u(x,y)\right)+2\,v(x,y)\left(\frac{\partial}{\partial y}v(x,y)\right)\right)}{\%1^{(3/2)}}+\frac{\frac{\partial}{\partial y}v(x,y)}{\sqrt{\%1}},0,0\Bigg]$$

$\%1 := u(x,y)^2 + v(x,y)^2$

$$DETJAC := 0$$

$$MINPPOLYNOMIAL := \alpha^2$$

$$Eigenvalues := 0, 0, 0, 0$$

The minimum polynomial indicates that all eigen values (curvatures) are zero

```
> #eigenvectors(JAC);
> GMN:=innerprod(transpose(JAC),JAC);
```

$GMN :=$

$$\Bigg[\frac{\left(\frac{\partial}{\partial x}u(x,y)\right)^2 v(x,y)^2-2\,u(x,y)\,v(x,y)\left(\frac{\partial}{\partial x}v(x,y)\right)\left(\frac{\partial}{\partial x}u(x,y)\right)+u(x,y)^2\left(\frac{\partial}{\partial x}v(x,y)\right)^2}{\%1^2},\Bigg(\left(\frac{\partial}{\partial x}u(x,y)\right)\left(\frac{\partial}{\partial y}u(x,y)\right)v(x,y)^2-u(x,y)\,v(x,y)\left(\frac{\partial}{\partial x}v(x,y)\right)\left(\frac{\partial}{\partial y}u(x,y)\right)-\left(\frac{\partial}{\partial x}u(x,y)\right)u(x,y)\,v(x,y)\left(\frac{\partial}{\partial y}v(x,y)\right)+u(x,y)^2\left(\frac{\partial}{\partial x}v(x,y)\right)\left(\frac{\partial}{\partial y}v(x,y)\right)\Bigg)\Big/\%1^2,0,0\Bigg]$$

$$\Bigg[\Bigg(\left(\frac{\partial}{\partial x}u(x,y)\right)\left(\frac{\partial}{\partial y}u(x,y)\right)v(x,y)^2-u(x,y)\,v(x,y)\left(\frac{\partial}{\partial x}v(x,y)\right)\left(\frac{\partial}{\partial y}u(x,y)\right)-\left(\frac{\partial}{\partial x}u(x,y)\right)u(x,y)\,v(x,y)\left(\frac{\partial}{\partial y}v(x,y)\right)+u(x,y)^2\left(\frac{\partial}{\partial x}v(x,y)\right)\left(\frac{\partial}{\partial y}v(x,y)\right)\Bigg)\Big/\%1^2,\frac{\left(\frac{\partial}{\partial y}u(x,y)\right)^2 v(x,y)^2-2\,u(x,y)\,v(x,y)\left(\frac{\partial}{\partial y}v(x,y)\right)\left(\frac{\partial}{\partial y}u(x,y)\right)+u(x,y)^2\left(\frac{\partial}{\partial y}v(x,y)\right)^2}{\%1^2},0,0\Bigg]$$

$[0, 0, 0, 0]$

$[0, 0, 0, 0]$

$\%1 := u(x,y)^2 + v(x,y)^2$

```
> MEAN_CURVATURE:=factor(subs(trace(JAC)));
```

$$MEAN\_CURVATURE := 0$$

```
> S2:=factor(trace(innerprod(JAC,JAC))):
  Gauss:=factor(subs((-(1/2)*(-trace(JAC)*trace(JAC)+S2))));
```

$$Gauss := 0$$

```
> ADJAC:=adjoint(JAC):
```

```
> ADJOINT_CURVATURE:=factor(subs(trace(ADJAC)));
```

$$ADJOINT\_CURVATURE := 0$$

```
>
> CurrentJ:=innerprod(ADJAC,NA);Interaction:=innerprod(CurrentJ,NA);DivJ:=factor(d
  iverge(CurrentJ,[x,y,z,t]));rho:=CurrentJ[4];
```

$$CurrentJ := [0, 0, 0, 0]$$

$$Interaction := 0$$

$$DivJ := 0$$

$$\rho := 0$$

```
>
```

**The charge current density vanishes identically. BUT THE TORSION vector is NOT ZERO. The Adjoint curvature is zero. All the curvature coefficients are zero. Hence the surface is FLAT.**

```
>
> Adjointcurv:=simplify(factor(trace(ADJAC))):Mean:=factor(trace(JAC)):Net:=factor
  (Adjointcurv-Interaction):Gnet:=factor(subs(c=1,n=1,Net));
```

$$Gnet := 0$$

Next, compute Maxwell-Faraday properties for normalized Action made homogeneous of degree zero.

```
> NB:=simplify(curl([NA[1],NA[2],NA[3]],[x,y,z]));NEP:=evalm(-grad(NA[4],[x,y,z]))
  ;NEV:=-[diff(NA[1],t),diff(NA[2],t),diff(NA[3],t)];NE:=simplify(evalm(NEV+NEP));
  NParity:=2*innerprod(NE,NB);NHelicity:=innerprod([NA[1],NA[2],NA[3]],NB);
```

$$NB := \left[\frac{\mathrm{v}(x,y)\left(-\mathrm{u}(x,y)\left(\frac{\partial}{\partial y}\mathrm{v}(x,y)\right)+\mathrm{v}(x,y)\left(\frac{\partial}{\partial y}\mathrm{u}(x,y)\right)\right)}{(\mathrm{u}(x,y)^2+\mathrm{v}(x,y)^2)^{(3/2)}}, \frac{\mathrm{v}(x,y)\left(\mathrm{u}(x,y)\left(\frac{\partial}{\partial x}\mathrm{v}(x,y)\right)-\mathrm{v}(x,y)\left(\frac{\partial}{\partial x}\mathrm{u}(x,y)\right)\right)}{(\mathrm{u}(x,y)^2+\mathrm{v}(x,y)^2)^{(3/2)}}, 0\right]$$

$$NEP := \left[\frac{1}{2}\frac{\mathrm{v}(x,y)\left(2\,\mathrm{u}(x,y)\left(\frac{\partial}{\partial x}\mathrm{u}(x,y)\right)+2\,\mathrm{v}(x,y)\left(\frac{\partial}{\partial x}\mathrm{v}(x,y)\right)\right)}{\%1^{(3/2)}}-\frac{\frac{\partial}{\partial x}\mathrm{v}(x,y)}{\sqrt{\%1}},\right.$$
$$\left.\frac{1}{2}\frac{\mathrm{v}(x,y)\left(2\,\mathrm{u}(x,y)\left(\frac{\partial}{\partial y}\mathrm{u}(x,y)\right)+2\,\mathrm{v}(x,y)\left(\frac{\partial}{\partial y}\mathrm{v}(x,y)\right)\right)}{\%1^{(3/2)}}-\frac{\frac{\partial}{\partial y}\mathrm{v}(x,y)}{\sqrt{\%1}}, 0\right]$$

$$\%1 := \mathrm{u}(x,y)^2+\mathrm{v}(x,y)^2$$

$$NEV := [0, 0, 0]$$

$$NE := \left[-\frac{\mathrm{u}(x,y)\left(\mathrm{u}(x,y)\left(\frac{\partial}{\partial x}\mathrm{v}(x,y)\right)-\mathrm{v}(x,y)\left(\frac{\partial}{\partial x}\mathrm{u}(x,y)\right)\right)}{(\mathrm{u}(x,y)^2+\mathrm{v}(x,y)^2)^{(3/2)}}, \frac{\mathrm{u}(x,y)\left(-\mathrm{u}(x,y)\left(\frac{\partial}{\partial y}\mathrm{v}(x,y)\right)+\mathrm{v}(x,y)\left(\frac{\partial}{\partial y}\mathrm{u}(x,y)\right)\right)}{(\mathrm{u}(x,y)^2+\mathrm{v}(x,y)^2)^{(3/2)}}, 0\right]$$

