```
> restart: with (linalg):with(liesymm):with(difforms):
> setup(x,y,z,t):defform(x=0,y=0,z=0,t=0,Vx=0,Vy=0,Vz=0,D1=0,D2=0,D3=0,Ax=0,Ay=0,A
  z=0,C=0,Phi=0,phi=0,theta=0,r=0,tau=0,a=const,b=const,c=const,aa=const,bb=const,
  M=const,cc=const,ee=const,Lx=0,Ly=0,Lz=0,vx=const,vy=const,vz=const);
```
Warning, new definition for norm
Warning, new definition for trace
Warning, new definition for close
Warning, new definition for `&^`
Warning, new definition for d
Warning, new definition for mixpar
Warning, new definition for wdegree

Christoffel metric based connections, Cartan frame based connections, their curvatures and the Principle of Equivalence.

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INTRODUCTION

Given a connection [C] as a Cartan matrix of 1-forms, the standard formula
Cartan matrix of curvature 2-forms ,
 [OMEGA] = [dC] + [C] ^ [C]
 is used to define the curvatures of the space.
*
However, a connection can be computed in several distinct ways.
Given a Frame field, [F], mapping contravectors on the initial state to contravectors on the final state,
the Right Cartan Connection can be computed from the formula [C] = [F inverse][dF].
*
Given a metric, [g], the Christoffel connections coefficients [Gamma] will be computed from the functions
that define the metric. The method is independent from the Frame. Both methods yield matrices of
1-forms defined on the initial state.
However, a compatibility condition between the metric [g] and Frame field [F] requires that the metric [g]
on the initial state be related to the metric [eta] on the final state by the formula [g] =
[Ftranspose][eta][F],
*
The two connections are identical if their exists a map of coordiantes from the initial state to the final
state, and the metric on the final state is a set of constants. The Frame can then be taken to be the
Jacobian matrix of the coordinate mapping. The Frame field is said to be integrable. The matrix of
curvature 2-forms is zero in this special case.

IN all cases, the Cartan matrix of connection 1-forms can be decomposed into two parts according to the
formula
[C] = [Gamma] - [T]
The standard formula for curvature will be used to yield
[dC] + [C]^[C] ={ [dGamma] + [C]^[Gamma]} - {[dT] + [C]^[T]} = {metrical curvature} - {inertial
curvature}.

For all Frame fields (with det not zero) the far left side vanishes, leaving the result
metric curvature = inertial curvature.
Thus a principle of equivalence is established for parallelizable spaces.

The classic coordinate map from spherical to Cartesian Coordinates

is given by the expressions:

> `x:=r*sin(theta)*cos(phi);y:=r*sin(theta)*sin(phi);z:=r*cos(theta);t:=tau;`

>

$$x := r \sin(\theta) \cos(\phi)$$
$$y := r \sin(\theta) \sin(\phi)$$
$$z := r \cos(\theta)$$
$$t := \tau$$

The Coordinate map generates a Frame field in terms of the Jacobian of the mapping

Compute the Frame matrix from the Jacobian matrix, so that the induced 1-forms are well defined.

>

> `FF11:=getcoeff(d(x)&^d(theta)&^d(phi)&^d(tau));FF12:=getcoeff(d(x)&^d(phi)&^d(tau)&^d(r));FF13:=getcoeff(d(x)&^d(tau)&^d(r)&^d(theta));FF14:=getcoeff(d(x)&^d(r)&^d(theta)&^d(phi));`

$$FF11 := \sin(\theta) \cos(\phi)$$
$$FF12 := r \cos(\phi) \cos(\theta)$$
$$FF13 := -r \sin(\theta) \sin(\phi)$$
$$FF14 := 0$$

>

> `FF21:=getcoeff(d(y)&^d(theta)&^d(phi)&^d(t));FF22:=getcoeff(d(y)&^d(phi)&^d(t)&^d(r));FF23:=getcoeff(d(y)&^d(t)&^d(r)&^d(theta));FF24:=getcoeff(d(y)&^d(r)&^d(theta)&^d(phi));`

$$FF21 := \sin(\theta) \sin(\phi)$$
$$FF22 := r \sin(\phi) \cos(\theta)$$
$$FF23 := r \sin(\theta) \cos(\phi)$$
$$FF24 := 0$$

