Linking Numbers and Braid Integrals vs. Torsion_Helicity and Spin Integrals

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Abstract

Cartan's methods of exterior differential forms are used to construct the integrals of Links, Braids, Torsion_Helicity and Spin, and to demonstrate their topological properties as deformation invariants.

1. Introduction

Closed tensor fields In both hydrodynamics and electromagnetism, there is interest in the possibility that divergence-free direction field lines (usually frozen in lines of vorticity or magnetic field) are linked or knotted. The evolution, or the creation, of such a topological state is an unsolved problem. The concept of a direction field in the Cartan calculus is represented by a N-1 form on a space of N dimensions. If the N-1 form is closed (implying that the direction field is divergence free) then the closed integrals of such N-1 forms are deformation invariants (hence represent topological properties) of all evolutionary processes that can be represented by a single parameter semi-group. If the N-1 form is closed, then either the lines that represent the direction field begin and terminate on boundary points of the domain, or are cyclic and close upon themselves. A divergence free direction field never stops or starts in the topological interior. The lines which stop and start on a boundary are of two types: those that stop and start on the same boundary component, and those the stop and start on a different boundary components. The fundamental idea starts with the concept of divergence free vector fields. However, the arguments of deformation of closed integrals extends to p-forms which are closed.

Herein, the discussion will be at first restricted to divergence free 3 dimensional vector fields, and then the concept of closed p-forms in 4 dimensions will be discussed. In particular a method for constructing certain classes of divergence free vector fields, and p-forms will be developed. These fields will form the basis of certain Linking and Braid integrals.

Divergence Free Vector fields in 3 dimensions

Definition 1.1. *The 3-volume element on a variety* {x,y,z} *is given by the n-form,*

$$\Omega_3 = dx\hat{}dy\hat{}dz \tag{1.1}$$

Definition 1.2. *Any direction field,* $\mathbf{X}$, *of contravariant components, generates a 2 form,* D :

$$D = i(\mathbf{X})\Omega_3 = \{\mathbf{X}^x dy\hat{}dz - \mathbf{X}^y dx\hat{}dz + \mathbf{X}^z dx\hat{}dy\} \tag{1.2}$$

The exterior derivative of D produces the N-form $\rho(x, y, z...)\Omega_N$ with the measure function $\rho(x, y, z...)$ equal to the divergence of the direction field,

$$\rho(x, y, z...) = \sum_{k=1}^{3} (\partial\mathbf{X}^k/\partial x^k) \tag{1.3}$$

Lemma 1.3. *Any 2 form,* D, *is an evolutionary deformation invariant of the flow,* $\mathbf{V}$, *iff* $L_{(\beta\mathbf{V})}D = 0$, *for arbitrary reparametrization functions,* $\beta(x, y, x...)$. *See P. Liebermann*

Definition 1.4. D_c *is defined as the set of N-1-forms which are closed.* $dD_c =$ 0.

Theorem 1.5. *As the direction field* D_c *has zero divergence, then the closed integrals of* D_c *are deformation invariants for any evolutionary process that can be described by a C2 (flow) vector field,* $\beta(x, y, z..)\mathbf{V}$, *on the N dimensional variety*

Proof:

$$\begin{aligned} L_{(\beta\mathbf{V})} \iint_{closed} D_c &= \iint_{closed} i(\beta\mathbf{V})dD_c + \iint_{closed} d(i(\beta\mathbf{V})D_c) \\ &= \iint_{closed} 0 + \iint_{closed} d(i(\beta\mathbf{V})D_c) = 0 + 0. \end{aligned} \tag{1.4}$$

The first integral vanishes as $dD_c = 0$, (the zero divergence condition). The second integral vanishes as the integral of an exact form, $d(i(\beta\mathbf{V})D_c)$, over a closed chain is zero. In other words, D_c is a relative integral invariant for any parametrization β of the *arbitrary* evolutionary direction field, or flow, generated by $\mathbf{V}$. The *relative* condition is that the 2-dimensional integration chain be closed (like the surface of a torus).

The result for closed integration chains is to be compared with the formulas for open integration chains (where Stokes's formula is used to compute the exact part in terms of a boundary integral):

$$\begin{aligned} L_{(\beta\mathbf{V})} \iint_{open} D_c &= \iint_{open} i(\beta\mathbf{V})dD_c + \iint_{open} d(i(\beta\mathbf{V})D_c) \\ &= \iint_{open} 0 + \int_{boundary} (i(\beta\mathbf{V})D_c). \end{aligned} \tag{1.5}$$

For integration domains that are open (like a disc) then deformation invariance of the open integral requires that $i(\beta\mathbf{V})D_c \Rightarrow 0$ on the one dimensional boundary. However this is a special (but useful) case that is sensitive to boundary conditions. (All of the preceeding development is easily extended to N dimensions.

Closed p-forms in N dimensions. The arguements presented above were built on the concept of a divergence free vector field. However, the concept of a divergence free vector field is a specialization of a more general concept, the concept of a closed differential form. What will be shown below is that given a vector field consisting of p independent functions, it is possible to produce a p-1 form G on a space of arbitrary dimension $M > p$, along with integrating divisors (functions), λ, such that $d(G/\lambda) = 0$. The integrating factors will be defined as "Holder norms".

For example, on a space of 4 dimensions, two independent differentiable functions may be used to construct a closed (but not exact) 1-form of 4 components. Three independent functions may be used to construct a closed but not exact

2-form of six components. Four independent functions may be used to construct a closed but not exact 3-form of 4 components. The closed but not exact 1-form has components that behave as a covariant vector field on 4 dimensions. The four components of the closed 3-form behave like a contravariant vector field.