$$NParity := 0$$

$$NHelicity := 0$$

```
> NTorsion_current:=simplify(evalm(crossprod(NE,[NA[1],NA[2],NA[3]],NB)+evalm(NB*N
  A[4])));
```

$$NTorsion\_current := \left[\frac{-\mathrm{u}(x,y)\left(\frac{\partial}{\partial y}\mathrm{v}(x,y)\right)+\mathrm{v}(x,y)\left(\frac{\partial}{\partial y}\mathrm{u}(x,y)\right)}{\mathrm{u}(x,y)^2+\mathrm{v}(x,y)^2}, -\frac{-\mathrm{u}(x,y)\left(\frac{\partial}{\partial x}\mathrm{v}(x,y)\right)+\mathrm{v}(x,y)\left(\frac{\partial}{\partial x}\mathrm{u}(x,y)\right)}{\mathrm{u}(x,y)^2+\mathrm{v}(x,y)^2}, 0\right]$$

```
> NParity:=2*innerprod(NE,NB);Diss:=innerprod([CurrentJ[1],CurrentJ[2],CurrentJ[3]
  ],NE);
```

$$NParity := 0$$

$$Diss := 0$$

>

>

```
> NAction2:=(NA[1]*d(x)+NA[2]*d(y)+NA[3]*d(z)+NA[4]*d(t));NFields:=wcollect(factor(d(NAction2)));NTTORSION:=wcollect(factor(NAction2&^d(NAction2)));NTPARITY:=simplify(factor(d(NAction2)&^d(NAction2)));
```

$$NAction2 := \frac{u(x,y)\,d(z)}{\sqrt{u(x,y)^2+v(x,y)^2}} + \frac{v(x,y)\,d(t)}{\sqrt{u(x,y)^2+v(x,y)^2}}$$

$$NFields := -\frac{\left(u(x,y)\,v(x,y)\left(\frac{\partial}{\partial y}v(x,y)\right) - \left(\frac{\partial}{\partial y}u(x,y)\right)v(x,y)^2\right)(d(y)\ \&\wedge\ d(z))}{\%1^{(3/2)}}$$
$$-\frac{\left(u(x,y)\,v(x,y)\left(\frac{\partial}{\partial x}v(x,y)\right) - \left(\frac{\partial}{\partial x}u(x,y)\right)v(x,y)^2\right)(d(x)\ \&\wedge\ d(z))}{\%1^{(3/2)}}$$
$$-\frac{\left(v(x,y)\,u(x,y)\left(\frac{\partial}{\partial y}u(x,y)\right) - \left(\frac{\partial}{\partial y}v(x,y)\right)u(x,y)^2\right)(d(y)\ \&\wedge\ d(t))}{\%1^{(3/2)}}$$
$$-\frac{\left(v(x,y)\,u(x,y)\left(\frac{\partial}{\partial x}u(x,y)\right) - \left(\frac{\partial}{\partial x}v(x,y)\right)u(x,y)^2\right)(d(x)\ \&\wedge\ d(t))}{\%1^{(3/2)}}$$

$$\%1 := u(x,y)^2 + v(x,y)^2$$

$$NTTORSION := -\frac{\left(-u(x,y)\left(\frac{\partial}{\partial x}v(x,y)\right) + v(x,y)\left(\frac{\partial}{\partial x}u(x,y)\right)\right)\&\wedge(d(z), d(x), d(t))}{u(x,y)^2+v(x,y)^2}$$
$$-\frac{\left(-u(x,y)\left(\frac{\partial}{\partial y}v(x,y)\right) + v(x,y)\left(\frac{\partial}{\partial y}u(x,y)\right)\right)\&\wedge(d(z), d(y), d(t))}{u(x,y)^2+v(x,y)^2}$$

$$NTPARITY := 0$$

>

Note that the Holder divisor creates an integrating factor for the topological parity of the rescaled Action and does not depend upon signature,

# A Minimal "Heisenberg" surface in 4D (Pfaff dimension 2 )

The 1-form is modeled with a constant a to produce a minimal surface in 4D of the classic variety such that the Gauss curvature is negative.

```
> A1:=y;A2:=-x;A3:=Gamma^2-(x^2+y^2);A4:=0;A:=[A1,A2,A3];phi:=-A4;
>
```

$$A1 := y$$

$$A2 := -x$$

$$A3 := \Gamma^2 - x^2 - y^2$$

$$A4 := 0$$

$$A := [y, -x, \Gamma^2 - x^2 - y^2]$$

$$\phi := 0$$

```
> lambda:=factor(subs(A[1]^2+A[2]^2+A[3]^2+phi^2))^(1/2);
```

$$\lambda := \sqrt{y^2 + x^2 + \Gamma^4 - 2\,\Gamma^2\,x^2 - 2\,\Gamma^2\,y^2 + x^4 + 2\,x^2\,y^2 + y^4}$$

```
> Action:=A1*d(x)+A2*d(y)+A3*d(z)+A4*d(t);Fields:=d(Action);TTORSION:=Action&^d(Action);TPARITY:=d(Action)&^d(Action);
```

$$Action := y\,\mathrm{d}(x) - x\,\mathrm{d}(y) + (\Gamma^2 - x^2 - y^2)\,\mathrm{d}(z)$$

$$Fields := 2\,(\mathrm{d}(y)\ \&\wedge\ \mathrm{d}(x)) - 2\,y\,(\mathrm{d}(y)\ \&\wedge\ \mathrm{d}(z)) - 2\,x\,(\mathrm{d}(x)\ \&\wedge\ \mathrm{d}(z))$$

$$TTORSION := -2\,\Gamma^2\ \&\wedge(\mathrm{d}(x), \mathrm{d}(y), \mathrm{d}(z))$$

$$TPARITY := 0$$

```
> B:=curl([A[1],A[2],A[3]],[x,y,z]);EP:=evalm(-grad(phi,[x,y,z]));EV:=-[diff(A[1],t),diff(A[2],t),diff(A[3],t)];E:=evalm(EV+EP);Parity:=innerprod(E,B);Helicity:=innerprod(A,B);
```

Values for the inhomogeneous 1-form

$$B := [-2\,y, 2\,x, -2]$$

$$EP := [0, 0, 0]$$

$$EV := [0, 0, 0]$$

$$E := [0, 0, 0]$$

$$Parity := 0$$

$$Helicity := -2\,\Gamma^2$$

The potentials produce a non-zero, but static B field. with finite helicity

```
> lambda:=factor(subs(A[1]^2+A[2]^2+A[3]^2+phi^2))^(1/2);
>
```

$$\lambda := \sqrt{y^2 + x^2 + \Gamma^4 - 2\,\Gamma^2\,x^2 - 2\,\Gamma^2\,y^2 + x^4 + 2\,x^2\,y^2 + y^4}$$

```
> NA:=evalm([A1,A2,A3,-phi]/lambda);
```

$$NA := \left[\frac{y}{\sqrt{\%1}}, -\frac{x}{\sqrt{\%1}}, \frac{\Gamma^2 - x^2 - y^2}{\sqrt{\%1}}, 0\right]$$

$$\%1 := y^2 + x^2 + \Gamma^4 - 2\,\Gamma^2\,x^2 - 2\,\Gamma^2\,y^2 + x^4 + 2\,x^2\,y^2 + y^4$$

```
>
> JAC:=jacobian(NA,[x,y,z,t]);DETJAC:=det(JAC);MINPOLYNOMIAL:=minpoly(JAC,alpha);`Eigenvalues`:=eigenvalues(JAC);
```

