

Linking Numbers and Braid Integrals

vs.

Torsion_Helicity and Spin Integrals

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Abstract: Cartan's methods of exterior differential forms are used to construct the integrals of Links, Braids, Torsion_Helicity and Spin, and to demonstrate their topological properties as deformation invariants.

0. Introduction

In both hydrodynamics and electromagnetism, there is interest in the possibility that divergence-free direction field lines (usually frozen in lines of vorticity or magnetic field) are linked or knotted. The evolution or the creation of such a topological state is an unsolved problem. The concept of a direction field in the Cartan calculus is represented by a N-1 form on a space of N dimensions. If the N-1 form is closed (implying that the direction field is divergence free) then the closed integrals of such N-1 forms are deformation invariants (hence represent topological properties) of all evolutionary processes that can be represented by a single parameter semi-group.

Definition1: The N-volume element on a variety $\{x,y,z..\}$ is

$$\Omega_N = dx^{\wedge}dy^{\wedge}dz^{\wedge}...$$

Definition2: Any direction field, \mathbf{X} , generates an N-1 form, D :

$$D = i(\mathbf{X})\Omega_N = \{\mathbf{X}^x dy^{\wedge}dz^{\wedge}... - \mathbf{X}^y dx^{\wedge}dz^{\wedge}... + \mathbf{X}^z dx^{\wedge}dy^{\wedge}...\}$$

The exterior derivative of D produces the N-form $\rho(x,y,z...) \Omega_N$ with the measure function $\rho(x,y,z...)$ equal to the divergence of the direction field,

$$\rho(x,y,z...) = \sum_{k=1}^N (\partial \mathbf{X}^k / \partial x^k)$$

Lemma3: Any N-1 form, D , is an evolutionary deformation invariant of the flow, \mathbf{V} , iff $L_{(\beta\mathbf{V})}\mathbf{A} = 0$, for arbitrary reparametrization functions, $\beta(x,y,x...)$. See P. Liebermann

Theorem: If the direction field has zero divergence, then the closed integrals of D_c are deformation invariants for *any* evolutionary process that can be described by a C2 (flow)

vector field, $\beta(x, y, z..) \mathbf{V}$, on the N dimensional variety

Proof:

$$\begin{aligned} L_{(\beta \mathbf{V})} \int_{N-1 \text{ closed}} \cdots \int D_c &= \int_{N-1 \text{ closed}} \cdots \int i(\beta \mathbf{V}) dD_c + \int_{N-1 \text{ closed}} \cdots \int d(i(\beta \mathbf{V}) D_c) \\ &= \int_{N-1 \text{ closed}} \cdots \int 0 + \int_{N-1 \text{ closed}} \cdots \int d(i(\beta \mathbf{V}) D_c) = 0 + 0. \end{aligned}$$

The first integral vanishes as $dD_c = 0$, (the zero divergence condition). The second integral vanishes as the integral of an exact form, $i(\beta \mathbf{V}) D_c$, over a closed chain is zero. In other words, D_c is a relative integral invariant for any parametrization β of the *arbitrary* evolutionary direction field, or flow, generated by \mathbf{V} . If the closed integration chain has a boundary, then from Stokes theorem, the integral is an evolutionary invariant if $i(\beta \mathbf{V}) D_c = 0$ on the $N - 2$ dimensional boundary. However this is a special case.

The work below will demonstrate how Torsion, Helicity, Links and Braid integrals all stem from these basic ideas about certain 2-forms and 3-forms on a 4 dimensional variety.

1. The Gauss Integrals (2-forms)

The basic issue is that not all divergence free fields are exact. It is true that all divergence free fields in a 3 dimensional *euclidean topology* are exact, but that is precisely where the topological features enter into the picture. A euclidean topology is simply connected and without obstructions.

In 3-dimensions (and with euclidean dogma) there are still two species of 3 component fields. Even in euclidean domains every one learns that the 3 vector of angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ is never to be added to the 3 vector of momentum, \mathbf{p} . They are "two different species" of vectors. However, with regard to non-euclidean topological domains there is also another concept defined as a vector density. There is a topological difference even in 3D between a covariant tensor and a contravariant tensor density, but not detectable if volume deforming processes are excluded (the typical non-dissipative case). To demonstrate these ideas, consider a C1 map onto a euclidean domain:

$$\phi : \{\xi^1, \xi^2, \xi^3\} \Rightarrow \{x, y, z\} = \phi^j(\xi^k)$$

with a Jacobian matrix defined by the equations

$$d\phi : \{d\xi^1, d\xi^2, d\xi^3\} \Rightarrow \{dx, dy, dz\} = [\mathbf{J}] \circ |d\xi\rangle = [\partial\phi^j/\partial\xi^k] |d\xi^k\rangle$$