Define an ordered independent set of n functions by the vectorial symbol,

$$\mathbf{Z} = [U, V, W...] \tag{1.6}$$

Each component function of $\mathbf{Z}$ is a function of several independent variables, say $\{x, y, z, t...\}$. From the Vector construct the n dimensional form

$$\Omega(\mathbf{Z}) = dU\hat{}dV\hat{}dW.... \tag{1.7}$$

Next construct the $p = n - 1$ form, Γ, defined as

$$\Gamma = i(\mathbf{Z})dU\hat{}dV\hat{}dW... = UdV\hat{}dW... - VdU\hat{}dW... + WdU\hat{}dV.... \tag{1.8}$$

Consider the function, λ,

$$\lambda = (aU^m + bV^m + cW^m + ...)^{n/m} \tag{1.9}$$

with arbitrary constants $a, b, c, ...$, any integer m, and with the integer n equal to the number of independent components of Z. Then it is possible to prove that the rescaled p-form Γ/λ is closed:

$$d(\Gamma/\lambda) = 0. \tag{1.10}$$

By functional substitution using the differentiable map

$$\phi := \{x, y, z, t...\} \Rightarrow [U(x, y, z, t...), V(x, y, z, t...), W(x, y, z, t...)....] \tag{1.11}$$

it is possible to pull back each of the closed p-forms so constructed to the domain of variables $x, y, z, t....$

The closed integrals of these closed p-forms define the deRham "period integrals". If the closed integration domains do not enclosed the zeros of λ, then the value of the integral is zero. Otherwise the values of the integrals have rational ratios, depending on the domain of integration. For example, consider the closed 1-form constructed from the 2 functions $U(x, y, z)$ and $V(x, y, z)$. The closed 1-form is

$$\Gamma = \{UdV - VdU\}/\{U^2 + V^2), \quad d\Gamma = 0 \tag{1.12}$$

However, the integral of Γ, on a closed integration chain that encircles the zero set of U and V, has a value equal to 2π. Consider the map

$$\phi : U(x, y, z) = \sqrt{x^2 + y^2 - 1}, \quad V(x, y, z) = z \tag{1.13}$$

with

$$d\phi : dU(x, y, z) = \{xdx + ydy)/\sqrt{x^2 + y^2 - 1}, \quad dV(x, y, z) = dz \tag{1.14}$$

and

$$\lambda = (U^2 + V^2)^{2/2} \tag{1.15}$$

to yield the pullback 1-form on x,y,z:

$$\phi * \Gamma = ((x^2 + y^2 - 1)dz - zxdx - zydy)/\{\sqrt{x^2 + y^2 - 1}(x^2 + y^2 + z^2 - 1)\}. \tag{1.16}$$

This 1-form is closed, implying that the vector field on {x,y,z} with components

$$\mathbf{A} = [-zx, -zy, (x^2 + y^2 - 1)]/\{\sqrt{x^2 + y^2 - 1}(x^2 + y^2 + z^2 - 1)\}. \tag{1.17}$$

has zero curl. An integration contour in x, y, z that encloses the origin in U, V space, links the circle of radius =1 in the $z = 0$ plane, and the integral has a finite non-zero circulation for this close path of integration.

The work below will demonstrate how Torsion, Helicity, Links and Braid integrals all stem from these basic ideas about certain 2-forms and 3-forms on a 4 dimensional variety.

2. The Gauss Integrals (2-forms)

The basic issue is that not all divergence free fields (differential forms) are exact. It is true that all divergence free fields in a 3 dimensional *euclidean topology* are exact, but that is precisely where the topological features enter into the picture. A euclidean topology is simply connected and without obstructions.

Even in 3-dimensions (and with euclidean dogma) there are still two species of 3 component fields. Every one learns from Gibbs vector analysis that the 3 vector of angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ is never to be added to the 3 vector of momentum, $\mathbf{p}$. The angular momentum and the linear momentum are "two different species" of vectors (direction fields). However, with regard to non-euclidean topological domains there is also another concept, defined as a vector density. There is a topological difference, even in 3D, between and a covariant tensor and a contravariant tensor density, but not detectable if volume deforming processes are excluded (the typical non-dissipative case).

To demonstrate these ideas, first consider ordinary 3 dimensional vector fields and a C1 map of a 3-space onto a euclidean domain of 3 dimensions:

$$\phi : \{x, y, z\} \Rightarrow \{U, V, W\} = \phi^j(x^k). \tag{2.1}$$

This map defines a vector field

$$\mathbf{Z} = [U(x, y, z), V(x, y, z), W(x, y, z)]. \tag{2.2}$$

The (square) Jacobian matrix is defined by the equations,

$$d\phi : \{dx, dy, dz\} \Rightarrow \{dU, dV, dW\} = [\mathbf{J}] \circ |d\mathbf{r}\rangle = \left[\partial\phi^j/\partial x^k\right] \left|dx^k\right\rangle \tag{2.3}$$

Next construct the volume element generated by these three functions.

$$\Omega_z = dU\hat{}dV\hat{}dW = \det[\mathbf{J}]dx\hat{}dy\hat{}dz, \tag{2.4}$$

and the $N - 1 = 2$ form

$$\begin{aligned} G &= i(\mathbf{Z})dU\hat{}dV\hat{}dW = UdV\hat{}dW - VdU\hat{}dW + WdU\hat{}dV && (2.5) \\ &= D^x dy\hat{}dz - D^y dx\hat{}dz + D^z dx\hat{}dy. && (2.6) \end{aligned}$$

The Vector $\mathbf{Z}$ induces a preimage, $\mathbf{D}$, on $\{x, y, z\}$. Formally, the vector , $\mathbf{D}$, is defined in terms of the adjoint mapping, by the matrix equation:

$$Contra-variant\ tensor\ \ density\ \ \left|\mathbf{D}^k(x^m)\right\rangle_{pb} = \left[\partial\phi^j/\partial x^k\right]^{adjoint} \circ \left|\mathbf{Z}(x^j)\right\rangle, \tag{2.7}$$