$$JAC := \begin{bmatrix} -\frac{1}{2}\frac{y\,\%3}{\%1^{(3/2)}} & -\frac{1}{2}\frac{y\,\%2}{\%1^{(3/2)}} + \frac{1}{\sqrt{\%1}} & 0 & 0 \\ \frac{1}{2}\frac{x\,\%3}{\%1^{(3/2)}} - \frac{1}{\sqrt{\%1}} & \frac{1}{2}\frac{x\,\%2}{\%1^{(3/2)}} & 0 & 0 \\ -\frac{1}{2}\frac{(\Gamma^2 - x^2 - y^2)\,\%3}{\%1^{(3/2)}} - 2\frac{x}{\sqrt{\%1}} & -\frac{1}{2}\frac{(\Gamma^2 - x^2 - y^2)\,\%2}{\%1^{(3/2)}} - 2\frac{y}{\sqrt{\%1}} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\%1 := y^2 + x^2 + \Gamma^4 - 2\,\Gamma^2\,x^2 - 2\,\Gamma^2\,y^2 + x^4 + 2\,x^2\,y^2 + y^4$$

$$\%2 := 2\,y - 4\,\Gamma^2\,y + 4\,x^2\,y + 4\,y^3$$

$$\%3 := 2\,x - 4\,\Gamma^2\,x + 4\,x^3 + 4\,x\,y^2$$

$$DETJAC := 0$$

$$MINPOLYNOMIAL := \frac{(\Gamma^4 - 2x^2y^2 - y^4 - x^4)\,\alpha}{(y^2 + x^2 + \Gamma^4 - 2\Gamma^2x^2 - 2\Gamma^2y^2 + x^4 + 2x^2y^2 + y^4)^2} + \alpha^3$$

$Eigenvalues :=$

$$0,\ 0,\ \frac{\sqrt{-\Gamma^4 + 2x^2y^2 + y^4 + x^4}}{y^2 + x^2 + \Gamma^4 - 2\Gamma^2x^2 - 2\Gamma^2y^2 + x^4 + 2x^2y^2 + y^4},\ -\frac{\sqrt{-\Gamma^4 + 2x^2y^2 + y^4 + x^4}}{y^2 + x^2 + \Gamma^4 - 2\Gamma^2x^2 - 2\Gamma^2y^2 + x^4 + 2x^2y^2 + y^4}$$

The minimum polynomial indicates that all eigen values (curvatures) are zero

```
> #eigenvectors(JAC);
> GMN:=innerprod(transpose(JAC),JAC);
  GMN2D:=array([[GMN[1,1],GMN[1,2]],[GMN[2,1],GMN[2,2]]]);DETGMN2D:=factor(det(GMN
  2D));factor(subs(r^2=x^2+y^2,DETGMN2D));
```

$$GMN := \begin{bmatrix} \frac{\Gamma^4 + 2\Gamma^2x^2 - 2\Gamma^2y^2 + y^4 + x^4 + y^2 + 2x^2y^2}{\%1^2} & \frac{y\,x\,(4\Gamma^2 - 1)}{\%1^2} & 0 & 0 \\ \frac{y\,x\,(4\Gamma^2 - 1)}{\%1^2} & \frac{\Gamma^4 - 2\Gamma^2x^2 + 2\Gamma^2y^2 + x^2 + 2x^2y^2 + x^4 + y^4}{\%1^2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\%1 := y^2 + x^2 + \Gamma^4 - 2\Gamma^2x^2 - 2\Gamma^2y^2 + x^4 + 2x^2y^2 + y^4$$

$$GMN2D := \begin{bmatrix} \frac{\Gamma^4 + 2\Gamma^2x^2 - 2\Gamma^2y^2 + y^4 + x^4 + y^2 + 2x^2y^2}{\%1^2} & \frac{y\,x\,(4\Gamma^2 - 1)}{\%1^2} \\ \frac{y\,x\,(4\Gamma^2 - 1)}{\%1^2} & \frac{\Gamma^4 - 2\Gamma^2x^2 + 2\Gamma^2y^2 + x^2 + 2x^2y^2 + x^4 + y^4}{\%1^2} \end{bmatrix}$$

$$\%1 := y^2 + x^2 + \Gamma^4 - 2\Gamma^2x^2 - 2\Gamma^2y^2 + x^4 + 2x^2y^2 + y^4$$

$$DETGMN2D := \frac{(\Gamma^2 + x^2 + y^2)^2}{(y^2 + x^2 + \Gamma^4 - 2\Gamma^2x^2 - 2\Gamma^2y^2 + x^4 + 2x^2y^2 + y^4)^3}$$

$$\frac{(\Gamma^2 + x^2 + y^2)^2}{(y^2 + x^2 + \Gamma^4 - 2\Gamma^2x^2 - 2\Gamma^2y^2 + x^4 + 2x^2y^2 + y^4)^3}$$

```
> MEAN_CURVATURE:=factor(subs(trace(JAC)));
```

$$MEAN\_CURVATURE := 0$$

```
> MEAN_CURVATURE := 0;
```

$$MEAN\_CURVATURE := 0$$

```
> S2:=factor(trace(innerprod(JAC,JAC))):
  Gauss:=factor(subs((-(1/2)*(-trace(JAC)*trace(JAC)+S2))));
```

$$Gauss := \frac{(\Gamma^2 + x^2 + y^2)(\Gamma^2 - x^2 - y^2)}{(y^2 + x^2 + \Gamma^4 - 2\Gamma^2x^2 - 2\Gamma^2y^2 + x^4 + 2x^2y^2 + y^4)^2}$$

This function  looks like a Rankine vortex inverted.  The Gauss current going to zero is the Spinodal line between different phases.

```
> ADJAC:=adjoint(JAC):

> ADJOINT_CURVATURE:=factor(subs(trace(ADJAC)));
```

$$ADJOINT\_CURVATURE := 0$$

```
>
```

```
> CurrentJ:=innerprod(ADJAC,NA);Interaction:=innerprod(CurrentJ,NA);DivJ:=factor(diverge(CurrentJ,[x,y,z,t]));rho:=CurrentJ[4];
```

$$CurrentJ := [0, 0, 0, 0]$$

$$Interaction := 0$$

$$DivJ := 0$$

$$\rho := 0$$

\>

**The charge current density vanishes identically. BUT THE TORSION vector is NOT ZERO. The Adjoint curvature is zero. The Gauss curvature is not zero. Hence the surface is not flat.**

\>

```
> Adjointcurv:=simplify(factor(trace(ADJAC))):Mean:=factor(trace(JAC)):Net:=factor(Adjointcurv-Interaction):Gnet:=factor(subs(c=1,n=1,Net));
```

$$Gnet := 0$$

Next, compute Maxwell-Farady properties for normalized Action.

```
> NB:=simplify(curl([NA[1],NA[2],NA[3]],[x,y,z]));NEP:=evalm(-grad(NA[4],[x,y,z]));NEV:=-[diff(NA[1],t),diff(NA[2],t),diff(NA[3],t)];NE:=simplify(evalm(NEV+NEP));NParity:=2*innerprod(NE,NB);NHelicity:=innerprod([NA[1],NA[2],NA[3]],NB);
```

$$NB := \left[ -\frac{y\,(\Gamma^2 + x^2 + y^2)}{\%1^{(3/2)}},\ \frac{x\,(\Gamma^2 + x^2 + y^2)}{\%1^{(3/2)}},\ -\frac{x^2 - 2\,\Gamma^2\,x^2 + y^2 + 2\,\Gamma^4 - 2\,\Gamma^2\,y^2}{\%1^{(3/2)}} \right]$$

$$\%1 := y^2 + x^2 + \Gamma^4 - 2\,\Gamma^2\,x^2 - 2\,\Gamma^2\,y^2 + x^4 + 2\,x^2\,y^2 + y^4$$

$$NEP := [0, 0, 0]$$

$$NEV := [0, 0, 0]$$

$$NE := [0, 0, 0]$$

$$NParity := 0$$

$$NHelicity := -2\,\frac{\Gamma^2}{y^2 + x^2 + \Gamma^4 - 2\,\Gamma^2\,x^2 - 2\,\Gamma^2\,y^2 + x^4 + 2\,x^2\,y^2 + y^4}$$

```
> NParity:=2*innerprod(NE,NB);Diss:=innerprod([CurrentJ[1],CurrentJ[2],CurrentJ[3]],E);
```