Construct a 3 component vector field \mathbf{Z} on $\{x, y, z\}$

$$\mathbf{Z} = [U(x, y, z), V(x, y, z), W(x, y, z)]$$

Now ask what is the pre-image of \mathbf{Z} on $\{\xi^1, \xi^2, \xi^3\}$. The functional form of the pre-image is not unique! There are two tensor fields whose functional forms are well defined by functional substitution; one is a contravariant tensor density, and the other is a covariant

tensor, on $\{\xi^1, \xi^2, \xi^3\}$. The two pre-images can be represented by the formulas,

$$\text{Contra - variant tensor density } |\mathbf{D}^k\rangle_{pb} = [\partial\phi^j/\partial\xi^k]^{adjoint} \circ |\mathbf{Z}\rangle,$$

and

$$\text{Co - variant tensor } |\mathbf{A}_k\rangle_{pb} = [\partial\phi^j/\partial\xi^k]^{transpose} \circ |\mathbf{Z}\rangle,$$

where *pb* means pullback from $\{x, y, z\}$ to $\{\xi^1, \xi^2, \xi^3\}$ by means of functional substitution (without the use of an inverse map, which may not exist globally). Note that the arguments made above are still valid even if the initial domain is of higher dimension than the final domain (say $\{\xi^1, \xi^2, \xi^3, \xi^4\} \Rightarrow \{x, y, z\}$ is a projection onto). The pre-images are well defined in a functional sense even if the topology of the initial state is different from the topology of the final state. The map ϕ need not have an inverse!

The notation above is deliberate, for it distinguishes the electromagnetic Intensities, \mathbf{E}, \mathbf{B} , (as components of a covariant tensor deduced from \mathbf{A}_k) from the electromagnetic Quantities (or excitations) \mathbf{D}, \mathbf{H} (as components of a contravariant tensor density).

The assumption of a euclidean domain masks these topological features. The topological closure of $|\mathbf{D}\rangle$ is the concept of zero divergence; the topological closure of $|\mathbf{A}\rangle$ is a zero curl concept. In 3D, for C^2 differentiable fields where $|\mathbf{B}\rangle = \text{curl } |\mathbf{A}\rangle$, it follows that the closure of the 2-form generated from the components of $|\mathbf{B}\rangle$ is always empty, in a global manner! The closure of $|\mathbf{D}\rangle$ is not globally empty!.

However it is possible to define integrating factors, λ , for *any* $|\mathbf{Z}\rangle$ such that

$$|\mathbf{D}_c^k\rangle = [\partial\phi^j/\partial\xi^k]^{adjoint} \circ |\mathbf{Z}/\lambda\rangle$$

is closed, *but not necessarily in a global manner*. That is, the two form with components \mathbf{D}_c is closed but not necessarily exact :

$$d(D_c) = d(D_c^z dy^x dz - D_c^y dx^x dz + D_c^x dx^x dy)/\lambda = (\text{div } \mathbf{D}_c^k) d\xi^1 \wedge d\xi^2 \wedge d\xi^3 = 0$$

Any homogeneous function of the Holder type and of degree n (with any anisotropic signature defined by constants a, b, c , and any exponent p) will act as integrating divisor for any 3D vector \mathbf{Z} when $n=3$. (The theorem is also valid for a ND vector in N space when $n=N$)

Theorem: If $\lambda(U, V, W) = \{aU^p + bV^p + cW^p\}^{3/p}$ then $\text{div}(\mathbf{Z}/\lambda) = 0$

Proof: Use the volume element $\Omega_N = dU \wedge dV \wedge dW$ and the vector field \mathbf{Z} to construct 2-form $\Gamma = i(\mathbf{Z}/\lambda)\Omega_N = \{UdV \wedge dW - VdU \wedge dW + WdU \wedge dV\}/\lambda(U, V, W)$ on N . Define $\lambda(U, V, W) = \{aU^p + bV^p + cW^p\}^{n/p}$. By direct computation, the "divergence" of Γ with respect to the coordinates $\{U, V, W\}$ on N is

$$d\Gamma = \{(3 - n)/\lambda\}\Omega_N$$

Hence, $d\Gamma = 0$ for $n = 3$, any p , and any signature for the anisotropic constants $\{a, b, c\}$.

In other words, the Holder type divisor, λ , acts as an integrating factor for the vector field, when $n=3$, any p , any a, b, c . The 2-form Γ on N has a well defined pullback G on M (a preimage based upon functional substitution :

$$G = \phi^*\Gamma.$$

or

$$G = i(\mathbf{D})dx^{\wedge}dy^{\wedge}dz \Leftarrow i(\mathbf{Z}/\lambda_N)dU^{\wedge}dV^{\wedge}dW = \Gamma$$

Moreover, if $d\Gamma = 0$ on N , then $dG = 0$ on M . (for proof, see Flanders, or Paulette Liebermann).

