NOTES

ON MINIMAL SURFACES AND PHYSICS

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Introduction

Besides the obvious application of minimal surface theory to the study of soap films, there are a number of other physical systems in which the theory of minimal surfaces has a sometimes surprising applicability. To this author, a rekindling of an interest in minimal surfaces came from the (recent in 1991) realization that the topological patterns associated with hydrodynamic instabilities and wakes can be related to minimal surfaces. Minimal surfaces can be interpreted as limit sets which are the results of a (perhaps irreversible) process decaying into a stationary state. A number of theories and hopefully interesting observations are presented below, utilizing the ideas of minimal surfaces in 4 dimensions.

Minimal Surfaces and Hydrodynamic Wakes

Recently, it has been established that the persistent scroll-like features of hydrodynamic wakes can be captured by a *special* subset of surfaces of tangential discontinuities [1]. These scroll patterns are epitomized experimentally by the Rayleigh-Taylor instability and the Kelvin-Helmoltz instability. To this author is was extraordinary that these *special* surfaces can be related to harmonic minimal surfaces, which, although locally unstable, are similar to soap films and enjoy a domain of global stability. The theory not only captures the observable features of hydrodynamic wakes, but also provides a process for the creation of these minimal surfaces of tangential discontinuities. The remarkable feature is that this process does not depend upon viscous friction explicitly. All that is needed is a domain of hyperbolicity for the partial differential equations that describe the flow. The simplest way to achieve these conditions is to admit of (slightly) compressible flow around a sharp corner, a condition always associated with real fluid flows.

As the minimal surfaces of hydrodynamic interest are defined in terms of those domains where the vector field is harmonic, the viscous contributions to the Navier-Stokes equations disappear on these sets! The flow is not dissipative and not diffusive on the minimal surface of tangential discontinuities, a fact the agrees with the long lived persistence and sharp definition of experimentally observed wakes. These wake patterns are known to be precursors [2] of the turbulent state, which will occur when the minimal surface loses its global stability.

After this observation, a reading of the excellent book by J.C.C. Nitsche [3] on minimal surfaces indicated that the foundations of these hydrodynamic ideas may go back to Weingarten [4]. Weingarten evidently had an interest in studying those special surfaces of constant pressure in a hydrodynamic flow which could simultaneously be described by either a potential function, or a set of stream lines. In two dimensions, this idea degenerates into those cases where a flow is described by either a potential function, or a stream function. The dual representation is a constraint that leads to Cauchy's conditions for complex analytic functions. However, Weingarten's ideas about such special surfaces were developed for three dimensional flows, not two. His results demonstrate that such special surfaces are indeed *minimal* surfaces. Without the power of the PC, the fact that these special surfaces are related to the complex spiral patterns [1] observed in hydrodynamic wakes was not available to Weingarten.

The basic notion of Weingarten has other extraordinary applications. The idea can be formulated in terms of a potential (or "energy") function, $H(p_x, p_x, p_x)$ whose partial derivatives,

$$V^x = \partial H/\partial p_x , \qquad V^y = \partial H/\partial p_y , \qquad V^z = \partial H/\partial p_z ,$$

can be combined by means of a Legendre transformation to produce the function:

$$L(V^x, V^y, V^z) = \mathbf{V} \bullet \mathbf{P} - H(p_x, p_x, p_x) = (\text{grad}_{\mathbf{P}} H) \bullet \mathbf{P} - H .$$

Now consider the algebraic variety given by the non-linear equation in terms of the coordinates given by the vector, **V**.

$$L(V^x, V^y, V^z) = 0. \quad \mathbf{P} = (\mathrm{grad}_{\mathbf{V}}\, L)$$

Such a result implies that the function H is homogeneous of degree 1 in the quantities (p_x , p_x , p_x). It follows that the vector field defined as **V** (p)= (grad $_{\mathbf{P}}$ H) is homogeneous of degree 0,

$$[\partial^2 H / \partial p_\mu \partial p_\nu]\, p_\nu = 0.$$

Example: if $H = (\mathbf{P}\bullet\mathbf{P})^{1/2}$ then $\mathbf{V} = (\mathrm{grad}_{\mathbf{P}}\, H) = \mathbf{P}/(\mathbf{P}\bullet\mathbf{P})^{1/2}$ and $\mathbf{V}\bullet\mathbf{V}=1$

When the Hessian determinant of H vanishes, then the functions given by $\{V^x, V^y, V^z\} = (\mathrm{grad}_{\mathbf{P}}\, H)$, may be considered as coordinates of a point on a surface, L= 0, in the space of variables $\{V^x, V^y, V^z\}$. This surface is always a minimal surface, and L satisfies the Beltrami equation for the surface given by the set $L(V^x, V^y, V^z) = 0$. As $L(V^x, V^y, V^z) = 0$ is a minimal surface, the function L is harmonic. The only non-linear real harmonic minimal surfaces are the helicoids [5], and it is these sets of minimal surfaces that lead to the spiral wakes of reference [1].