$$NParity := 0$$

$$Diss := 0$$

```
> NTorsion_current:=simplify(evalm(crossprod(NE,[NA[1],NA[2],NA[3]],NB)+evalm(NB*NA[4])));
```

$$NTorsion\_current := [0, 0, 0]$$

\>

\>

\>

```
> NAction2:=(NA[1]*d(x)+NA[2]*d(y)+NA[3]*d(z)+NA[4]*d(t));NFields:=wcollect(factor(d(NAction2)));NTTORSION:=wcollect(factor(NAction2&^d(NAction2)));NTPARITY:=simplify(factor(d(NAction2)&^d(NAction2)));
```

$$NAction2 := \frac{y\,d(x)}{\sqrt{\%1}} - \frac{x\,d(y)}{\sqrt{\%1}} + \frac{(\Gamma^2 - x^2 - y^2)\,d(z)}{\sqrt{\%1}}$$

$$\%1 := y^2 + x^2 + \Gamma^4 - 2\,\Gamma^2\,x^2 - 2\,\Gamma^2\,y^2 + x^4 + 2\,x^2\,y^2 + y^4$$

$$NFields := \frac{(-\Gamma^2 x - x^3 - x y^2)\,(d(x) \&\wedge d(z))}{\%1^{(3/2)}} + \frac{(-\Gamma^2 y - x^2 y - y^3)\,(d(y) \&\wedge d(z))}{\%1^{(3/2)}} + \frac{(x^2 - 2\Gamma^2 x^2 + y^2 + 2\Gamma^4 - 2\Gamma^2 y^2)\,(d(y) \&\wedge d(x))}{\%1^{(3/2)}}$$

$$\%1 := y^2 + x^2 + \Gamma^4 - 2\Gamma^2 x^2 - 2\Gamma^2 y^2 + x^4 + 2x^2 y^2 + y^4$$

$$NTTORSION := -2\frac{\Gamma^2 \&\wedge(d(x), d(y), d(z))}{y^2 + x^2 + \Gamma^4 - 2\Gamma^2 x^2 - 2\Gamma^2 y^2 + x^4 + 2x^2y^2 + y^4}$$

$$NTPARITY := 0$$

Helicity not zero, Minimal surface, Negative Gauss curvature for x^2+y^2 >Gamma^2. Positive Gauss curvature for x^2+y^2 <Gamma^2. Gauss curvature = zero at x^2+y^2 = Gamma^2. This is like the spinodal line !!!! for a Gibbs surface.

```
>
>
```

## A Hopf surface in 4D (Pfaff dimension 4) a MINIMAL surface with TWO imaginary curvatures

The 1-form is modeled after the Hopf map 1-form. In the example, constant coefficients are used to show that topological torsion and topological pairity need not be zero. By permuting variables and signs it is to be observed that there are 2 distinct pairs of triples of Hopf implicit surfaces. Three with positive parity and three with negative parity. (orientation) For more details, see http://www22.pair.com/csdc/pdf/vig2000.pdf

```
>
> A1:=a*y;A2:=-a*x;A3:=b*t;A4:=-b*z;A:=[A1,A2,A3];phi:=-A4;
```

$$A1 := a\,y$$

$$A2 := -a\,x$$

$$A3 := b\,t$$

$$A4 := -b\,z$$

$$A := [a\,y, -a\,x, b\,t]$$

$$\phi := b\,z$$

```
>
>
> lambda:=factor(subs(A[1]^2+A[2]^2+A[3]^2+phi^2))^(1/2);
```

$$\lambda := \sqrt{a^2 y^2 + a^2 x^2 + b^2 t^2 + b^2 z^2}$$

```
> Action:=A1*d(x)+A2*d(y)+A3*d(z)+A4*d(t);Fields:=d(Action);TTORSION:=Action&^d(Action);TPARITY:=d(Action)&^d(Action);
```

$$Action := a\,y\,d(x) - a\,x\,d(y) + b\,t\,d(z) - b\,z\,d(t)$$

$$Fields := 2\,a\,(d(y) \&\wedge d(x)) + 2\,b\,(d(t) \&\wedge d(z))$$

$$TTORSION := 2\,a\,y\,b \&\wedge(d(x), d(t), d(z)) - 2\,a\,x\,b \&\wedge(d(y), d(t), d(z)) + 2\,b\,t\,a \&\wedge(d(z), d(y), d(x)) - 2\,b\,z\,a \&\wedge(d(t), d(y), d(x))$$

$$TPARITY := 8\,a\,b \&\wedge(d(y), d(x), d(t), d(z))$$

```
> B:=curl([A[1],A[2],A[3]],[x,y,z]);EP:=evalm(-grad(phi,[x,y,z]));EV:=-[diff(A[1],t),diff(A[2],t),diff(A[3],t)];E:=evalm(EV+EP);Parity:=innerprod(E,B);Helicity:=innerprod(A,B);
>
```

$$B := [0, 0, -2\,a]$$
$$EP := [0, 0, -b]$$
$$EV := [0, 0, -b]$$
$$E := [0, 0, -2\,b]$$
$$Parity := 4\,a\,b$$
$$Helicity := -2\,b\,t\,a$$

The potentials produce a non-zero, but static E and B field. It is noteworthy that the two fields are PARALLEL or ANTI PARALLEL depending on the signs of A2 and A4.

```
> lambda:=factor(subs(A[1]^2+A[2]^2+A[3]^2+phi^2)^(1/2));
>
```

$$\lambda := \sqrt{a^2\,y^2 + a^2\,x^2 + b^2\,t^2 + b^2\,z^2}$$

```
> NA:=evalm([A1,A2,A3,-phi]/lambda);
```

$$NA := \left[\frac{a\,y}{\sqrt{\%1}}, -\frac{a\,x}{\sqrt{\%1}}, \frac{b\,t}{\sqrt{\%1}}, -\frac{b\,z}{\sqrt{\%1}}\right]$$
$$\%1 := a^2\,y^2 + a^2\,x^2 + b^2\,t^2 + b^2\,z^2$$

```
>
> JAC:=jacobian(NA,[x,y,z,t]);DETJAC:=det(JAC);MINP:=minpoly(JAC,alpha);eigenvalues(JAC);
```

$$JAC := \begin{bmatrix} -\frac{a^3\,y\,x}{\%1^{(3/2)}} & -\frac{a^3\,y^2}{\%1^{(3/2)}} + \frac{a}{\sqrt{\%1}} & -\frac{a\,y\,b^2\,z}{\%1^{(3/2)}} & -\frac{a\,y\,b^2\,t}{\%1^{(3/2)}} \\ \frac{a^3\,x^2}{\%1^{(3/2)}} - \frac{a}{\sqrt{\%1}} & \frac{a^3\,y\,x}{\%1^{(3/2)}} & \frac{a\,x\,b^2\,z}{\%1^{(3/2)}} & \frac{a\,x\,b^2\,t}{\%1^{(3/2)}} \\ -\frac{b\,t\,a^2\,x}{\%1^{(3/2)}} & -\frac{b\,t\,a^2\,y}{\%1^{(3/2)}} & -\frac{b^3\,t\,z}{\%1^{(3/2)}} & -\frac{b^3\,t^2}{\%1^{(3/2)}} + \frac{b}{\sqrt{\%1}} \\ \frac{b\,z\,a^2\,x}{\%1^{(3/2)}} & \frac{b\,z\,a^2\,y}{\%1^{(3/2)}} & \frac{b^3\,z^2}{\%1^{(3/2)}} - \frac{b}{\sqrt{\%1}} & \frac{b^3\,t\,z}{\%1^{(3/2)}} \end{bmatrix}$$
$$\%1 := a^2\,y^2 + a^2\,x^2 + b^2\,t^2 + b^2\,z^2$$
$$DETJAC := 0$$
$$MINP := \frac{b^2\,(x^2 + z^2 + y^2 + t^2)\,a^2\,\alpha^2}{(a^2\,y^2 + a^2\,x^2 + b^2\,t^2 + b^2\,z^2)^2} + \alpha^4$$
$$0, \frac{\sqrt{-x^2 - z^2 - y^2 - t^2}\;a\,b}{a^2\,y^2 + a^2\,x^2 + b^2\,t^2 + b^2\,z^2}, -\frac{\sqrt{-x^2 - z^2 - y^2 - t^2}\;a\,b}{a^2\,y^2 + a^2\,x^2 + b^2\,t^2 + b^2\,z^2}$$