On a space of three dimensions there are 2-forms that are exact, and 2-forms that are exact. Although the 2-form with covariant components $\langle \mathbf{B} \rangle$ constructed from the curl of a vector potential \mathbf{A} is closed and exact, the 2-form with tensor density components $\langle \mathbf{D}_c \rangle$ is closed, BUT NOT NECESSARILY EXACT. The fundamental idea is that for a non-bounding closed cycle (*nbcc*) (such as a closed twisted ribbon),

$$\iint_{nbcc} \mathbf{B} \circ d(\text{Area}) = 0, \quad \text{but} \quad \iint_{nbcc} \mathbf{D} \circ d(\text{Area}) \neq 0$$

where for a boundary (such as toroidal surface)

$$\iint_{boundary} \mathbf{B} \circ d(\text{Area}) = 0, \quad \text{and} \quad \iint_{boundary} \mathbf{D} \circ d(\text{Area}) = 0.$$

If the integration chain is a closed in the sense of cycle, and is not a boundary, then there must exist points of the integration domain which must be excluded. These points form the topological defects (the point charges in EM theory or "topological holes") or the topological obstructions that are of interest to the theory of Links and Braids. In particular, the theory of links depends upon such obstructions and is represented by integrals of the form:

$$Lk = \iint_{nbcc} (D^z dy^{\wedge} dz - D^y dx^{\wedge} dz + D^x dx^{\wedge} dy) / \lambda = \iint_{nbcc} G \neq 0, \quad dG = 0.$$

and should have nothing to do with magnetic flux,

$$\Phi_m = \iint_{nbcc} (B_x dy^{\wedge} dz - B_y dx^{\wedge} dz + B_z dx^{\wedge} dy) = \iint_{nbcc} F = 0, \quad dF = 0,$$

which has no obstructions, as the integrand is globally exact. If F was to have obstructions, the pre-images global postulate of potentials $F - dA = 0$ must fail, and the conservation of flux would not be true. Such a failure implies the existence of magnetic monopoles (the obstructions to F being globally exact). My personal view (along with E.J.Post and many others) subsumes that the failure to detect magnetic monopoles is proof that classical electromagnetism is defined by the postulate of potentials; i.e., $F - dA = 0$, globally. On the other hand, the 2-form of field excitations, G , is not exact.

Example 1. The Gauss Link Integral

As an example of the Gauss integral, Lk, consider the case where the displacement vector is the difference of two position vectors to two separate space curves. Define

$$\mathbf{Z} = (\mathbf{R}_2 - \mathbf{R}_1) \quad \mathbf{R}_2 = [x_2, y_2, z_2] \quad \mathbf{R}_1 = [x_1, y_1, z_1]$$

$$\lambda = (a(x_2 - x_1)^p + b(y_2 - y_1)^p + c(z_2 - z_1)^p)^{3/p}$$

where \mathbf{R}_1 defines the position vector to one field of space curves, and \mathbf{R}_2 defines the position vector to a second field of space curves. Space curves from different families can have different parametrizations. Hence, the vector \mathbf{Z} represents the vector difference of points on two different space curves which cannot be synchronized parametrically. Next assume that the displacements of interest are constrained by two parametric curves given by the exterior differential system

$$d\mathbf{R}_1 - \mathbf{V}_1 dt = 0 \quad \text{and} \quad d\mathbf{R}_2 - \mathbf{V}_2 dt' = 0,$$

where the parameters dt and dt' are not functionally related, such that

$$dt \wedge dt' \neq 0, \quad \text{but} \quad dt \wedge dt = 0 \quad \text{and} \quad dt' \wedge dt' = 0.$$

The vector \mathbf{D} can be interpreted as the displacement vector between points on the space curve C1 parametrized by t , and the points on another space curve C2 parametrized by t' . The integral to be evaluated is

$$\begin{aligned} Lk &= \iint_{\text{closed}} \Gamma = \iint_{\text{closed}} i(\mathbf{Z}/\lambda) d(x_2 - x_1) \wedge d(y_2 - y_1) \wedge d(z_2 - z_1) \\ &= \iint_{\text{closed}} (1/\lambda) (V_{21} - V_{11}) \wedge (V_{22} dt' - V_{12} dt) \wedge (V_{23} dt' - V_{13} dt) + \dots \\ &= \iint_{\text{closed}} (1/\lambda) (V_{21} - V_{11}) \wedge (V_{22} V_{13} - V_{12} V_{23}) dt \wedge dt' + \dots \end{aligned}$$

using $dt \wedge dt' \neq 0$, but $dt \wedge dt = 0$ and $dt' \wedge dt' = 0$. Rewriting the formula using the isotropic Gauss format, $a=b=c=1, p=2$ leads to the classic Gauss Linkage formula,

$$Lk = \iint_{\text{closed}} G = \oint_t \oint_{t'} \{(\mathbf{R}_2 - \mathbf{R}_1) \circ \mathbf{V}_1 \times \mathbf{V}_2\} dt \wedge dt' / \lambda$$

$$\lambda = (\mathbf{R}_1 \circ \mathbf{R}_1 - 2\mathbf{R}_1 \circ \mathbf{R}_2 + \mathbf{R}_2 \circ \mathbf{R}_2)^{3/2}.$$

However, the formula works for the anisotropic, any p , case.