Minimal Surfaces and Pfaff Dimension 3

For Navier-Stokes flows, the vorticity must be integrable:

$$\omega \bullet \mathbf{curl}\, \omega = 0.$$

Hence, there exists a representation in the form

$$\omega = \lambda(x,y,z,t)\, \mathrm{grad}\, \phi(x,y,z,t).$$

From another point of view there exists a map from {x,y,z,t} to {α,β} such that

$$\omega = \mathrm{grad}(\alpha) \times \mathrm{grad}(\beta) .$$

Consider the first case. As ω = curl **v**, the the spatial divergence of ω must vanish. Therefore,

$$\mathrm{grad}(\lambda) \bullet \mathrm{grad}(\phi) + \lambda \nabla^2\phi = 0.$$

But if φ(x,y,z,t) = 0 defines a minimal surface, then

$$\mathrm{grad}(n) \bullet \mathrm{grad}(\phi) - n \nabla^2\phi = 0 \quad \text{where } n = \{\nabla\phi \bullet \nabla\phi\}^{1/2}$$

It follows that

$$\mathrm{grad}(\phi) \bullet \nabla n\lambda = 0.$$

Either nλ is a constant, or the gradient of nλ resides on the minimal surface; that is nλ = nλ(α,β), the coordinates on the surface. Note that the zero div condition could also be used for the velocity field, but then, the assumption is one of incompressibility. The zero div condition must always be true for the vorticity case, and does not imply incompressibility!!!

The result is that curl $\mathbf{v} = \{f(\alpha,\beta)/ n(x,y,z,t)\}\nabla\phi(x,y,z,t)$
with grad f orthogonal to grad φ, the normal to the minimal surface.

There are certain questions that arise?

What is the map between λ,ϕ and α,β ?

Can the problem of Navier-Stokes be reduced to a time-dependent flow for the vorticity in two spatial dimensions (α,β) and time.??

Minimal Surfaces and Thermodynamics

The notation used in the preceeding paragraph was deliberate, for most students of physics immediately recognize the correspondence to Hamilton's principle in optics. Perhaps not so obvious, but certainly recognized in terms of light of Caratheodory's strong interest in the calculus of variations, the same minimal surface ideas apply to thermodynamics. Changing notation,

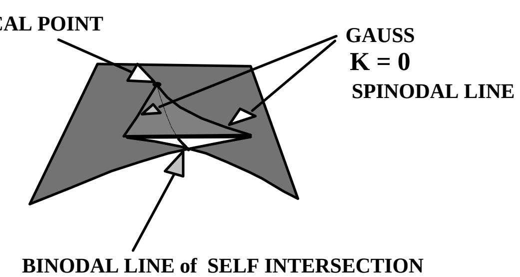
$L = - U, \; H = - G(T,P)$,

$\mathbf{V} = \{ S , V \} = (\mathrm{grad}_{\mathbf{P}}\, G) = - (\mathrm{grad}_{\mathbf{P}}\, U)$,

$\mathbf{P} = \{ T, P \} = - (\mathrm{grad}_{\mathbf{V}}\, U)$ leads to the Gibbs free energy relationship,

$$G(T, P) = - ST - VP + U(S, V) .$$

The objective is to find a surface function $\Phi = 0$ with a vanishing Hessian determinant. That surface is a minimal surface, and is Gibb's equilibrium surface. Of special interest are those non-linear analytic surface functions which are at least cubic (the third derivative does not vanish). On a space of three variables, such a function always can be generated by the Hamilton-Cayley function relating the three eigenvalues of the functional dyadic, or Jacobian determinant of Φ. The Gibb's surface for the Van der Waal's gas is a classic example, for it demonstrates that mininal surfaces can be stable or unstable. The Gibbs surface is a minimal equilibrium surface which for the Van der Waals gas has self-intersections and a critical point. The Gibbs minimal surface is not an embedding in 3-dimensions. An example of a dynamical system displaying these properties was given in reference [6].