The minimum polynomial indicates that all eigen values (curvatures) are zero or pure imaginary

```
> #eigenvectors(JAC);
> GMN:=innerprod(transpose(JAC),JAC);
```

$$GMN := \begin{bmatrix} \frac{a^2(a^2y^2+b^2t^2+b^2z^2)}{\%1^2} & -\frac{a^4yx}{\%1^2} & -\frac{a^2xb^2z}{\%1^2} & -\frac{a^2xb^2t}{\%1^2} \\ -\frac{a^4yx}{\%1^2} & \frac{a^2(a^2x^2+b^2t^2+b^2z^2)}{\%1^2} & -\frac{a^2yb^2z}{\%1^2} & -\frac{a^2yb^2t}{\%1^2} \\ -\frac{a^2xb^2z}{\%1^2} & -\frac{a^2yb^2z}{\%1^2} & \frac{b^2(a^2y^2+a^2x^2+b^2t^2)}{\%1^2} & -\frac{b^4zt}{\%1^2} \\ -\frac{a^2xb^2t}{\%1^2} & -\frac{a^2yb^2t}{\%1^2} & -\frac{b^4zt}{\%1^2} & \frac{b^2(a^2y^2+a^2x^2+b^2z^2)}{\%1^2} \end{bmatrix}$$

$$\%1 := a^2y^2+a^2x^2+b^2t^2+b^2z^2$$

```
> MEAN_CURVATURE:=factor(subs(trace(JAC)));
```

$$MEAN\_CURVATURE := 0$$

```
> S2:=factor(trace(innerprod(JAC,JAC))):
Gauss:=factor(subs((-(1/2)*(-trace(JAC)*trace(JAC)+S2))));
```

$$Gauss := \frac{a^2b^2(x^2+z^2+y^2+t^2)}{(a^2y^2+a^2x^2+b^2t^2+b^2z^2)^2}$$

```
> ADJAC:=adjoint(JAC):
```

```
> ADJOINT_CURVATURE:=factor(subs(trace(ADJAC)));
```

$$ADJOINT\_CURVATURE := 0$$

```
>
> CurrentJ:=innerprod(ADJAC,NA);Interaction:=innerprod(CurrentJ,NA);DivJ:=factor(diverge(CurrentJ,[x,y,z,t]));rho:=CurrentJ[4];Parity:=2*innerprod(E,B);Diss:=innerprod([CurrentJ[1],CurrentJ[2],CurrentJ[3]],E);
```

$$CurrentJ := \left[\frac{b^2xa^2}{\%1^2}, \frac{a^2b^2y}{\%1^2}, \frac{a^2b^2z}{\%1^2}, \frac{a^2b^2t}{\%1^2}\right]$$

$$\%1 := a^2y^2+a^2x^2+b^2t^2+b^2z^2$$

$$Interaction := 0$$

$$DivJ := 0$$

$$\rho := \frac{a^2b^2t}{(a^2y^2+a^2x^2+b^2t^2+b^2z^2)^2}$$

$$Parity := 8\,a\,b$$

$$Diss := -2\frac{a^2b^3z}{(a^2y^2+a^2x^2+b^2t^2+b^2z^2)^2}$$

```
>
```

**The charge current density is finite. THE TORSION current is NOT ZERO. The Adjoint curvature is zero.**

```
>
> Adjointcurv:=simplify(factor(trace(ADJAC))):Mean:=factor(trace(JAC)):Net:=factor(Adjointcurv-Interaction):Gnet:=factor(subs(c=1,n=1,Net));
```

$$Gnet := 0$$

Next, compute Maxwell-Farady properties for normalized Action.

```
> NB:=simplify(curl([NA[1],NA[2],NA[3]],[x,y,z]));NEP:=evalm(-grad(NA[4],[x,y,z]));NEV:=-[diff(NA[1],t),diff(NA[2],t),diff(NA[3],t)];NE:=simplify(evalm(NEV+NEP));NParity:=2*innerprod(NE,NB);NHelicity:=innerprod([NA[1],NA[2],NA[3]],NB);
```

$$NB := \left[ -\frac{a\,b\,(t\,a\,y + x\,b\,z)}{\%1^{(3/2)}}, \frac{a\,b\,(-y\,b\,z + t\,a\,x)}{\%1^{(3/2)}}, -\frac{a\,(a^2\,x^2 + a^2\,y^2 + 2\,b^2\,t^2 + 2\,b^2\,z^2)}{\%1^{(3/2)}} \right]$$

$$\%1 := a^2\,y^2 + a^2\,x^2 + b^2\,t^2 + b^2\,z^2$$

$$NEP := \left[ -\frac{b\,z\,a^2\,x}{\%1^{(3/2)}}, -\frac{b\,z\,a^2\,y}{\%1^{(3/2)}}, -\frac{b^3\,z^2}{\%1^{(3/2)}} + \frac{b}{\sqrt{\%1}} \right]$$

$$\%1 := a^2\,y^2 + a^2\,x^2 + b^2\,t^2 + b^2\,z^2$$

$$NEV := \left[ \frac{a\,y\,b^2\,t}{\%1^{(3/2)}}, -\frac{a\,x\,b^2\,t}{\%1^{(3/2)}}, \frac{b^3\,t^2}{\%1^{(3/2)}} - \frac{b}{\sqrt{\%1}} \right]$$

$$\%1 := a^2\,y^2 + a^2\,x^2 + b^2\,t^2 + b^2\,z^2$$

$$NE := \left[ -\frac{a\,b\,(-y\,b\,t + z\,a\,x)}{\%1^{(3/2)}}, -\frac{a\,b\,(x\,b\,t + z\,a\,y)}{\%1^{(3/2)}}, \frac{b^3\,(t^2 - z^2)}{\%1^{(3/2)}} \right]$$

$$\%1 := a^2\,y^2 + a^2\,x^2 + b^2\,t^2 + b^2\,z^2$$

$$NParity := -4\,\frac{a\,b^3\,(t^2 - z^2)}{(a^2\,y^2 + a^2\,x^2 + b^2\,t^2 + b^2\,z^2)^2}$$

$$NHelicity := -2\,\frac{b\,t\,a}{a^2\,y^2 + a^2\,x^2 + b^2\,t^2 + b^2\,z^2}$$

```
> NE := vector([-a*b*(-y*b*t+z*a*x)/((a^2*y^2+a^2*x^2+b^2*t^2+b^2*z^2)^(3/2)),
  -a*b*(x*b*t+z*a*y)/((a^2*y^2+a^2*x^2+b^2*t^2+b^2*z^2)^(3/2)),
  b^3*(t^2-z^2)/((a^2*y^2+a^2*x^2+b^2*t^2+b^2*z^2)^(3/2))]);
```

$$NE := \left[ -\frac{a\,b\,(-y\,b\,t + z\,a\,x)}{\%1^{(3/2)}}, -\frac{a\,b\,(x\,b\,t + z\,a\,y)}{\%1^{(3/2)}}, \frac{b^3\,(t^2 - z^2)}{\%1^{(3/2)}} \right]$$