At this point the Frame matrix will be modified to make the third 1-form sigmaz not exact and not closed.
A constant coefficient "aa" will scale the perturbation of the FRame. When aa = 0 , the Frame is the
Jacobian matrix of the mapping. The associated Frame matrix elements of the perturbed Frame are
then:

> `sigmaz:=wcollect(d(z)+aa*(y*d(x)-x*d(y)));dzz:=sigmaz;`
> `FF31:=getcoeff(dzz&^d(theta)&^d(phi)&^d(tau));FF32:=getcoeff(dzz&^d(phi)&^d(tau)&^d(r));FF33:=factor(simplify(getcoeff(dzz&^d(tau)&^d(r)&^d(theta))));FF34:=getcoeff(dzz&^d(r)&^d(theta)&^d(phi));`

>

>

$$sigmaz := \cos(\theta) \, d(r) - r \sin(\theta) \, d(\theta) + aa \, (-r^2 \sin(\theta)^2 \sin(\phi)^2 - r^2 \sin(\theta)^2 \cos(\phi)^2) \, d(\phi)$$
$$dzz := \cos(\theta) \, d(r) - r \sin(\theta) \, d(\theta) + aa \, (-r^2 \sin(\theta)^2 \sin(\phi)^2 - r^2 \sin(\theta)^2 \cos(\phi)^2) \, d(\phi)$$

$$FF31 := \cos(\theta)$$
$$FF32 := -r\sin(\theta)$$
$$FF33 := aa\ r^2\ (\cos(\theta) - 1)\ (\cos(\theta) + 1)$$
$$FF34 := 0$$

```
> 
> dtt:=d(t);
> FF41:=getcoeff(dtt&^d(theta)&^d(phi)&^d(tau));FF42:=simplify(getcoeff(dtt&^d(phi
  )&^d(tau)&^d(r)));FF43:=factor(simplify(getcoeff(dtt&^d(tau)&^d(r)&^d(theta))));
  FF44:=getcoeff(dtt&^d(r)&^d(theta)&^d(phi));
```

$$dtt := d(\tau)$$
$$FF41 := 0$$
$$FF42 := 0$$
$$FF43 := 0$$
$$FF44 := 1$$

```
> 
```

These matrix elements will be put into a perturbed Frame matrix format as:

```
> 
> R:=[x,y,z,t]:FFI:=simplify(subs(array([[FF11,FF12,FF13,FF14],[FF21,FF22,FF23,FF2
  4],[FF31,FF32,FF33,FF34],[FF41,FF42,FF43,FF44]])));
```

$$FFI := \begin{bmatrix} \sin(\theta)\cos(\phi) & r\cos(\phi)\cos(\theta) & -r\sin(\theta)\sin(\phi) & 0 \\ \sin(\theta)\sin(\phi) & r\sin(\phi)\cos(\theta) & r\sin(\theta)\cos(\phi) & 0 \\ \cos(\theta) & -r\sin(\theta) & -aa\ r^2 + aa\ r^2\cos(\theta)^2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

```
> dim:=4;coord:=[r,theta,phi,tau];GGI:=simplify(inverse(FFI));DET:=simplify(det(FF
  I));
```

$$dim := 4$$
$$coord := [r, \theta, \phi, \tau]$$

$GGI :=$

$\quad [\sin(\theta)\ (-r\sin(\phi)\cos(\theta)\ aa + \cos(\phi))\ ,\ \sin(\theta)\ (r\cos(\phi)\cos(\theta)\ aa + \sin(\phi))\ ,\ \cos(\theta)\ ,\ 0]$

$$\quad \left[\frac{\sin(\phi)\ aa\ r - \sin(\phi)\ aa\ r\cos(\theta)^2 + \cos(\theta)\cos(\phi)}{r}\ ,\ \frac{-\cos(\phi)\ aa\ r + \cos(\phi)\ aa\ r\cos(\theta)^2 + \cos(\theta)\sin(\phi)}{r}\ , \right.$$

$$\quad \left. -\frac{\sin(\theta)}{r}\ ,\ 0 \right]$$

$$\quad \left[-\frac{\sin(\phi)}{r\sin(\theta)}\ ,\ \frac{\cos(\phi)}{r\sin(\theta)}\ ,\ 0\ ,\ 0 \right]$$

$$\quad [0\ ,\ 0\ ,\ 0\ ,\ 1]$$

$$DET := \sin(\theta)\ r^2$$

The right Cartan Connection is based on the perturbed Frame: [C] = [G] [dF]= - [dG][F]