GIBBS MINIMAL SURFACE H = 0

Is the proper relation

$$L(N,P,T) = (NU+PV+TS) - H(U,V,S) \approx N(g - f(P,T))$$

$$\partial H/\partial U = N,\ \partial H/\partial V = P,\ \partial H/\partial U = N$$
$$\partial L \partial N = U,\ \partial L/\partial P = V,\ \partial L/\partial T = S.$$

Then the components of the position vector, N, P, and T determine a minimal surface.

Minimal Surfaces and Fractals

The theory of hydrodynamic wakes in terms of minimal surfaces is motivated by an extraordinary theorem in minimal surface theory, which states that the set of 4-dimensional vector fields which belong to the equivalence class of complex holomorphic curves will generate minimal surfaces in 4-dimensions [3]. The position vector describing this minimal surface can be generated from the Weierstrass representation in terms of the holomorphic function, $R(w) = R(\sigma + i\tau)$:

$$\mathbf{V}(\sigma + i\tau) = \begin{array}{l} V^x = \mathrm{Re}\int (1 - w^2) R(w) dw \\ V^y = \mathrm{Re}\int i(1 + w^2) R(w) dw \\ V^z = \mathrm{Re}\int 2wR(w) dw \end{array}$$

The notation used is in terms of a vector V in velocity space in deference to the hydrodynamic applications.

As a first example consider the 4-dimensional vector field written in terms of complex variables as:

$$\mathbf{V_n} = \{ z , u_n(z) \}$$

where u_n is the nth functional iterate of the analytic function $u_1 = \lambda z(1-z)$. Each functional iterate produces another holomorphic function, and therefore another minimal surface, but with more and more singularities developed from the zero sets of the functional iterates of u_n. In the limit, a Julia set is determined and it follows that this set must act as a boundary of a minimal surface of some connected domain. For a given λ, say $\lambda = 1.64+ i\ 0.96$, the values of z for which the sequence of functional maps extends to infinity is associated with a minimal surface with a fractal boundary. This fractal boundary is the famous "fractal dragon" that appears on the cover of Madelbrot's book [4].

As a second example, consider

$$\mathbf{V_n} = \{ z , u_n(z) \} = \{ z, z^2 - \mu\}; \{ z, (z^2-\mu)^2 - \mu \}; ....$$

According to the minimal surface theorem, this vector field represents a one (complex) parameter family of minimal surfaces in 4-dimensions. It follows that the Mandelbrot set, which is given by the values of μ for which the function u(z) fails to iterate the origin (z = 0) to infinity [4], is the fractal envelope of a family of minimal surfaces in 4-dimensions parameterized by $\mu = a + ib$. The compliment to the Madelbrot set is a minimal surface with a fractal boundary where all functional sequences iterate to infinity. Hence the minimal surface is complete.

As another example, consider those functions of a complex variable such that $R(w) = \partial^3F(w)/\partial w^3$. All functions F(w) that have the form

$$F(w) = A\ w \ln w + B + Cw + Dw^2$$

$$= A\ w \ln w + B + (C+D)w + Dw(w-1)$$

generate the same function $R(w) = \partial^3F(w)/\partial w^3 = -A/w^2$. Rewriting R(w) in the form

$$R(w) = (b-ia)/2w^2 \text{ , with } w = -i \exp(\eta+i\xi)$$

and substituiting into the Weirstrauss formulas yields the position vector to a family of minimal surfaces of the form

$$x = a \sinh\eta \cos\xi - b \cosh\eta \sin \xi$$
$$y = a \sinh\eta \sin \xi - b \cosh\eta \cos\xi$$
$$z = a\eta + b\xi + c$$

For a = 0 the surface is a catenoid; for b = 0 the surface is a helicoid. (Nitshce p.70). The interesting features are

1. All wave functions are related to a minimal surface by this technique.
2. The primitive function F(w) is related to the Helmholtz free energy, and it is the entropy term Aw ln w that generates the family of minimal surfaces.
3. The resulting minimal surface is independent of the Mandelbrot term, Dw(w-1)

Minimal Surfaces and the Onset of Turbulence

Consider a space-variety {x,y,z,t} and a real 1-form of action, $A = A_\mu dx^\mu$, defined on this domain (no metric has yet to be assigned). Construct the real 2-form F = dA by exterior differentiation. The resultant 6-vector has space-space components, **B** and space-time components **E**. The 4-form $F\wedge F = 2\, \mathbf{E}\bullet\mathbf{B}\ dx\wedge dy\wedge dz\wedge dt$. Define the complex 3 vector as

$$\mathbf{M} = (\varepsilon)^{1/2} (\mathbf{E} + i\, c(x,y,z,t)\, \mathbf{B})$$

where the arbitrary functions, ε(x,y,z,t) and c(x,y,z,t) are to be chosen such that $\varepsilon\{\mathbf{E}\bullet\mathbf{E} - c^2\, \mathbf{B}\bullet\mathbf{B}\} = 0$. Then the complex square of **M** is equal to