The functional substitution and pullback (pb) construction works even though the Jacobian map does not have an inverse. In this respect the retrodictive process resembles the pull back of a 1-form, where the covariant tensor field is functionally well behaved with respect to the transpose mapping:

$$Co-variant\ tensor \quad |\mathbf{A}_k(x^m)\rangle_{pb} = \left[\partial\phi^j/\partial x^k\right]^{transpose} \circ \left|\mathbf{Z}(x^j)\right\rangle, \tag{2.8}$$

Now, the extraordinary result is that if $\mathbf{Z}$ is rescaled by the divisor

$$\lambda(U,V,W) = \{aU^p + bV^p + cW^p\}^{3/p} \tag{2.9}$$

then the 2-form

$$\widehat{G} = i(\mathbf{Z}/\lambda)dU\hat{}dV\hat{}dW \tag{2.10}$$

$$= i(\mathbf{D}/\lambda)dx\hat{}dy\hat{}dz, \qquad d\widehat{G} = 0 \tag{2.11}$$

is closed. This result implies that the rescaled vector field, $\widehat{\mathbf{D}} = \mathbf{D}/\lambda$, has zero divergence.

The notation above is deliberate, for in 4 dimensions it distinguishes the electromagnetic Intensities, $\mathbf{E}, \mathbf{B}$, (as components of a covariant tensor deduced from $\mathbf{A}_k$) from the electromagnetic Quantities (or excitations) $\mathbf{D}, \mathbf{H}$ (as components of a contravariant tensor density).

The assumption of a euclidean domain masks these topological features. The topological closure of $|\mathbf{D}\rangle$ is the concept of zero divergence; the topological closure of $|\mathbf{A}\rangle$ is a zero curl concept. In 3D, for c2 differentiable fields where $|\mathbf{B}\rangle = curl\ \ |\mathbf{A}\rangle$, it follows that the closure of the 2-form generated from the components of $|\mathbf{B}\rangle$ is always empty, in a global manner! However, the closure of $|\mathbf{D}\rangle$ need not be globally empty!

The fundamental result can be generalized to $N-1$ forms in any dimension.

Theorem 2.1. *If for N functions, such that*

$$\begin{aligned} G &= i[U,V,W..])dU\hat{}dV\hat{}dW... \\ and\ \lambda &= \{aU^p + bV^p + cW^p...\}^{N/p} \\ then\quad div(G/\lambda) &= 0\ for\ any\ p\ and\ any\ \{a,b,c...\} \end{aligned}$$

Proof: Consider the map $\{x, y, z...\} \Rightarrow \{U, V, W...\}$,the volume element $\Omega_N = dU\hat{}dV\hat{}dW,$ and the vector field $\mathbf{Z}$ to construct the N-1=2-form, G, on $\{U, V, W\}$ space:

$$G = i(\mathbf{Z}/\lambda)\Omega_N = \{UdV\hat{}dW - VdU\hat{}dW + WdU\hat{}dV\}/\lambda(U, V, W) \tag{2.12}$$

on N. Define the divisor, $\lambda(U, V, W) = \{aU^p + bV^p + cW^p\}^{n/p}$. By direct computation, the "divergence" of G with respect to the coordinates $\{U, V, W\}$ on N is

$$dG = \{(N - n)/\lambda\}\Omega_N \tag{2.13}$$

Hence, $dG = 0$ for $n = N$, any p, and any signature for the anisotropic constants $\{a, b, c...\}$. In other words, the Holder type divisor, λ, acts as an integrating factor for the vector field, when n=3, any p, any a,b,c. The "excluded" points are the zero sets of λ.

On a space of three dimensions there are 2-forms of three components that are exact, and there are 2-forms of three components that are not exact. Although the 2-form with covariant components $|\mathbf{B}\rangle$ constructed from the curl of a vector potential $\mathbf{A}$ is closed and exact, the 2-form with tensor density components $|\mathbf{D}/\lambda\rangle$ is closed, BUT NOT NECESSARILY EXACT. The fundamental idea is that for a non-bounding closed cycle (*nbcc*) (such as formed by a closed twisted ribbon),

$$\iint_{nbcc} \mathbf{B}\circ d(Area) = 0, \qquad but \iint_{nbcc} \mathbf{D}\circ d(Area) \neq 0 \tag{2.14}$$

where for a boundary (such as toroidal surface)

$$\iint_{boundary} \mathbf{B}\circ d(Area) = 0, \qquad and \iint_{boundary} \mathbf{D}\circ d(Area) = 0. \tag{2.15}$$

If the integration chain is a closed in the sense of cycle, and is not a boundary, then there must exist points of the integration domain which must be excluded. These points form the topological defects (the point charges in EM theory or "topological holes") or the topological obstructions that are of interest to the theory of Links and Braids. In particular, the theory of links depends upon such obstructions and is represented by integrals of the form:

$$Lk = \iint_{nbcc} (D^z dy\hat{}dz - D^y dx\hat{}dz + D^z dx\hat{}dy)/\lambda = \iint_{nbcc} G \neq 0, \quad dG = 0. \tag{2.16}$$

and should have nothing to do with magnetic flux,

$$\Phi_m = \iint_{nbcc} (B_x dy\hat{}dz - B_y dx\hat{}dz + B_z dx\hat{}dy) = \iint_{nbcc} F = 0, \quad dF = 0, \tag{2.17}$$

which has no obstructions, as the integrand is globally exact. If F was to have obstructions, the pre-images global postulate of potentials $F - dA = 0$ must fail, and the conservation of flux would not be true. Such a failure implies the existence of magnetic monopoles (the obstructions to F being globally exact). The authors personal view (along with E.J.Post and many others) subsumes that the failure to detect magnetic monopoles is proof that classical electromagnetism is defined by the postulate of potentials; i.e., $F - dA = 0$, globally. On the other hand, the 2-form of field excitations, G, is not exact.