From Stokes theorem, if the closed 2 dimensional integration domain is a boundary of a 3 dimensional domain, then the Link integral vanishes. However, if a particular integration chain is a closed cycle (not a boundary of a 3 dimensional domain) then the linking integral has values with rational ratios. These closed integrals are deRham period integrals in two dimensions. Points where \mathbf{D} vanishes are excluded.

When the two curves are distinct, the integration is over the two bounding cycles of a

closed ribbon. The ribbon surface is closed but it is not a boundary of any volume. Then the two non-intersecting cycles (that form the boundary of the ribbon area) are defined by the two distinct parameters, dt , and dt' . When integrations are computed along these closed curves whose tangent vectors are \mathbf{V}_1 and \mathbf{V}_2 , then the integer values of the closed integral may be interpreted as how many times the two curves are linked. The interpretation of the closed surface integral as a orientable ribbon works if the triple product divided by lambda does not change sign as t and t' are varied. If the integrand changes sign, then the ribbon is non-orientable.

The constraint that $dt \wedge dt' \neq 0$ implies that the "motion" along the curve generated by \mathbf{R}_1 is independent of the "motion" along the curve generated by \mathbf{R}_2 . If the curve generated by \mathbf{R}_1 is a conic in the xy plane and the curve generated by \mathbf{R}_2 is a conic in the xz plane, then the surface swept out by the vector \mathbf{D} is a Dupin cyclide. Such surfaces have application to the propagation of waves in electromagnetic systems.

Example 2: Flat tangential developables

From another point of view, consider the ruled surface defined by the vector field of two parameters, $\{t, \mu\}$ (isotropic, $a=b=c=1$, $p=2$). The ruled surface will be defined by the position vector $\mathbf{R}(t)$ to a space curve and a ruling parameter μ times the tangent Velocity vector to the space curve, $\mathbf{V}(t)$.

Use the general methods above to create the doubly parametrized divergence free vector field:

$$\mathbf{Z}(\mu, t) = \{ \mathbf{R}(t) \pm \mu \mathbf{V}(t) \}$$

$$\lambda(\mu, t) = (\mathbf{R} \circ \mathbf{R} \pm 2\mu \mathbf{R} \circ \mathbf{V} + \mu \mathbf{V} \circ \mu \mathbf{V})^{3/2}.$$

Vector fields of this type are primitive examples of "strings" for fixed values of the parameter, t , and string parameter, μ . Direct substitution of the physical constraints, $d\mathbf{R} - \mathbf{V}dt = 0$, and $d\mathbf{V} - \mathbf{A}dt = 0$, such that $d\mathbf{Z} = d\{ \mathbf{R}(t) \pm \mu \mathbf{V}(t) \}$ into the definition of the linking integral

$$\iint_{\text{closed on } N} i(\mathbf{Z}/\lambda) dZ^1 \wedge dZ^2 \wedge dZ^3$$

leads to yet another realization and interpretation of the Gauss formula:

$$Q = \iint_{\text{closed}} G = \iint_{\text{closed on } \mu t} \{ \mathbf{R} \circ \mu \mathbf{V} \times \mathbf{A} \} dt \wedge d\mu / \lambda$$

$$= \iint_{\text{closed}} \{ \mathbf{A} \circ \mathbf{R} \times \mu \mathbf{V} \} dt \wedge d\mu / (\mathbf{R} \circ \mathbf{R} \pm 2\mu \mathbf{R} \circ \mathbf{V} + \mu \mathbf{V} \circ \mu \mathbf{V})^{3/2}.$$

It is apparent that the interaction of the "angular" momentum, $\mathbf{L} = \mathbf{R} \times \mu \mathbf{V}$, and the acceleration, \mathbf{A} , produces a topological invariant whose values are "quantized" (in the sense that the ratios of the closed integrals are rational). Note that the triple vector product of the

integrand numerator is proportional to the Frenet torsion of the orbit. For an orbit that is planar the Frenet torsion is zero everywhere, and the Gauss integral vanishes.

Recall that if a the space curve is an edge of regression, then the ruled surfaces associated with to the forward and backward motions (the \pm signs in the formula) are not same to second order. Such a result demonstrates an obvious distinction between forward and backward motion that breaks time reversal symmetry. Linear rulings in one direction are on 1 sheet of the ruled surface, and rulings in the opposite direction are on the other surface. The two surfaces meet at an edge of regression. Similar time reversal symmetry breaking effects have been observed macroscopically in dual polarized ring lasers,

Example 3. Scrolls

The two parameter surface described above is closely related to the ruled surface known as the tangential developable. Such ruled surfaces (parametrized by arc length s rather than time, t , and with the directrix of the ruling in the direction of the unit tangent vector, and multiplied by μ) have zero Gauss curvature. Though bent, such surfaces can be rolled out flat. By constructing the ruled surfaces in terms of the normal and/or binormal to a space curve, other forms of ruled surfaces yield negative values for the Gauss curvature of the surface, and are not "flat". They are defined as Scrolls.