$$\mathbf{M} \bullet \mathbf{M} = \varepsilon\{\mathbf{E}\bullet\mathbf{E} - c^2\, \mathbf{B}\bullet\mathbf{B}\} + i\ 2\, (\varepsilon/\mu)^{1/2}\, \mathbf{E}\bullet\mathbf{B} = 0 + i\, (\varepsilon/\mu)^{1/2} *(F\wedge F),$$

and

$$\mathbf{M}^{*}\bullet \mathbf{M} = \varepsilon\{\mathbf{E}\bullet\mathbf{E} + c^2\, \mathbf{B}\bullet\mathbf{B}\}.$$

Every student of physics recognizes that $\varepsilon\{\mathbf{E}\bullet\mathbf{E} - c^2\, \mathbf{B}\bullet\mathbf{B}\} \approx \{\mathbf{D}\bullet\mathbf{E} - \mathbf{B}\bullet\mathbf{H}\}$ is the first Poincare invariant (or Lagrangian of the Field), and that $2\, (\varepsilon/\mu)^{1/2}\, \mathbf{E}\bullet\mathbf{B}$ is the second Poincare invariant of a Lorentz transformation. Moreover, the coefficient, $(\varepsilon/\mu)^{1/2}$, is the radiation impedance, which for vacuum is equal to 377 ohms. The function, $\varepsilon\{\mathbf{E}\bullet\mathbf{E} + c^2\, \mathbf{B}\bullet\mathbf{B}\} \approx \{\mathbf{D}\bullet\mathbf{E} + \mathbf{B}\bullet\mathbf{H}\}$ is usually interpreted as (twice) the energy density of the electromagnetic field (in situations where $\varepsilon\mu c^2 = 1$).

When $\varepsilon\{\mathbf{E}\bullet\mathbf{E} - c^2\, \mathbf{B}\bullet\mathbf{B}\}= 0$, the energy density in the electric and magnetic fields are equal. When the second Poincare invariant vanishes, then the **E** vector is orthogonal to the **B** vector. Such a situation is the standard case for electromagnetic waves propagating with the characteristic speed c. However, the entire analysis applies to any 1-form of action, A, whether it describes an electromagnetic system or a fluid.

According to Osserman, if there exists a map

$$(\alpha,\beta) \to \varepsilon\mathbf{E}(\alpha,\beta) \qquad \text{and}$$
$$(\alpha,\beta) \to \varepsilon c\mathbf{B}(\alpha,\beta),$$

such that **M** is holomorphic in (α + iβ), then there exists harmonic coordinates, x^k, such that

$$\varepsilon\mathbf{E} = \partial x^k/\partial\alpha \quad \text{and} \quad \varepsilon c\mathbf{B} = -\partial x^k/\partial\beta .$$

These coordinates, x^k, define a *minimal surface, subject to the constraint* $\mathbf{M} \bullet \mathbf{M} = 0$. If $\mathbf{M}^{*}\bullet \mathbf{M} \neq 0$ then the minimal surface is regular. In regions where the electromagnetic energy density is not finite, the associated minimal surface will have singularities. Note that it is necessary (for c ≠ 0) that the 1-form A must satisfy the condition, $F\wedge F = dA \wedge dA = 0$, otherwise $\mathbf{M} \bullet \mathbf{M} \neq 0$. That is, the 1-form, A, must be defined in a domain of Pfaff dimension less than 4, if a minimal surface is to exist. Turning the argument around, if $F\wedge F = 0$, then find a characteristic speed function, c(x,y,z,t), such that $\mathbf{M} \bullet \mathbf{M} = 0$, and then note the existence of a minimal surface. The minimal surface need not be regular; it can have self intersections and isolated singularities.

This remarkable result implies that as long as $F\hat{}F = 0$, a characteristic speed, c, can be chosen such that the dynamical system modeled by the Action, A, can be associated with a *minimal surface.* The dynamical system then, like soap films, can be stabilized globally, even though it may have local instabilities. However, when

$$F\hat{}F = 2\mathbf{E} \bullet \mathbf{B}\, dx\hat{}dy\hat{}dz\hat{}dt \neq 0,$$

the possibility of global stability via a minimal surface is lost, and the minimal surface (or persistent hydrodynamic wake) is no longer possible. In the hydrodynamic situation, the flow with a minimal surface wake becomes turbulent in domains where $F\hat{}F \neq 0$. At such points the minimal surface is destroyed; the vector field **M** is no longer analytic. Full turbulence is to be associated with domains of Pfaff Dimension 4, a result derived from another argument in reference [5]. Chaotic fractal domains can be associated with minimal surfaces, and therefore these domains for real fields, although chaotic, are not of Pfaff dimension 4.