Example 1. The Gauss Link Integral The first application is to the divergence free vector field on 3 dimensions which is not exact, but is closed, and requires three functions for its description. The generic form for the integral of interest is given by the expression

$$Lk = \iint i(Z)\Omega_z = \iint_{closed} (UdV\hat{}dW - VdU\hat{}dW + WdU\hat{}dV)/\lambda \tag{2.18}$$

As an example of the Gauss integral, Lk, consider the case where the displacement vector is the difference of two position vectors to two separate space curves. Define

$$\begin{aligned} \mathbf{Z} &= (\mathbf{R}_2 - \mathbf{R}_1) \quad \mathbf{R}_2 = [x_2, y_2, z_2] \quad \mathbf{R}_1 = [x_1, y_1, z_1] && (2.19) \\ \lambda &= (a(x_2 - x_1)^p + b(y_2 - y_1)^p - +c(z_2 - z_1)^p)^{3/p} && (2.20) \end{aligned}$$

where $\mathbf{R}_1$ defines the position vector to one field of space curves, and $\mathbf{R}_2$ defines the position vector to a second field of space curves. Space curves from different families can have different parametrizations. Hence, the vector $\mathbf{Z}$ represents the vector difference of points on two different space curves which cannot be synchronized parametrically. Next assume that the displacements of interest are constrained by two parametric curves given by the exterior differential system

$$d\mathbf{R}_1 - \mathbf{V}_1 dt \;= 0 \quad and \quad d\mathbf{R}_2 - \mathbf{V}_2 dt' = 0, \tag{2.21}$$

where the parameters dt and $dt\prime$ are not functionally related, such that

$$dt\hat{}dt\prime \neq 0, \; but \; dt\hat{}dt = 0 \; and \; dt'\hat{}dt' = 0. \tag{2.22}$$

The vector $\mathbf{D}$ can be interpreted as the displacement vector between points on the space curve C1 parametrized by t, and the points on another space curve C2 parametrized by $t\prime$. The integral to be evaluated is

$$\begin{aligned} Lk &= \iint_{closed} \Gamma = \iint_{closed} i(\mathbf{Z}/\lambda) d(x_2 - x_1)\hat{} d(y_2 - y_1)\hat{} d(z_2 - z_1) \\ &= \iint_{closed} (1/\lambda)(V_{21} - V_{11})\hat{}(V_{22}dt' - V_{12}dt)\hat{}(V_{23}dt' - V_{13}dt) + ... \\ &= \iint_{closed} (1/\lambda)(V_{21} - V_{11})\hat{}(V_{22}V_{13} - V_{12}V_{23}) dt\hat{}dt' + ... \end{aligned} \tag{2.23}$$

using $dt\hat{}dt' \neq 0$, but $dt\hat{}dt = 0$ and $dt\prime\hat{}dt\prime = 0$. Rewriting the formula using the isotropic Gauss format, a=b=c=1,p=2 leads to the classic Gauss Linkage formula,

$$Lk = \underset{closed}{\iint} G = \oint_t \oint_{t'} \{(\mathbf{R}_2 - \mathbf{R}_1) \circ \mathbf{V}_1 \times \mathbf{V}_2\} dt\hat{}dt'/\lambda \tag{2.24}$$

$$\lambda = (\mathbf{R}_1 \circ \mathbf{R}_1 - 2\mathbf{R}_1 \circ \mathbf{R}_2 + \mathbf{R}_2 \circ \mathbf{R}_2)^{3/2}. \tag{2.25}$$

However, the zero divergence formula works for the anisotropic case, for any a,b,c and for any exponent p.

From Stokes theorem, if the closed 2 dimensional integration domain is a boundary of a 3 dimensional domain, then the Link integral vanishes. However, if a particular integration chain is a closed cycle (not a boundary of a 3 dimensional domain) then the linking integral has values with rational ratios. These closed integrals are deRham period integrals in two dimensions. Points where $\mathbf{D}$ vanishes are excluded.

When the two curves are distinct, the integration is over the two bounding cycles of a closed ribbon. The ribbon surface is closed but it is not a boundary of any volume. Then the two non-intersecting cycles (that form the boundary of the ribbon area) are defined by the two distinct parameters, dt, and dt'. When integrations are computed along these closed curves whose tangent vectors are $\mathbf{V}_1$ and $\mathbf{V}_2$, then the integer values of the closed integral may be interpreted as how many times the two curves are linked. The interpretation of the closed surface integral as a orientable ribbon works if the triple product divided by lambda does not change sign as t and t' are varied. If the integrand changes sign, then the ribbon in non-orientable.

The constraint that $dt\hat{}dt' \neq 0$ implies that the "motion" along the curve generated by $\mathbf{R}_1$ is independent of the "motion" along the curve generated by $\mathbf{R}_2$. If the curve generated by $\mathbf{R}_1$ is a conic in the xy plane and the curve generated by $\mathbf{R}_2$ is a conic in the xz plane, then the surface swept out by the vector $\mathbf{D}$ is a Dupin cyclide. Such surfaces have application to the propagation of waves in electromagnetic systems.

Example 2: Flat tangential developables From another point of view, consider the ruled surface defined by the vector field of two parameters, $\{t, \mu\}$ (isotropic, a=b=c=1 , p=2). The ruled surface will be defined by the position vector $\mathbf{R}(t)$ to a space curve and a ruling parameter μ times the tangent Velocity vector to the space curve, $\mathbf{V}(t)$.