Of particular interest to physics are those ruled surfaces of negative Gauss curvature, which are also minimal surfaces. They have application in describing hydrodynamic wakes. These surfaces can be viewed as double edged ribbons for given values of μ . The equations for the ruled surface of a scroll, with a directrix in the direction of the binormal, $\mathbf{b}(s)$, are :

$$\mathbf{D}(\mu, s) = \{\mathbf{R}(s) \pm \mu \mathbf{b}(s)\} / \lambda$$

$$\lambda = (\mathbf{R} \circ \mathbf{R} \pm 2\mu \mathbf{R} \circ \mathbf{b} + \mu \mathbf{b} \circ \mu \mathbf{b})^{3/2}.$$

When the parameter μ takes on the constant values $\mu = \kappa/\tau^3$, (with κ = the Frenet curvature, and τ = the torsion of the space curve) then the ruled surface is a minimal surface, and the binormal field twists about the space curve generated by $\mathbf{R}(s)$.

Another interesting scroll is that generated by the Darboux vector.

$$\mathbf{D}(\theta, s) = \{\mathbf{R}(s) \pm (\mathbf{n}(s) \cos(\theta) + \mathbf{b}(s) \sin(\theta))\} / \lambda$$

$$\lambda = (\mathbf{D} \circ \mathbf{D})^{3/2}.$$

which seems to be of interest to Longcope.

2. Braids, Spin and Torsion_Helicity (3-forms)

Braids

For $n = 4$ the same procedures described above may be used to produce a period integral over a closed 3-dimensional domain. The technique is to define a 4 dimensional vector field,

$\mathbf{Z} = [Z_1, Z_2, Z_3, Z_4]$. Use the general renormalization function,

$$\lambda = \{\alpha Z_1^p + \beta Z_2^p + \gamma Z_3^p + \epsilon Z_4^p\}^{n/p}$$

and set $n=4$, for zero four divergence. Construct the closed 3 -form,

$$\Gamma = i(\mathbf{Z}/\lambda) dZ_1 \wedge dZ_2 \wedge dZ_3 \wedge dZ_4$$

Assume the 4 component vector has a realization as $\mathbf{Z} = \mathbf{P}_1 + \mathbf{P}_2 + \mathbf{P}_3$, where the three independent fields \mathbf{P} represent three space-time curves that obey the kinematic constraints:

$$d\mathbf{P}_1 - \mathbf{f}_1 ds = 0, \quad d\mathbf{P}_2 - \mathbf{f}_2 ds' = 0, \quad d\mathbf{P}_3 - \mathbf{f}_3 ds'' = 0.$$

Substitute for each of the differentials in Γ (and further assume that the domain $\{x,y,z,t\}$ of interest is further constrained such that $dt = 0$) to yield the three form

$$G = \{\mathbf{f}_1 \circ (\mathbf{f}_2 \times \mathbf{f}_3)\} ds \wedge ds' \wedge ds'' / \lambda$$

$$\lambda = \{\alpha Z_1^p + \beta Z_2^p + \gamma Z_3^p + \epsilon Z_4^p\}^{4/p}$$

The spatial braid integral becomes equal to

$$Br := \oint_t \oint_{t'} \oint_{t''} \mathbf{f}_1 \circ (\mathbf{f}_2 \times \mathbf{f}_3) ds \wedge ds' \wedge ds'' / \lambda$$

The integrations are now over three closed curves whose tangents are the "Newtonian forces", \mathbf{f} , on three "particles". Where in the two dimensional Gauss integral, of the previous section, the evaluation was along the closed curves of a ribbon, in this case the integrations are along the closed trajectories of three "particles" which form the vertices of a triangle.

The idea that three "lines" are used to form the integral (whose values form rational ratios) is the reason that this topological integral in the format given above is defined as the braid integral. Of course the three form of topological torsion is a variant of the braid integral, but applies to those topologies where the system is not reducible to three factors dt, dt' and dt'' . An example of a period 3 braid with Braid integral zero (chaotic) and Braid integral 2 (non-chaotic) is given in Figure 5. It is illuminating to construct the two braids by wrapping a long flat ribbon of paper smoothly around the palm of your hand. Close the ribbon surface by pasting the ends together. Then make another example, where this time thread the loose end underneath the middle wrap, rather than over the middle wrap, before gluing the ends together. Take the two examples from your hand and note that one is continuously deformable into a closed cylinder ($Tw = 0$) while the other has a 4 pi twist ($Tw = 2$). What is surprising is that it is the $Tw = 0$ configuration that has a chaotic neighborhood, while the $Tw = 2$ structure is not chaotic. To test for chaos construct the equivalent of the closed braid form copper tubes. Then link any pair of tubes with a large loop of elastic or thread. Push the looping thread around the period three copper tube, and note that for a $Tw=2$ configuration, the looping thread becomes untangled after 6 pi revolutions about the central axis. For the $Tw = 0$ configuration, the looping thread never

unwinds, but becomes more and more twisted and complex.