$$\%1 := a^2\,y^2 + a^2\,x^2 + b^2\,t^2 + b^2\,z^2$$

```
> NTorsion_current:=simplify(evalm(crossprod(NE,[NA[1],NA[2],NA[3]],NB)+evalm(NB*N
  A[4])));
```

$$NTorsion\_current := \left[ 0, 0, 2\,\frac{b\,z\,a}{a^2\,y^2 + a^2\,x^2 + b^2\,t^2 + b^2\,z^2} \right]$$

```
>
>
>
> NAction2:=(NA[1]*d(x)+NA[2]*d(y)+NA[3]*d(z)+NA[4]*d(t));NFields:=wcollect(factor
  (d(NAction2)));NTTORSION:=wcollect(factor(NAction2&^d(NAction2)));NTPARITY:=simp
  lify(factor(d(NAction2)&^d(NAction2)));
```

$$NAction2 := \frac{a\,y\,d(x)}{\sqrt{\%1}} - \frac{a\,x\,d(y)}{\sqrt{\%1}} + \frac{b\,t\,d(z)}{\sqrt{\%1}} - \frac{b\,z\,d(t)}{\sqrt{\%1}}$$

$$\%1 := a^2\,y^2 + a^2\,x^2 + b^2\,t^2 + b^2\,z^2$$

$$NFields := \frac{(-a\,b^2\,y\,z + a^2\,b\,t\,x)\,(d(z)\,\&\!\wedge\,d(x))}{\%1^{(3/2)}} + \frac{(-a^2\,b\,z\,x - a\,b^2\,y\,t)\,(d(t)\,\&\!\wedge\,d(x))}{\%1^{(3/2)}}$$
$$+ \frac{(2\,a\,b^2\,z^2 + a^3\,x^2 + a^3\,y^2 + 2\,a\,b^2\,t^2)\,(d(y)\,\&\!\wedge\,d(x))}{\%1^{(3/2)}} + \frac{(a^2\,b\,t\,y + a\,b^2\,x\,z)\,(d(z)\,\&\!\wedge\,d(y))}{\%1^{(3/2)}}$$
$$+ \frac{(a\,b^2\,x\,t - a^2\,b\,z\,y)\,(d(t)\,\&\!\wedge\,d(y))}{\%1^{(3/2)}} + \frac{(2\,b\,a^2\,x^2 + b^3\,z^2 + 2\,b\,a^2\,y^2 + b^3\,t^2)\,(d(t)\,\&\!\wedge\,d(z))}{\%1^{(3/2)}}$$

$$\%1 := a^2 y^2 + a^2 x^2 + b^2 t^2 + b^2 z^2$$

$$NTTORSION := 2\,\frac{b\,t\,a\,\&\hat{}(\mathrm{d}(x), \mathrm{d}(z), \mathrm{d}(y))}{\%1} - 2\,\frac{b\,z\,a\,\&\hat{}(\mathrm{d}(x), \mathrm{d}(t), \mathrm{d}(y))}{\%1} + 2\,\frac{a\,y\,b\,\&\hat{}(\mathrm{d}(x), \mathrm{d}(t), \mathrm{d}(z))}{\%1} - 2\,\frac{a\,x\,b\,\&\hat{}(\mathrm{d}(y), \mathrm{d}(t), \mathrm{d}(z))}{\%1}$$

$$\%1 := a^2 y^2 + a^2 x^2 + b^2 t^2 + b^2 z^2$$

$$NTPARITY := -4\,\frac{a\,b\,\&\hat{}(\mathrm{d}(z), \mathrm{d}(x), \mathrm{d}(t), \mathrm{d}(y))}{a^2 y^2 + a^2 x^2 + b^2 t^2 + b^2 z^2}$$

>

Note that the Holder divisor creates an integrating factor for the topological parity of the rescaled Action and does not depend upon signature,

>
>
>

**What is remarkable about the Hopf map is that the Jacobian matrix has two zero curvatures and two imaginary equal and opposite curvatures. Yet the Gauss sectional curvaure is real and positive. The surface is an "imaginary 2 surface" in 4D. IT can have both a right handed and a left handed realization. The right and left handed concepts relate to the parity (whether E and B are parallel or anti-parallel) and to orientation of the 4 form, F^F.**

>
>

When a=b the charge density is positive but time dependent, parity is zero and the torsion current is not equal to the adjoint current. The E field is zero, but the magnetic field is not zero.
The interaction energy is zero, the odd curvatures are zero, and the gauss curvature is that of a 4D euclidean sphere.

If a = -b non of the curvatures are zero, the parity is not zero, and the charge density is negative, and the torsion current is proportional to the charge-current density which is related to an expansion - contraction.
The topologicalTOrsion and the Topological Parity are not zero

>
>
>

# A Hopf surface which is a Real Minimal Surface in 4D (Pfaff Dimension 2)

The 1-form is modeled after the Hopf map 1-form.

>

```
> A1:=((z^2+t^2))*y;A2:=-((z^2+t^2))*x;A3:=(1-(x^2+y^2))*t;A4:=-(1-(x^2+y^2))*z;
```

$$A1 := (t^2 + z^2)\,y$$

$$A2 := -(t^2 + z^2)\,x$$

$$A3 := (1 - y^2 - x^2)\,t$$

$$A4 := -(1 - y^2 - x^2)\,z$$

>

```
> A:=evalm([A1,A2,A3]);phi:=-A4;
```

$$A := [(t^2+z^2)\,y, -(t^2+z^2)\,x, (1-y^2-x^2)\,t]$$

$$\phi := (1-y^2-x^2)\,z$$

```
> A:=evalm([A1,A2,A3]);phi:=-A4;Action:=A1*d(x)+A2*d(y)+A3*d(z)+A4*d(t);Fields:=wcollect(factor(d(Action)));TTORSION:=factor(Action&^d(Action));TPARITY:=factor(d(Action)&^d(Action));
```

$$A := [(t^2+z^2)\,y, -(t^2+z^2)\,x, (1-y^2-x^2)\,t]$$

$$\phi := (1-y^2-x^2)\,z$$

$$Action := (t^2+z^2)\,y\,\mathrm{d}(x) - (t^2+z^2)\,x\,\mathrm{d}(y) + (1-y^2-x^2)\,t\,\mathrm{d}(z) - (1-y^2-x^2)\,z\,\mathrm{d}(t)$$

$$Fields := (2\,y\,z + 2\,x\,t)\,(\mathrm{d}(z)\ \&\wedge\ \mathrm{d}(x)) + (-2\,x\,z + 2\,y\,t)\,(\mathrm{d}(t)\ \&\wedge\ \mathrm{d}(x)) + (2\,t^2 + 2\,z^2)\,(\mathrm{d}(y)\ \&\wedge\ \mathrm{d}(x)) + (-2\,x\,z + 2\,y\,t)\,(\mathrm{d}(z)\ \&\wedge\ \mathrm{d}(y)) + (-2\,x\,t - 2\,y\,z)\,(\mathrm{d}(t)\ \&\wedge\ \mathrm{d}(y)) + (2 - 2\,y^2 - 2\,x^2)\,(\mathrm{d}(t)\ \&\wedge\ \mathrm{d}(z))$$

$$TTORSION := -2\,(t^2+z^2)\,(-\&\wedge(\mathrm{d}(x), \mathrm{d}(z), \mathrm{d}(y))\,t + \&\wedge(\mathrm{d}(x), \mathrm{d}(t), \mathrm{d}(y))\,z)$$

$$TPARITY := -8\,(t^2+z^2)\ \&\wedge(\mathrm{d}(z), \mathrm{d}(x), \mathrm{d}(t), \mathrm{d}(y))$$

```
>
> B:=curl([A[1],A[2],A[3]],[x,y,z]);EP:=evalm(-grad(phi,[x,y,z]));EV:=-[diff(A[1],t),diff(A[2],t),diff(A[3],t)];E:=evalm(EV+EP);Parity:=2*factor(innerprod(E,B));
>
```