Use the general methods above to create the doubly parametrized divergence free vector field:

$$\begin{aligned} \mathbf{Z}(\mu, t) &= \{\mathbf{R}(t) \pm \mu\mathbf{V}(t)\ \} && (2.26) \\ \lambda(\mu, t) &= (\mathbf{R}(t) \circ \mathbf{R}(t) \pm 2\mu\mathbf{R}(t) \circ \mathbf{V}(t) + \mu\mathbf{V}(t) \circ \mu\mathbf{V}(t))^{3/2}. && (2.27) \end{aligned}$$

Vector fields of this type are primitive examples of "strings" for fixed values of the parameter, t, and string parameter, μ. Direct substitution of the physical constraints, $d\mathbf{R} - \mathbf{V}dt = 0$, and $d\mathbf{V} - \mathbf{A}dt = 0$, such that $d\mathbf{Z} = d\{\mathbf{R}(t) \pm \mu\mathbf{V}(t)\ \}$ into the definition of the linking integral

$$\iint_{closed\, on\, N} i(\mathbf{Z}/\lambda) dZ^1\hat{}dZ^2\hat{}dZ^3 \qquad (2.28)$$

leads to yet another realization and interpretation of the Gauss formula:

$$\begin{aligned} Q &= \iint_{closed} G = \iint_{closed\, on\, \mu t} \{\mathbf{R}\circ\mu\mathbf{V} \times \mathbf{A}\} dt\hat{}d\mu/\lambda \\ &= \iint_{closed} \{\mathbf{A} \circ \mathbf{R}\times\mu\mathbf{V}\} dt\hat{}d\mu/(\mathbf{R} \circ \mathbf{R} \pm 2\mu\mathbf{R} \circ \mathbf{V} + \mu\mathbf{V} \circ \mu\mathbf{V})^{3/2}. \quad (2.29) \end{aligned}$$

It is apparent that the interaction of the "angular" momentum, $\mathbf{L} = \mathbf{R} \times \mu\mathbf{V}$, and the acceleration, $\mathbf{A}$, produces a topological invariant whose values are "quantized" (in the sense that the ratios of the closed integrals are rational). Note that the triple vector product of the integrand numerator is proportional to the Frenet torsion of the orbit. For an orbit that is planar the Frenet torsion is zero everywhere, and the Gauss integral vanishes.

Recall that if a the space curve is an edge of regression, then the ruled surfaces associated with to the forward and backward motions (the $\pm$ signs in the formula) are not same to second order. Such a result demonstrates an obvious distinction between forward and backward motion that breaks time reversal symmetry. Linear rulings in one direction are on 1 sheet of the ruled surface, and rulings in the opposite direction are on the other surface. The two surfaces meet at an edge of regression. Similar time reversal symmetry breaking effects have been observed macroscopically in dual polarized ring lasers,

Example 3. Scrolls The two parameter surface described above is closely related to the ruled surface known as the tangential developable. Such ruled surfaces (parametrized by arc length s rather than time, t, and with the directrix of the ruling in the direction of the unit tangent vector, and multiplied by μ) have zero Gauss curvature. Though bent, such surfaces can be rolled out flat. By constructing the ruled surfaces in terms of the normal and/or binormal to a space curve, other forms of ruled surfaces yield negative values for the Gauss curvature of the surface, and are not "flat". They are defined as Scrolls.

Of particular interest to physics are those ruled surfaces of negative Gauss curvature, which are also minimal surfaces. They have application in describing hydrodynamic wakes. These surfaces can be viewed as double edged ribbons for given values of μ. The equations for the ruled surface of a scroll, with a directrix in the direction of the binormal, $\mathbf{b}(s)$, are :

$$\mathbf{D}(\mu, s) = \{\mathbf{R}(s) \pm \mu\mathbf{b}(s)\}/\lambda \tag{2.30}$$

$$\lambda = (\mathbf{R}(s) \circ \mathbf{R}(s) \pm 2\mu\mathbf{R}(s) \circ \mathbf{b}(s) + \mu\mathbf{b}(s) \circ \mu\mathbf{b}(s))^{3/2}. \tag{2.31}$$

When the parameter μ takes on the constant values $\mu = \kappa/\tau^3$, (with $\kappa =$ the Frenet curvature, and $\tau =$ the torsion of the space curve) then the ruled surface is a minimal surface, and the binormal field twists about the space curve generated by $\mathbf{R}(s)$.

Another interesting scroll is that generated by the Darboux vector.

$$\mathbf{D}(\theta, s) = \{\mathbf{R}(s) \pm (\mathbf{n}(s)\cos(\theta) + \mathbf{b}(s)\sin(\theta)\}/\lambda \tag{2.32}$$
$$\lambda = (\mathbf{D} \circ \mathbf{D})^{3/2}. \tag{2.33}$$

which seems to be of interest to Longcope.

3. Braids, Spin and Torsion_Helicity (3-forms)

Braids For n = 4 the same procedures described above may be used to produce a period integral over a closed 3-dimensional domain. The technique is to define a 4 dimensional vector field, $\mathbf{Z} = [Z_1, Z_2, Z_3, Z_4]$. Use the general renormalization function,

$$\lambda = \{\alpha Z_1^p + \beta Z_2^p + \gamma Z_3^p + \epsilon Z_4^p\}^{n/p} \tag{3.1}$$

and set n=4, for zero four divergence. Construct the closed 3 -form,

$$\Gamma = i(\mathbf{Z}/\lambda) dZ_1 \hat{}\, dZ_2 \hat{}\, dZ_3 \hat{}\, dZ_4 \tag{3.2}$$