The equivalent to Figure 5, and the fact that there are two distinct period 3 configurations one chaotic and one non-chaotic, was brought to my attention during a stimulating lecture given by J. Los at the August, 1991, Pedagogical Workshop on Topological Fluid Mechanics held at the Institute for Theoretical Physics, Santa Barbara UCSB.

It is to be noted that the 3-form of topological torsion is related to the braid integral, a three dimensional thing in four dimensions, and not the Gauss linkage integral, which is a two dimensional thing in three dimensions.

(I am sure you are familiar with the Figure below, although I don't remember if you mentioned the chaotic distinctions in your Physics Today article.)

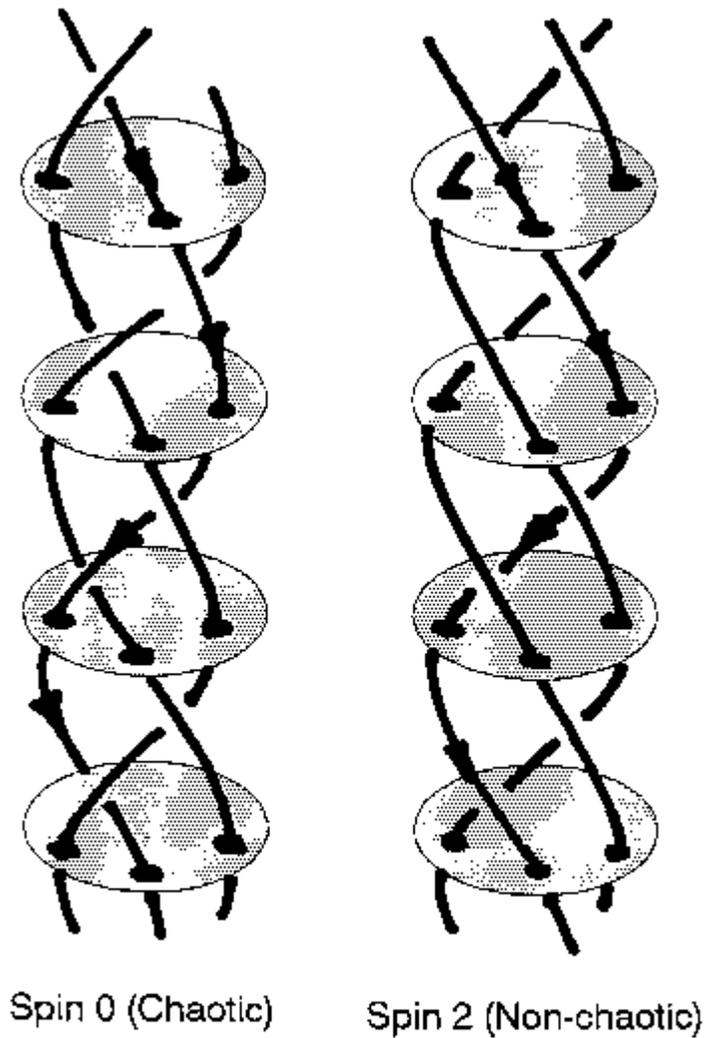


Figure 5. Period 3 Braids

Magnetic Helicity (a 3-form)

First, the definition of Magnetic Helicity with which I am comfortable is that given by the 3D (volume) integral of the vector potential dotted with its curl:

Definition 1:

$$Magnetic\ Helicity := \iiint \mathbf{A} \circ curl\mathbf{A} dVol = \iiint \mathbf{A} \circ \mathbf{B} dVol = \iiint A^{\wedge}F = \iiint A^{\wedge}dA$$

For me this object is a well defined object on any variety $\{x,y,z\}$. On every 3 dimensional *closed* manifold for which $\mathbf{A} = \mathbf{A}(x,y,z)$, it is easy to show that the *closed* integrals of

Magnetic Helicity density, $\mathbf{A} \circ \mathbf{B}$, are deformation invariants of **any** continuous evolutionary process that can be described by a singly parametrized vector direction (or flow) field, $\mathbf{V}(x,y,z)$. In the language of Poincare-Cartan, the closed integral is a relative integral invariant. All that is required is that the points which make up the integration chain in the initial state remain on the same flow fiber in the final state.