First compute the differentials of the inverse matrix [GG]

```
> for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do d1GG[i,j,k]
  := (diff(GGI[i,j],coord[k])) od od od:
```

Compute the elements of the matrix product of - d[G][F]= G[dF]

```
> for b from 1 to dim do for a from 1 to dim do for k from 1 to dim do  ss:=0;for
  m from 1 to dim do ss := ss+(d1GG[a,m,k]*FFI[m,b]); CC[a,b,k]:=simplify(-ss) od
  od od od ;
```

Right Cartan connection coefficients:

```
>
> for b from 1 to dim do for a from 1 to dim do for k from 1 to dim do if
  CC[a,b,k]=0 then else print(`Cartan_RIGHT`(a,b,k)=factor(CC[a,b,k])) fi od od od
  ;
```

$$\text{Cartan\_RIGHT}(1, 1, 3) = -r \cos(\theta)\, aa\, (\cos(\theta) - 1)\, (\cos(\theta) + 1)$$

$$\text{Cartan\_RIGHT}(2, 1, 2) = \frac{1}{r}$$

$$\text{Cartan\_RIGHT}(2, 1, 3) = \sin(\theta)\, aa\, (\cos(\theta) - 1)\, (\cos(\theta) + 1)$$

$$\text{Cartan\_RIGHT}(3, 1, 3) = \frac{1}{r}$$

$$\text{Cartan\_RIGHT}(1, 2, 2) = -r$$

$$\text{Cartan\_RIGHT}(1, 2, 3) = \sin(\theta)\, r^2 \cos(\theta)^2\, aa$$

$$\text{Cartan\_RIGHT}(2, 2, 1) = \frac{1}{r}$$

$$\text{Cartan\_RIGHT}(2, 2, 3) = r \cos(\theta)\, aa\, (\cos(\theta) - 1)\, (\cos(\theta) + 1)$$

$$\text{Cartan\_RIGHT}(3, 2, 3) = \frac{\cos(\theta)}{\sin(\theta)}$$

$$\text{Cartan\_RIGHT}(1, 3, 1) = r \cos(\theta)\, aa\, (\cos(\theta) - 1)\, (\cos(\theta) + 1)$$

$$\text{Cartan\_RIGHT}(1, 3, 2) = -\sin(\theta)\, r^2 \cos(\theta)^2\, aa$$

$$\text{Cartan\_RIGHT}(1, 3, 3) = r\, (\cos(\theta) - 1)\, (\cos(\theta) + 1)$$

$$\text{Cartan\_RIGHT}(2, 3, 1) = -\sin(\theta)\, aa\, (\cos(\theta) - 1)\, (\cos(\theta) + 1)$$

$$\text{Cartan\_RIGHT}(2, 3, 2) = -r \cos(\theta)\, aa\, (\cos(\theta) - 1)\, (\cos(\theta) + 1)$$

$$\text{Cartan\_RIGHT}(2, 3, 3) = -\sin(\theta) \cos(\theta)$$

$$\text{Cartan\_RIGHT}(3, 3, 1) = \frac{1}{r}$$

$$\text{Cartan\_RIGHT}(3, 3, 2) = \frac{\cos(\theta)}{\sin(\theta)}$$

Matrix elements of the right Cartan matrix of connection 1-forms.

```
> CGamma11:=CC[1,1,1]*d(r)+CC[1,1,2]*d(theta)+CC[1,1,3]*d(phi)+CC[1,1,4]*d(tau);
> CGamma12:=CC[1,2,1]*d(r)+CC[1,2,2]*d(theta)+CC[1,2,3]*d(phi)+CC[1,2,4]*d(tau);
> CGamma13:=CC[1,3,1]*d(r)+CC[1,3,2]*d(theta)+CC[1,3,3]*d(phi)+CC[1,3,4]*d(tau);
> CGamma14:=CC[1,4,1]*d(r)+CC[1,4,2]*d(theta)+CC[1,4,3]*d(phi)+CC[1,4,4]*d(tau);
```