$$B := [-2\,y\,t + 2\,x\,z, 2\,y\,z + 2\,x\,t, -2\,t^2 - 2\,z^2]$$

$$EP := [2\,x\,z, 2\,y\,z, -1 + y^2 + x^2]$$

$$EV := [-2\,y\,t, 2\,x\,t, -1 + y^2 + x^2]$$

$$E := [-2\,y\,t + 2\,x\,z, 2\,y\,z + 2\,x\,t, -2 + 2\,y^2 + 2\,x^2]$$

$$Parity := 8\,t^2 + 8\,z^2$$

The potentials produce a non-zero, but static E and B field. It ie noteworthy that the two fields are PARALLEL or ANTI PARALLEL depending on the signs of A3 and A4, or A1 and A2. That is A1= -y, A2 = +x has negative parity compared to the example under study.

```
> lambda:=(A[1]^2+A[2]^2+A[3]^2+phi^2)^(1/2);
```

$$\lambda := \sqrt{(t^2+z^2)^2\,y^2 + (t^2+z^2)^2\,x^2 + (1-y^2-x^2)^2\,t^2 + (1-y^2-x^2)^2\,z^2}$$

```
> NA:=evalm([A1,A2,A3,-phi]/lambda);
```

$$NA := \left[\frac{(t^2+z^2)\,y}{\sqrt{\%2}}, -\frac{(t^2+z^2)\,x}{\sqrt{\%2}}, \frac{(1-y^2-x^2)\,t}{\sqrt{\%2}}, -\frac{(1-y^2-x^2)\,z}{\sqrt{\%2}}\right]$$

$$\%1 := (1-y^2-x^2)^2$$

$$\%2 := (t^2+z^2)^2\,y^2 + (t^2+z^2)^2\,x^2 + \%1\,t^2 + \%1\,z^2$$

```
>
> JAC:=jacobian(NA,[x,y,z,t]):#DET:=factor(det(JAC));#minpoly(JAC,alpha);#eigenvalues(JAC);
```

The minimum polynomial indicates that two eigen values (curvatures) are zero and two are pure real numbers of opposite signs.

The sum of the two real curvatures gives zero (a Minimal Surface) and the Gauss curvature is negative as is expected for real sufaces.

```
> MEAN_CURVATURE:=factor(subs(trace(JAC)));
```

$$MEAN\_CURVATURE := 0$$

```
> S2:=factor(trace(innerprod(JAC,JAC))):
Gauss:=factor(subs((-(1/2)*(-trace(JAC)*trace(JAC)+S2))));
```

$$Gauss := -\frac{(-1+y^2+x^2)(x^4-x^2+x^2z^2+x^2t^2+2x^2y^2+y^2t^2-y^2+y^2z^2+y^4+z^2+t^2)}{(y^2t^2+x^2t^2+1+y^4+2x^2y^2-2y^2-2x^2+x^2z^2+y^2z^2+x^4)^2}$$

```
> ADJAC:=adjoint(JAC):
```

```
> ADJOINT_CURVATURE:=factor(subs(trace(ADJAC)));
```

$$ADJOINT\_CURVATURE := 0$$

```
>
```

```
> CurrentJ:=innerprod(ADJAC,NA);Interaction:=innerprod(CurrentJ,NA);DivJ:=factor(d
iverge(CurrentJ,[x,y,z,t]));rho:=CurrentJ[4];
```

$$CurrentJ := \left[-\frac{(-1+y^2+x^2)^2 x}{\%1^2}, -\frac{(-1+y^2+x^2)^2 y}{\%1^2}, -\frac{z(x^4+2x^2y^2+y^4-1)}{\%1^2}, -\frac{t(x^4+2x^2y^2+y^4-1)}{\%1^2}\right]$$

$$\%1 := y^2t^2+x^2t^2+1+y^4+2x^2y^2-2y^2-2x^2+x^2z^2+y^2z^2+x^4$$

$$Interaction := 0$$

$$DivJ := 0$$

$$\rho := -\frac{t(x^4+2x^2y^2+y^4-1)}{(y^2t^2+x^2t^2+1+y^4+2x^2y^2-2y^2-2x^2+x^2z^2+y^2z^2+x^4)^2}$$

```
> NB:=(curl([NA[1],NA[2],NA[3]],[x,y,z]));NEP:=evalm(-grad(NA[4],[x,y,z]));NEV:=-[
diff(NA[1],t),diff(NA[2],t),diff(NA[3],t)];NE:=simplify(evalm(NEV+NEP));NParity:
=2*innerprod(NE,NB);NHelicity:=innerprod([NA[1],NA[2],NA[3]],NB);
```

$$NB := \left[-\frac{1}{2}\frac{(1-y^2-x^2)\,t\,(2(t^2+z^2)^2y-4(1-y^2-x^2)t^2y-4(1-y^2-x^2)z^2y)}{\%2^{(3/2)}}-2\frac{t\,y}{\sqrt{\%2}}\right.$$
$$-\frac{1}{2}\frac{(t^2+z^2)\,x\,(4(t^2+z^2)y^2z+4(t^2+z^2)x^2z+2\,\%1\,z)}{\%2^{(3/2)}}+2\frac{z\,x}{\sqrt{\%2}},$$
$$-\frac{1}{2}\frac{(t^2+z^2)\,y\,(4(t^2+z^2)y^2z+4(t^2+z^2)x^2z+2\,\%1\,z)}{\%2^{(3/2)}}+2\frac{z\,y}{\sqrt{\%2}}$$
$$+\frac{1}{2}\frac{(1-y^2-x^2)\,t\,(2(t^2+z^2)^2x-4(1-y^2-x^2)t^2x-4(1-y^2-x^2)z^2x)}{\%2^{(3/2)}}+2\frac{t\,x}{\sqrt{\%2}},$$
$$\frac{1}{2}\frac{(t^2+z^2)\,x\,(2(t^2+z^2)^2x-4(1-y^2-x^2)t^2x-4(1-y^2-x^2)z^2x)}{\%2^{(3/2)}}-2\frac{t^2+z^2}{\sqrt{\%2}}$$

$$+\frac{1}{2}\frac{(t^2+z^2)\,y\,(2\,(t^2+z^2)^2\,y-4\,(1-y^2-x^2)\,t^2\,y-4\,(1-y^2-x^2)\,z^2\,y)}{\%2^{(3/2)}}\Bigg]$$

$$\%1 := (1-y^2-x^2)^2$$

$$\%2 := (t^2+z^2)^2\,y^2+(t^2+z^2)^2\,x^2+\%1\,t^2+\%1\,z^2$$

$$NEP := \Bigg[-\frac{1}{2}\frac{(1-y^2-x^2)\,z\,(2\,(t^2+z^2)^2\,x-4\,(1-y^2-x^2)\,t^2\,x-4\,(1-y^2-x^2)\,z^2\,x)}{\%2^{(3/2)}}-2\frac{z\,x}{\sqrt{\%2}},$$

$$-\frac{1}{2}\frac{(1-y^2-x^2)\,z\,(2\,(t^2+z^2)^2\,y-4\,(1-y^2-x^2)\,t^2\,y-4\,(1-y^2-x^2)\,z^2\,y)}{\%2^{(3/2)}}-2\frac{z\,y}{\sqrt{\%2}},$$

$$-\frac{1}{2}\frac{(1-y^2-x^2)\,z\,(4\,(t^2+z^2)\,y^2\,z+4\,(t^2+z^2)\,x^2\,z+2\,\%1\,z)}{\%2^{(3/2)}}+\frac{1-y^2-x^2}{\sqrt{\%2}}\Bigg]$$