This topological conservation law has nothing to do with electromagnetism, per se, but for classical definitions of the Magnetic Field intensity, such as $\mathbf{B} = \text{curl } \mathbf{A}$, it works for electromagnetism. It is also true for fluids or any other system that can be described by a 1-form of Action, A , and can be extrapolated to higher dimensions. The proof is easy:

$$\begin{aligned} L_{(\beta\mathbf{V})} \iiint_{\text{closed}} \mathbf{A} \circ \mathbf{B} dVol &= \iiint_{\text{closed}} \{d(i(\beta\mathbf{V})A^F) + i(\beta\mathbf{V})d(A^F)\} = \\ &\iiint_{\text{closed}} \{d(i(\beta\mathbf{V})A^F) + i(\beta\mathbf{V})F^F\} \\ &= 0 + 0 \supset \text{evolutionary invariance for any } \beta\mathbf{V} \end{aligned}$$

The first integral vanishes because the integral of an exact form over a closed integration chain vanishes. The second integral vanishes as $F^F = 0$ on a 3 dimensional manifold; all 3-dimensional volume elements are closed. As $\beta(x,y,z)$ is arbitrary, it acts as a possible deformation parameter. The result is true for any gauge, does not depend upon metric, and is independent of any geometrical connection, and certainly does not depend upon a constitutive constraint. A true topological quantity. Note that the closed 3-dimensional integration domain can be a cycle and does not have to be a boundary (of a higher dimensional space). It is important to realize (for application to domains that do not have a euclidean topology) that the 3-form A^F has components that transform as a covariant tensor of rank 3.

Perhaps it is more important that the theorem of Topological deformation invariance of the closed integrals of the 3-form is also valid in **any** dimension for which F^F is zero. In higher dimensions the integral must be evaluated over 3-sub_manifolds that need not be space-like. For 4 dimensions the 3-form A^F has 4 components and can be constructed as

$$\text{Topological Torsion: } A^F = i(\mathbf{T}_4)dx^dy^dz^dt$$

where

$$\mathbf{T}_4 = [\mathbf{E} \times \mathbf{A} + \phi\mathbf{B}, \mathbf{A} \circ \mathbf{B}]$$

From this formulation it must be remembered that the components of this "vector" transform as a third rank covariant tensor field; $\mathbf{A} \circ \mathbf{B}$ is merely the fourth component. In the literature such objects are often described as pseudo vectors.

The criteria $F^F = 0$ for topological invariance with respect to any continuous evolutionary process is equivalent to the statement that $\mathbf{E} \circ \mathbf{B} = 0$. The condition is sufficient, but not necessary, for topological invariance of the Helicity integral, even when

the fields are explicitly time dependent. The relative integral invariant in 4D is:

$$H = \iiint_{closed} A^{\wedge} F = \iiint_{closed} \mathbf{T}^x dy^{\wedge} dz^{\wedge} dt - \mathbf{T}^y dx^{\wedge} dz^{\wedge} dt + \mathbf{T}^z dx^{\wedge} dy^{\wedge} dt - \mathbf{A} \circ \mathbf{B} dx^{\wedge} dy^{\wedge} dz$$

In all cases the 3 divergence of \mathbf{B} vanishes, and the 2-form F is closed, for it is exact, $dF = 0$. For isochronous domains, $dt = 0$, and the integral reduces to the standard spatial format of plasma physics. However, there are example electromagnetic fields for which $\mathbf{A} \circ \mathbf{B} = 0$, and yet \mathbf{T}_4 is not zero.

Example: Define $\mathbf{A} = [0, 0, (x^2 + y^2)/2]$, $\phi = z$. Then

$$\mathbf{B} = [-y, x, 0]$$

$$\mathbf{E} = [0, 0, -1]$$

$$\mathbf{A} \circ \mathbf{B} = 0, \quad \mathbf{E} \circ \mathbf{B} = 0,$$

$$\mathbf{T}_4 = z\mathbf{B}$$

Note that in 4 dimensions (that admit time dependent fields), the frozen in lines are the line of the Torsion vector, which can be dominated by the \mathbf{B} field but also have a component due to $\mathbf{E} \times \mathbf{A}$. I suspect that measurements are made on the lines of \mathbf{T}_4 , and only indirectly on the "lines" of " \mathbf{B} " (for remember the electromagnetic field has six components which can transform into one another). The implication is that the lines of \mathbf{T}_4 can continuously evolve, with most regions in a plasma dominated by $\phi\mathbf{B} \neq 0$, $\mathbf{E} \times \mathbf{A} \rightarrow 0$. Then as \mathbf{B} lines approach one another, induction causes $\mathbf{E} \times \mathbf{A} \neq 0$, $\phi\mathbf{B} \rightarrow 0$. The " \mathbf{B} lines" terminate on a null (a boundary point), break apart, and then possibly reconnect different segments, after which $\phi\mathbf{B} \neq 0$, $\mathbf{E} \times \mathbf{A} \rightarrow 0$. (I believe this mechanism is that which Hornig is developing).

I call $A^{\wedge} F$ the topological torsion of the field, and $d(A^{\wedge} F) = F^{\wedge} F$, the topological parity.