Assume the 4 component vector has a realization as $\mathbf{Z} = \mathbf{P_1} + \mathbf{P_2} + \mathbf{P_3}$, where the three independent fields $\mathbf{P}$ represent three space-time curves that obey the kinematic constraints:

$$d\mathbf{P_1} - \mathbf{f_1} ds = 0, \quad d\mathbf{P_2} - \mathbf{f_2} ds' = 0, \quad d\mathbf{P_3} - \mathbf{f_3} ds'' = 0. \tag{3.3}$$

Substitute for each of the differentials in Γ (and further assume that the domain {x,y,z,t} of interest is further constrained such that $dt = 0$) to yield the three form

$$G = \{\, \mathbf{f}_1 \circ (\mathbf{f}_2 \times \mathbf{f}_3)\} ds \hat{}\, ds' \hat{}\, ds''/\lambda \tag{3.4}$$

$$\lambda = \{\alpha Z_1^p + \beta Z_2^p + \gamma Z_3^p + \epsilon Z_4^p\}^{4/p} \tag{3.5}$$

The spatial braid integral becomes equal to

$$Br := \oint_t \oint_{t'} \oint_{t''} \mathbf{f}_1 \circ (\mathbf{f}_2 \times \mathbf{f}_3)\} ds \hat{}\, ds' \hat{}\, ds''/\lambda \tag{3.6}$$

The integrations are now over three closed curves whose tangents are the "Newtonian forces", **f**, on three "particles". Where in the two dimensional Gauss integral, of the previous section, the evaluation was along the closed curves of a ribbon, in this case the integrations are along the closed trajectories of three "particles" which form the vertices of a triangle.

The idea that three "lines" are used to form the integral (whose values form rational ratios) is the reason that this topological integral in the format given above is defined as the braid integral. Of course the three form of topological torsion is a variant of the braid integral, but applies to those topologies where the system is not reducible to three factors $dt, dt\prime$ and $dt"$. An example of a period 3 braid with Braid integral zero (chaotic) and Braid integral 2 (non-chaotic) is given in Figure 5. It is illuminating to construct the two braids by wrapping a long flat ribbon of paper smoothly around the palm of your hand. Close the ribbon surface by pasting the ends together. Then make another example, where this time thread the loose end underneath the middle wrap, rather than over the middle wrap, before gluing the ends together. Take the two examples from your hand and note that one is continuously deformable into a closed cylinder (Tw = zero) while the other has a 4 pi twist (Tw =2). What is surprising is that it is the Tw =0 configuration that has a chaotic neighborhood, while the Tw = 2 structure is not chaotic. To test for chaos construct the equivalent of the closed braid from copper tubes. Then link any pair of tubes with a large loop of elastic or thread. Push the looping thread around the period three copper tube, and note that for a Tw=2 configuration, the looping thread becomes untangled after 6 pi revolutions about the central axis. For the Tw = 0 configuration, the looping thread never unwinds, but becomes more and more twisted and complex.

The equivalent to Figure 5, and the fact that there are two distinct period 3 configurations, one chaotic and one non-chaotic, was brought to the author's attention during a stimulating lecture given by J. Los at the August, 1991, Pedagogical Workshop on Topological Fluid Mechanics held at the Institute for Theoretical Physics, Santa Barbara UCSB.

It is to be noted that the 3-form of topological torsion is related to the braid integral, a three dimensional thing in four dimensions, and not the Gauss linkage integral, which is a two dimensional thing in three dimensions.

3.1. Magnetic Helicity (a 3-form)

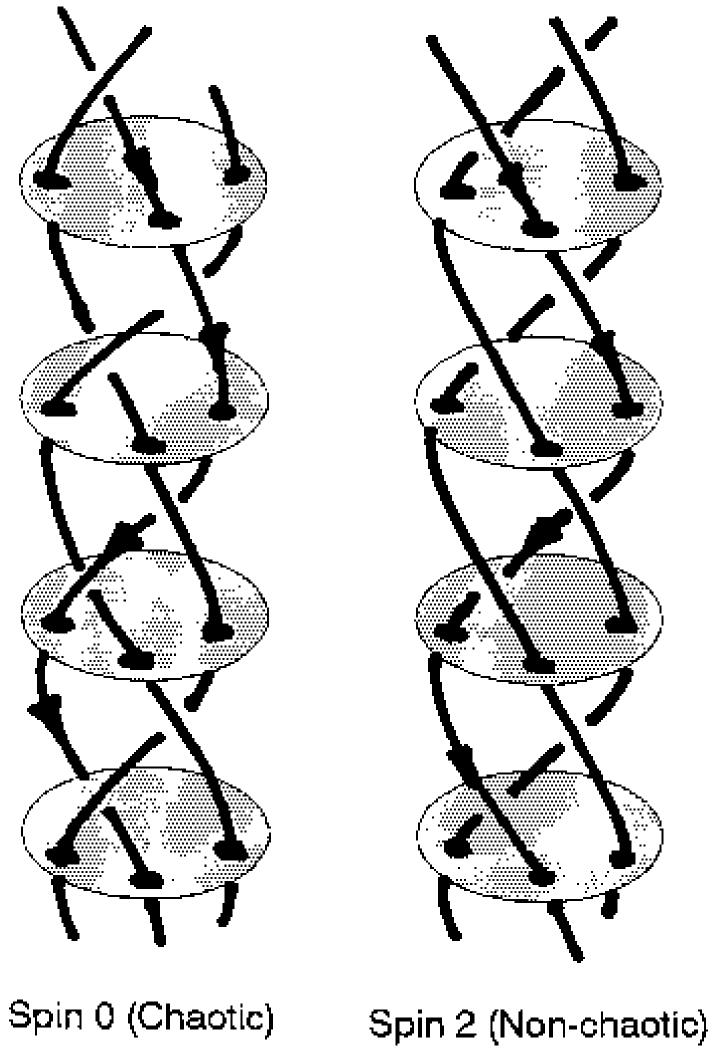

Figure 5. Period 3 Braids

Figure 3.1:

First, the definition of Magnetic Helicity with which I am comfortable is that given by the 3D (volume) integral of the vector potential dotted with its curl:

Definition 3.1.

$$Magnetic\ Helicity := \iiint \mathbf{A} \circ curl \mathbf{A}\, dVol = \iiint \mathbf{A} \circ \mathbf{B}\, dVol = \iiint A\hat{}F = \iiint A\hat{}dA \tag{3.7}$$

For me this object is a well defined object on any variety $\{x, y, z\}$. On every 3 dimensional *closed* manifold for which $\mathbf{A} = \mathbf{A}(x, y, z)$, it is easy to show that the *closed* integrals of Magnetic Helicity density, $\mathbf{A} \circ \mathbf{B}$, are deformation invariants of **any** continuous evolutionary process that can be described by a singly parametrized vector direction (or flow) field, $\mathbf{V}(x, y, z)$. In the language of Poincare-Cartan, the closed integral is a relative integral invariant. All that is required is that the points which make up the integration chain in the initial state remain on the same flow fiber in the final state.

This topological conservation law has nothing to do with electromagnetism, per se, but for classical definitions of the Magnetic Field intensity, such as $\mathbf{B} = curl\ \mathbf{A}$, it works for electromagnetism. It is also true for fluids or any other system that can be described by a 1-form of Action, A, and can be extrapolated to higher dimensions. The proof is easy:

$$\begin{aligned} L_{(\beta\mathbf{V})} \iiint_{closed} \mathbf{A} \circ \mathbf{B}\, dVol &= \iiint_{closed} \{d(i(\beta\mathbf{V})A\hat{}F) + i(\beta\mathbf{V})d(A\hat{}F)\} = \\ &\quad \iiint_{closed} \{d(i(\beta\mathbf{V})A\hat{}F) + i(\beta\mathbf{V})F\hat{}F\} \\ &= 0 + 0 \supset\ evolutionary\ invariance\ for\ any\ \beta\mathbf{V} \end{aligned} \tag{3.8}$$

The first integral vanishes because the integral of an exact form over a closed integration chain vanishes. The second integral vanishes as $F\hat{}F = 0$ on a 3 dimensional manifold; all 3-dimensional volume elements are closed. As $\beta(x, y, z)$ is arbitrary, it acts as a possible deformation parameter. The result is true for any gauge, does not depend upon metric, and is independent of any geometrical connection, and certainly does not depend upon a constitutive constraint. A true topological quantity. Note that the closed 3-dimensional integration domain can be a cycle and does not have to be a boundary (of a higher dimensional space). It is important to realize (for application to domains that do not have a euclidean topology) that the 3-form $A\hat{}F$ has components that transform as a covariant tensor of rank 3.

Perhaps it is more important that the theorem of Topological deformation invariance of the closed integrals of the 3-form is also valid in **any** dimension for which $F\hat{}F$ is zero. In higher dimensions the integral must be evaluated over 3-sub_manifolds that need not be space-like. For 4 dimensions the 3-form $A\hat{}F$ has 4 components and can be constructed as

$$\text{Topological Torsion:} \quad A\hat{}F = i(\mathbf{T}_4)dx\hat{}dy\hat{}dz\hat{}dt \tag{3.9}$$

where

$$\mathbf{T}_4 = [\mathbf{E} \times \mathbf{A} + \phi\mathbf{B}, \mathbf{A} \circ \mathbf{B}]$$

From this formulation it must be remembered that the components of this "vector" transform as a third rank covariant tensor field; $\mathbf{A} \circ \mathbf{B}$ is merely the fourth component. In the literature such objects are often described as pseudo vectors.

The criteria $F\hat{}F = 0$ for topological invariance with respect to any continuous evolutionary process is equivalent to the statement that $\mathbf{E} \circ \mathbf{B} = 0$. The condition is sufficient, but not necessary, for topological invariance of the Helicity integral, even when the fields are explicitly time dependent. The relative integral invariant in 4D is:

$$H = \iiint_{closed} A\hat{}F = \iiint_{closed} \mathbf{T}^x dy\hat{}dz\hat{}dt - \mathbf{T}^y dx\hat{}dz\hat{}dt + \mathbf{T}^z dx\hat{}dy\hat{}dt - \mathbf{A} \circ \mathbf{B} dx\hat{}dy\hat{}dz \tag{3.10}$$

In all cases the 3 divergence of $\mathbf{B}$ vanishes, and the 2-form F is closed, for it is exact, $dF = 0$. For isochronous domains, $dt = 0$, and the integral reduces to the standard spatial format of plasma physics. However, there are example electromagnetic fields for which $\mathbf{A} \circ \mathbf{B} = 0$, and yet $\mathbf{T}_4$ is not zero.

Example: Define

$$\mathbf{A} = [0, 0, (x^2 + y^2)/2], \quad \phi = z. \tag{3.11}$$

Then

$$\mathbf{B} = [-y, x, 0] \tag{3.12}$$

$$\mathbf{E} = [0, 0, -1] \tag{3.13}$$

$$\mathbf{A} \circ \mathbf{B} = 0, \quad \mathbf{E} \circ \mathbf{B} = 0, \tag{3.14}$$

$$\mathbf{T}_4 = [z\mathbf{B}, 0] \tag{3.15}$$

Note that in 4 dimensions (that admit time dependent fields), the frozen in lines are the line of the Torsion vector, which can be dominated by the $\mathbf{B}$ field but also have a component due to $\mathbf{E} \times \mathbf{A}$. I suspect that measurements are made on the lines of $\mathbf{T}_4$, and only indirectly on the "lines" of "$\mathbf{B}$" (for remember the electromagnetic field has six components which can transform into one another). The implication is that the lines of $\mathbf{T}_4$ can continuously evolve, with most regions in a plasma dominated by $\phi\mathbf{B} \neq 0$, $\mathbf{E} \times \mathbf{A} \to 0$. Then as $\mathbf{B}$ lines approach one another, induction causes $\mathbf{E} \times \mathbf{A} \neq 0$, $\phi\mathbf{B} \to 0$. The "$\mathbf{B}$ lines" terminate on a null (a boundary point), break apart, and then possibly reconnect different segments, after which $\phi\mathbf{B} \neq 0$, $\mathbf{E} \times \mathbf{A} \to 0$. (I believe this mechanism is that which Hornig is developing).