$$\%1 := (1-y^2-x^2)^2$$

$$\%2 := (t^2+z^2)^2\,y^2+(t^2+z^2)^2\,x^2+\%1\,t^2+\%1\,z^2$$

$$NEV := \left[\frac{1}{2}\frac{(t^2+z^2)\,y\,\%3}{\%2^{(3/2)}}-2\frac{t\,y}{\sqrt{\%2}},-\frac{1}{2}\frac{(t^2+z^2)\,x\,\%3}{\%2^{(3/2)}}+2\frac{t\,x}{\sqrt{\%2}},\frac{1}{2}\frac{(1-y^2-x^2)\,t\,\%3}{\%2^{(3/2)}}-\frac{1-y^2-x^2}{\sqrt{\%2}}\right]$$

$$\%1 := (1-y^2-x^2)^2$$

$$\%2 := (t^2+z^2)^2\,y^2+(t^2+z^2)^2\,x^2+\%1\,t^2+\%1\,z^2$$

$$\%3 := 4\,(t^2+z^2)\,y^2\,t+4\,(t^2+z^2)\,x^2\,t+2\,\%1\,t$$

$$NE := \Bigg[-\frac{x\,z\,t^2+x^3\,z\,t^2+x\,z\,y^2\,t^2+y\,t+2\,t\,y^3\,x^2-2\,t\,x^2\,y-2\,t\,y^3+t\,y\,x^4+t\,y^5+x\,z^3+x\,z^3\,y^2+x^3\,z^3}{\sqrt{\%1}\,(y^2\,t^2+x^2\,t^2+1+y^4+2\,x^2\,y^2-2\,y^2-2\,x^2+x^2\,z^2+y^2\,z^2+x^4)},$$

$$-\frac{t^2\,y\,z+t^2\,y^3\,z+t^2\,y\,z\,x^2+2\,t\,x\,y^2-2\,t\,x^3\,y^2-x\,t-t\,x^5-t\,x\,y^4+2\,t\,x^3+y\,z^3+y^3\,z^3+y\,z^3\,x^2}{\sqrt{\%1}\,(y^2\,t^2+x^2\,t^2+1+y^4+2\,x^2\,y^2-2\,y^2-2\,x^2+x^2\,z^2+y^2\,z^2+x^4)},$$

$$-\frac{(-1+y^2+x^2)\,(2\,y^2\,t^2+2\,y^2\,z^2+2\,x^2\,t^2+2\,x^2\,z^2+1+y^4+2\,x^2\,y^2-2\,y^2-2\,x^2+x^4)\,(t^2-z^2)}{\%1^{(3/2)}}\Bigg]$$

$$\%1 := y^2\,t^4+2\,y^2\,z^2\,t^2+y^2\,z^4+x^2\,t^4+2\,x^2\,z^2\,t^2+x^2\,z^4+t^2-2\,y^2\,t^2-2\,x^2\,t^2+t^2\,y^4+2\,t^2\,x^2\,y^2+t^2\,x^4+z^2-2\,y^2\,z^2$$
$$-2\,x^2\,z^2+z^2\,y^4+2\,z^2\,x^2\,y^2+z^2\,x^4$$

$$NParity := 4\,(x^4\,t^4-x^2\,t^4+2\,y^2\,t^4\,x^2+y^4\,t^4-y^2\,t^4-t^2\,x^4+2\,x^2\,t^2-2\,t^2\,x^2\,y^2-t^2+2\,y^2\,t^2-t^2\,y^4+z^2\,x^4-x^4\,z^4$$
$$+2\,z^2\,x^2\,y^2+x^2\,z^4-2\,x^2\,z^2-2\,y^2\,z^4\,x^2-y^4\,z^4+z^2+y^2\,z^4-2\,y^2\,z^2+z^2\,y^4)\,\big/\,($$
$$(y^2\,t^2+x^2\,t^2+1+y^4+2\,x^2\,y^2-2\,y^2-2\,x^2+x^2\,z^2+y^2\,z^2+x^4)\,(y^2\,t^4+2\,y^2\,z^2\,t^2+y^2\,z^4+x^2\,t^4+2\,x^2\,z^2\,t^2+x^2\,z^4+t^2$$
$$-2\,y^2\,t^2-2\,x^2\,t^2+t^2\,y^4+2\,t^2\,x^2\,y^2+t^2\,x^4+z^2-2\,y^2\,z^2-2\,x^2\,z^2+z^2\,y^4+2\,z^2\,x^2\,y^2+z^2\,x^4))$$

$$NHelicity := -2\frac{t}{y^2\,t^2+x^2\,t^2+1+y^4+2\,x^2\,y^2-2\,y^2-2\,x^2+x^2\,z^2+y^2\,z^2+x^4}$$

The torsion current is zero, yet the adjoint current is not zero!!!! Expansion contraction seems to produce a charge current density!!!????

**What is remarkable about the Hopf map is that the Jacobian matrix has two zero curvatures and two imaginary equal and opposite curvatures. Yet the Gauss sectional curvaure is real and positive. The surface is an "imaginary 2 surface" in 4D. IT can have both a right handed and a left handed realization. The right and left handed concepts relate to the parity (whether**

**E and B are parallel or anti-parallel) and to orientation of the 4 form, F^F.**

```
>
> Adjointcurv:=simplify(factor(trace(ADJAC))):Mean:=factor(trace(JAC)):Net:=factor
  (Adjointcurv-Interaction):Gnet:=factor(subs(c=1,n=1,Net));
```

$$Gnet := 0$$

```
>
```

When a=b the charge density is positive but time dependent, parity is zero and the torsion current is zero. The E field is zero, but the magnetic field is not zero.
The interaction energy is zero, the odd curvatures are zero, and the gauss curvature is that of a 4D euclidean sphere.

If a = -b non of the curvatures are zero, the parity is not zero, and the charge density is negative, and the torsion current is proportional to the charge-current density.

```
> GMN:=innerprod(transpose(JAC),JAC);DETGMN:=det(GMN);minpoly(GMN,alpha);
Error, (in innerprod) object too large
```

$$DETGMN := 0$$

$$-\frac{a^4 b^4 (x^2+z^2+y^2+t^2)\,\alpha}{\%1^4}+\frac{a^2 b^2 (2 a^2 x^2+a^2 t^2+a^2 z^2+2 a^2 y^2+b^2 x^2+b^2 y^2+2 b^2 t^2+2 b^2 z^2)\,\alpha^2}{\%1^3}$$
$$-\frac{(a^4 y^2+a^4 x^2+2 a^2 b^2 z^2+2 a^2 b^2 t^2+2 b^2 a^2 y^2+2 b^2 a^2 x^2+b^4 z^2+b^4 t^2)\,\alpha^3}{\%1^2}+\alpha^4$$

$$\%1 := a^2 y^2+a^2 x^2+b^2 t^2+b^2 z^2$$

```
> Action2:=(A1*d(x)+A2*d(y)+A3*d(z)+A4*d(t))/lambda;Fields:=wcollect(factor(d(Acti
  on2)));TTORSION:=factor(Action2&^d(Action2));TPARITY:=factor(d(Action2)&^d(Actio
  n2));
```

The induced metric is singular and is rank 2, not 3.

**YET THE principal CURVATURES ARE IMAGINARY IN ONE EXAMPLE AND REAL IN THE OTHER.**

**IS THERE A SET OF INiTIAL CONDITIONS SUCH THAT A FLOW IN THE DIRECTION OF AN EXPANSION OR CONTRACTION STARTS IN THE STATE OF REAL GAUSS CURVATURE WITH IMAGINARY BENDING CURVATURES, AND WINDS UP AS NEGATIVE GAUSS CURVATURE AND REAL CURVATURES.**
**WILL THIS EXPLAIN THE CREATION OF FALACO SOLITONS?? with visual negative gauss curvature and presumed to me a minimal surface, FROM THE RANKINE VORTEX, which is visually associated with a positive Gauss curvature,.**

```
>
>
```