Special situations become evident when the integration domain is compact with a boundary, for then Stokes law may be applied, and deformation invariance requires that $i(\beta\mathbf{V})A^{\wedge} F = 0$ on the boundary. These are interesting but special cases which are invariants of only a special choice of boundary conditions. For example, if $\beta(x, y, z) = 0$ defines the boundary, then for deformation invariance of the integral it must be true that the function β also must be an evolutionary invariant, such that $L_{(\mathbf{V})}d\beta = i(\mathbf{V})d\beta = 0$. Classically the function β which is used to define the boundary, is not arbitrary, but must be a first integral of the evolutionary vector field. For such special cases, the field on the boundary need not be tangential! There are other special situations as well.

For integration domains which are open, the criteria for absolute integral invariance is much more severe, and requires that $d(i(\beta\mathbf{V})A^{\wedge} F) = 0$. This constraint is to be recognized as the criteria that the evolutionary vector field $\beta\mathbf{V}$ be an element of the symplectic group. I have demonstrated that all such evolutionary processes are thermodynamically reversible.

Spin (a 3-form)

To relate the above definition of a 3-form of Helicity with the six dimensional formulation involving the Biot-Savart substitution (to me) is an extraordinary constraint on the topology of the domain. The substitution effectively mixes a tensor and a tensor density, where definition 1 above mixes a tensor with a tensor, A^F . There is however, another well defined electromagnetic 3-form, A^G , which mixes a tensor (A) and a tensor density (G). I call A^G –with physical dimensions of angular-momentum– the Spin 3-form. (The 3-form A^F has physical dimensions of Angular momentum divided by Ohms.)

In electrical engineering notation,

$$\text{Spin} : \mathbf{S}_4 = [\mathbf{A} \times \mathbf{H} + \phi \mathbf{D}, \mathbf{A} \circ \mathbf{D}]$$

$$\text{Torsion_Helicity} : \mathbf{T}_4 = [\mathbf{E} \times \mathbf{A} + \phi \mathbf{B}, \mathbf{A} \circ \mathbf{B}]$$

The closure of A^G defines a measure known as the first Poincare invariant,

$$P1 := (\mathbf{B} \circ \mathbf{H} - \mathbf{D} \circ \mathbf{E}) - (\mathbf{A} \circ \mathbf{J} - \rho \phi) \equiv d(A^G)$$

while the closure of A^F yields the second Poincare invariant.

$$P2 := 2(\mathbf{E} \circ \mathbf{B}) \equiv d(A^F).$$

When these measures vanish (the divergences of the 4-vectors vanish), then there exist separate topological conservation laws (of Spin and Helicity).

It is extraordinary to me that the Solar - Plasma community does not seem to make use this other topological property which is conserved when the first Poincare measure is zero. The Spin 3-form is of equal importance as the 3-form of Torsion-Helicity, yet no one in the Plasma community seems to use it.

For the above example, with the added assumption that $\mathbf{D} = \epsilon \mathbf{E}$ and $\mathbf{B} = \mu \mathbf{H}$

$$\mathbf{A} = [0, 0, (x^2 + y^2)/2], \quad \phi = z. \text{ Then}$$

$$\mathbf{B} = [y, -x, 0]$$

$$\mathbf{E} = [0, 0, -1]$$

$$\mathbf{A} \circ \mathbf{B} = 0,$$

$$\mathbf{T}_4 = [zy, -zx, 0]$$

$$P2 = \mathbf{E} \circ \mathbf{B} = 0,$$

$$\mathbf{A} \circ \mathbf{D} = -\epsilon(x^2 + y^2)/2$$

$$\mathbf{S}_4 = [x(x^2 + y^2)/2, y(x^2 + y^2)/2, -\epsilon\mu z, -\epsilon\mu(x^2 + y^2)/2]/\mu$$

$$P1 = (2(x^2 + y^2) - \epsilon\mu)/\mu$$

$$\mathbf{J} = [0, 0, -2/\mu]$$

$$\rho = 0$$

which implies that Helicity is conserved but Spin is not conserved in the example field.

For more details about Torsion vs. Spin see

<http://www.uh.edu/~rkiehn/pdf/classice.pdf>

You might also be interested in something that I did after the Chapman conference stimulated me. I think I have found a *raison d'etre* for the formation of an accretion disk in a otherwise central field problem. Using Maple, I have found a time dependent solution to Maxwell's equations that makes the $z=0$ plane of a rotating plasma a chiral attractor. The force of attraction from the top and the bottom is due to a Lorentz $\mathbf{J} \times \mathbf{B}$ term. This the first time I have ever seen a simple mechanism that might explain the formation of accretion disks around stars and planets. The Solution is just a model field, but exhibits both Helicity and Spin effects.

see

<http://www.uh.edu/~rkiehn/pdf/diracch.pdf>

REGARDS

RMK