For a dynamical system constrained by the Navier-Stokes equations [5], the Pfaff dimension 4 requirement for full turbulence implies that the vorticity vector, curl **V**, does not satisfy the conditions of Frobenius integrability in the turbulent state. The 4-form $F\hat{}F$ can be computed for the Navier-Stokes fluid to be equal to the expression,

$$F\hat{}F = 2\,\nu\, \mathrm{curl}\, \mathbf{V} \bullet \mathrm{curl\, curl}\, \mathbf{V}\, dx\hat{}dy\hat{}dz\hat{}dt\,.$$

When $F\hat{}F = 0$ a minimal surface representation can be found, which implies the existence of a minimal surface (and perhaps an associated characteristic speed function c) in those domains where the vorticity vector can be represented in terms of (at most) 2 scalar functions -- the Pfaff requirement for complete integrability of curl **V** in N dimensions. In the turbulent state, the flow is without a persistent (minimal surface) wake, and $F\hat{}F \neq 0$. It takes more that 2 functions to describe the vorticity field in such domains, when the evolution is constrained by the Navier-Stokes equations.

An early attempt by the present author (1967) to understand the transition to turbulence used the idea that the domain of laminar flow was to be associated with integrable dynamical systems, and, as turbulence was the antithesis of the turbulent state, the domain of turbulence must be associated with non-integrable systems. The transition to turbulence must involve a dynamical topological evolution from a domain of $A\hat{}dA \neq 0$ to $A\hat{}dA = 0$. This idea was sharpened about 1987 by the recognition that a deterministic chaotic state could be reversible, and hence on physical arguments, not turbulent. To get irreversibility it is necessary that the evolutionary orbits must intersect, an argument that leads to the idea the Euler characteristic of space-time must not be zero. As the integral of $F\hat{}F$ determines the Euler characteristic, then if $F\hat{}F = 0$ its integral must be zero, and the Euler characteristic of such space times must be zero. It follow that the transition to turbulence proceeds from a laminar state, to a chaotic state, to a turbulent state following the sequence of Pfaff dimensions $2 \rightarrow 3 \rightarrow 4$. The minimal surfaces are of Pfaff dimension less than 4, and represent a generalized conservation law: $d(A\hat{}dA) = \mathrm{div}\, \mathbf{T} + \partial h/\partial t = 0$. The transition to turbulence is a transition from a connected topology to a disconnected topology [5].

Mininal Surfaces and the States of Matter

A rather remarkable feature of Julia sets is that they are either connected "repellors" and form the boundary of an attracting finite point, or they are disconnected "repellors" and have the properties of a "Fatou dust". In the interior of the connected or "filled-in" attractive domain, there are other connected sets that have periodic (or smooth) boundaries. All of these sets are related to minimal surfaces, through the theorem that a holomorphic curve generates a minimal surface in 4-dimensions. There appears to be three classifications: two classes of sets within the connected domain, and one class of sets in the exterior, of the boundary formed by the Julia set. In the connected domain, certain minimal surfaces have a periodic smooth boundary under successive iteration of the holomorphic function, where other minimal surfaces that have a Julia (not-smooth) set as a boundary. When the parametric values exceed critical values, the Julia sets form disconnected sets (the Fatou dust).

With these observations in mind, hypothecate on physical grounds that thermodynamic "equilibrium" systems are represented by either integrable or minimal surfaces in space-time. Then it would follow that there are three basic classes of matter:

Gases 1. Disconnected Fatou dusts
Liquids 2. Connected domains with a fractal Julia boundary
Solids 3. Connected domains with periodic interiors

The idea that an equilibrium surface is an integrable minimal surface goes back to Gibbs and Caratheodory. The simplest thermodynamic equilibrium surfaces are "two dimensional" -hence integrable - systems for which there exist neighboring points that are not reachable from the equilibrium foliation. The Gibbs free energy is a minimum for the equilibrium state. Such systems are represented by a 1-form which is integrable in the sense of Frobenious ($A\wedge dA = 0$). It would appear that this idea of equilibrium can be generalized to include cases of Pfaff dimension 3, but not Pfaff dimension 4. The criteria $dA\wedge dA = 0$ must be satisfied for a minimal surface, but the surface need not be integrable.

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