I call $A\hat{}F$ the topological torsion of the field, and $d(A\hat{}F) = F\hat{}F$, the topological parity.

Special situations become evident when the integration domain is compact with a boundary, for then Stokes law may be applied, and deformation invariance requires that $i(\beta\mathbf{V})A\hat{}F{=}0$ *on the boundary.* These are interesting but special cases which are invariants of only a special choice of boundary conditions. For example, if $\beta(x, y, z) = 0$ defines the boundary, then for deformation invariance of the integral it must be true that the function β also must be an evolutionary invariant, such that $L_{(\mathbf{V})}d\beta = i(\mathbf{V})d\beta = 0$. Classically the function β which is used to define the boundary, is not arbitrary, but must be a first integral of the evolutionary vector field. For such special cases, the field on the boundary need not be tangential! There are other special situations as well.

For integration domains which are open, the criteria for absolute integral invariance is much more severe, and requires that $d(i(\beta\mathbf{V})A\hat{}F) = 0$. This constraint is to be recognized as the criteria that the evolutionary vector field $\beta\mathbf{V}$ be an element of the symplectic group. I have demonstrated that all such evolutionary processes are thermodynamically reversible.

Spin (a 3-form) To relate the above definition of a 3-form of Helicity with the six dimensional formulation involving the Biot-Savart substitution (to me) is an extraordinary constraint on the topology of the domain. The substitution effectively mixes a tensor and a tensor density, where definition 1 above mixes a tensor with a tensor, $A\hat{}F$. There is however, another well defined electromagnetic 3-form, $A\hat{}G$, which mixes a tensor (A) and a tensor density (G). I call $A\hat{}G$ – with physical dimensions of angular-momentum– the Spin 3-form. (The 3-form $A\hat{}F$ has physical dimensions of Angular momentum divided by Ohms.)

In electrical engineering notation,

$$Spin : \mathbf{S}_4 = [\mathbf{A} \times \mathbf{H} + \phi \mathbf{D}, \mathbf{A} \circ \mathbf{D}]$$

$$Torsion\_Helicity : \mathbf{T}_4 = [\mathbf{E} \times \mathbf{A} + \phi \mathbf{B}, \mathbf{A} \circ \mathbf{B}]$$

The closure of $A\hat{}G$ defines a measure known as the first Poincare invariant,

$$P1 := (\mathbf{B} \circ \mathbf{H} - \mathbf{D} \circ \mathbf{E}) - (\mathbf{A} \circ \mathbf{J} - \rho\phi) \equiv d(A\hat{}G)$$

while the closure of $A\hat{}F$ yields the second Poincare invariant.

$$P2 := 2(\mathbf{E} \circ \mathbf{B}) \equiv d(A\hat{}F).$$

When these measures vanish (the divergences of the 4-vectors vanish), then there exist separate topological conservation laws (of Spin and Helicity).

It is extraordinary to me that the Solar - Plasma community does not seem to make use of this other topological property which is conserved when the first Poincare measure is zero. The Spin 3-form is of equal importance as the 3-form of Torsion-Helicity, yet no one in the Plasma community seems to use it.

For the above example, with the added assumption that $\mathbf{D} = \varepsilon \mathbf{E}$ and $\mathbf{B} = \mu \mathbf{H}$

$$\begin{aligned}
\mathbf{A} &= [0, 0, (x^2 + y^2)/2], \quad \phi = z. \text{ Then} \\
\mathbf{B} &= [y, -x, 0] \\
\mathbf{E} &= [0, 0, -1] \\
\mathbf{A} \circ \mathbf{B} &= 0, \\
\mathbf{T}_4 &= [zy, -zx, 0, 0] \\
P2 &= \mathbf{E} \circ \mathbf{B} = 0, \\
\mathbf{A} \circ \mathbf{D} &= -\varepsilon(x^2 + y^2)/2 \\
\mathbf{S}_4 &= [x(x^2+y^2)/2, y(x^2+y^2)/2, -\varepsilon\mu z, -\varepsilon\mu(x^2+y^2)/2]/\mu \\
P1 &= (2(x^2 + y^2) - \varepsilon\mu)/\mu \\
\mathbf{J} &= [0, 0, -2/\mu] \\
\rho &= 0
\end{aligned}$$

which implies that Helicity is conserved but Spin is not conserved in the example field.

For more details about Torsion vs. Spin see http://www.uh.edu/~rkiehn/pdf/classice.pdf

You might also be interested in something that I did after the Chapman conference stimulated me. I think I have found a *raison d'etre* for the formation of an accretion disk in a otherwise central field problem. Using Maple, I have found a time dependent solution to Maxwell's equations that makes the z=0 plane of a rotating plasma a chiral attractor. The force of attraction from the top and the bottom is due to a Lorentz $\mathbf{J} \times \mathbf{B}$ term. This the first time I have ever seen a simple mechanism that might explain the formation of accretion disks around stars and planets. The Solution is just a model field, but exhibits both Helicity and Spin effects.

see

http://www.uh.edu/~rkiehn/pdf/diracch.pdf