$$CGamma11 := (r \cos(\theta)\, aa - r \cos(\theta)^3\, aa)\, d(\phi)$$

$$CGamma12 := -r\, d(\theta) + \sin(\theta)\, r^2 \cos(\theta)^2\, aa\, d(\phi)$$

$$CGamma13 := (-r \cos(\theta)\, aa + r \cos(\theta)^3\, aa)\, d(r) - \sin(\theta)\, r^2 \cos(\theta)^2\, aa\, d(\theta) + (-r + r \cos(\theta)^2)\, d(\phi)$$

$$CGamma14 := 0$$

```
> CGamma21:=CC[2,1,1]*d(r)+CC[2,1,2]*d(theta)+CC[2,1,3]*d(phi)+CC[2,1,4]*d(tau);
> CGamma22:=CC[2,2,1]*d(r)+CC[2,2,2]*d(theta)+CC[2,2,3]*d(phi)+CC[2,2,4]*d(tau);
> CGamma23:=CC[2,3,1]*d(r)+CC[2,3,2]*d(theta)+CC[2,3,3]*d(phi)+CC[2,3,4]*d(tau);
```

```
> CGamma24:=CC[2,4,1]*d(r)+CC[2,4,2]*d(theta)+CC[2,4,3]*d(phi)+CC[2,4,4]*d(tau);
```

$$CGamma21 := \frac{\mathrm{d}(\theta)}{r} + (-\sin(\theta)\, aa + \sin(\theta)\, aa \cos(\theta)^2)\, \mathrm{d}(\phi)$$

$$CGamma22 := \frac{\mathrm{d}(r)}{r} + (-r\cos(\theta)\, aa + r\cos(\theta)^3\, aa)\, \mathrm{d}(\phi)$$

$$CGamma23 := (\sin(\theta)\, aa - \sin(\theta)\, aa \cos(\theta)^2)\, \mathrm{d}(r) + (r\cos(\theta)\, aa - r\cos(\theta)^3\, aa)\, \mathrm{d}(\theta) - \sin(\theta)\cos(\theta)\, \mathrm{d}(\phi)$$

$$CGamma24 := 0$$

```
> CGamma31:=CC[3,1,1]*d(r)+CC[3,1,2]*d(theta)+CC[3,1,3]*d(phi)+CC[3,1,4]*d(tau);
> CGamma32:=CC[3,2,1]*d(r)+CC[3,2,2]*d(theta)+CC[3,2,3]*d(phi)+CC[3,2,4]*d(tau);
> CGamma33:=CC[3,3,1]*d(r)+CC[3,3,2]*d(theta)+CC[3,3,3]*d(phi)+CC[3,3,4]*d(tau);
> CGamma34:=CC[3,4,1]*d(r)+CC[3,4,2]*d(theta)+CC[3,4,3]*d(phi)+CC[3,4,4]*d(tau);
```

$$CGamma31 := \frac{\mathrm{d}(\phi)}{r}$$

$$CGamma32 := \frac{\cos(\theta)\, \mathrm{d}(\phi)}{\sin(\theta)}$$

$$CGamma33 := \frac{\mathrm{d}(r)}{r} + \frac{\cos(\theta)\, \mathrm{d}(\theta)}{\sin(\theta)}$$

$$CGamma34 := 0$$

```
> CGamma41:=CC[4,1,1]*d(r)+CC[4,1,2]*d(theta)+CC[4,1,3]*d(phi)+CC[4,1,4]*d(tau);
> CGamma42:=CC[4,2,1]*d(r)+CC[4,2,2]*d(theta)+CC[4,2,3]*d(phi)+CC[4,2,4]*d(tau);
> CGamma43:=CC[4,3,1]*d(r)+CC[4,3,2]*d(theta)+CC[4,3,3]*d(phi)+CC[4,3,4]*d(tau);
> CGamma44:=CC[4,4,1]*d(r)+CC[4,4,2]*d(theta)+CC[4,4,3]*d(phi)+CC[4,4,4]*d(tau);
```

$$CGamma41 := 0$$
$$CGamma42 := 0$$
$$CGamma43 := 0$$
$$CGamma44 := 0$$

```
> CartanC:=array([[CGamma11,CGamma12,CGamma13,CGamma14],[CGamma21,CGamma22,CGamma2
  3,CGamma24],[CGamma31,CGamma32,CGamma33,CGamma34],[CGamma41,CGamma42,CGamma43,CG
  amma44]]);
> simpform(d(CartanC));
>
```

$CartanC :=$

$[(r\cos(\theta)\, aa - r\cos(\theta)^3\, aa)\, \mathrm{d}(\phi)\,,\, -r\, \mathrm{d}(\theta) + \sin(\theta)\, r^2 \cos(\theta)^2\, aa\, \mathrm{d}(\phi)\,,$

$(-r\cos(\theta)\, aa + r\cos(\theta)^3\, aa)\, \mathrm{d}(r) - \sin(\theta)\, r^2 \cos(\theta)^2\, aa\, \mathrm{d}(\theta) + (-r + r\cos(\theta)^2)\, \mathrm{d}(\phi)\,,\, 0]$

$\left[\frac{\mathrm{d}(\theta)}{r} + (-\sin(\theta)\, aa + \sin(\theta)\, aa \cos(\theta)^2)\, \mathrm{d}(\phi)\,,\, \frac{\mathrm{d}(r)}{r} + (-r\cos(\theta)\, aa + r\cos(\theta)^3\, aa)\, \mathrm{d}(\phi)\,, \right.$

$\left. (\sin(\theta)\, aa - \sin(\theta)\, aa \cos(\theta)^2)\, \mathrm{d}(r) + (r\cos(\theta)\, aa - r\cos(\theta)^3\, aa)\, \mathrm{d}(\theta) - \sin(\theta)\cos(\theta)\, \mathrm{d}(\phi)\,,\, 0 \right]$

$\left[\frac{\mathrm{d}(\phi)}{r}\,,\, \frac{\cos(\theta)\, \mathrm{d}(\phi)}{\sin(\theta)}\,,\, \frac{\mathrm{d}(r)}{r} + \frac{\cos(\theta)\, \mathrm{d}(\theta)}{\sin(\theta)}\,,\, 0 \right]$

$[0\,,\, 0\,,\, 0\,,\, 0]$

$[(\cos(\theta)\, aa - \cos(\theta)^3\, aa)\, (\mathrm{d}(r)\, \&^\wedge\, \mathrm{d}(\phi)) + (-r\, aa\sin(\theta) + 3\, r\cos(\theta)^2\, aa\sin(\theta))\, (\mathrm{d}(\theta)\, \&^\wedge\, \mathrm{d}(\phi))\,,$

$-(\mathrm{d}(r)\, \&^\wedge\, \mathrm{d}(\theta)) + (r^2 \cos(\theta)^3\, aa - 2\sin(\theta)^2\, r^2 \cos(\theta)\, aa)\, (\mathrm{d}(\theta)\, \&^\wedge\, \mathrm{d}(\phi))$

$+ 2\sin(\theta)\, r\cos(\theta)^2\, aa\, (\mathrm{d}(r)\, \&^\wedge\, \mathrm{d}(\phi))\,,\, (-1 + \cos(\theta)^2)\, (\mathrm{d}(r)\, \&^\wedge\, \mathrm{d}(\phi)) - 2\, r\cos(\theta)\sin(\theta)\, (\mathrm{d}(\theta)\, \&^\wedge\, \mathrm{d}(\phi))$

$$+ (r \, aa \, \sin(\theta) - r \cos(\theta)^2 \, aa \, \sin(\theta)) \, (\mathrm{d}(\theta) \,\&^\wedge\, \mathrm{d}(r)) , 0]$$

$$\left[-\frac{\mathrm{d}(r) \,\&^\wedge\, \mathrm{d}(\theta)}{r^2} + (-\cos(\theta) \, aa + \cos(\theta)^3 \, aa - 2 \sin(\theta)^2 \, aa \cos(\theta)) \, (\mathrm{d}(\theta) \,\&^\wedge\, \mathrm{d}(\phi)) , \right.$$

$$(-\cos(\theta) \, aa + \cos(\theta)^3 \, aa) \, (\mathrm{d}(r) \,\&^\wedge\, \mathrm{d}(\phi)) + (r \, aa \, \sin(\theta) - 3 \, r \cos(\theta)^2 \, aa \, \sin(\theta)) \, (\mathrm{d}(\theta) \,\&^\wedge\, \mathrm{d}(\phi)) ,$$

$$\left. (-\cos(\theta)^2 + \sin(\theta)^2) \, (\mathrm{d}(\theta) \,\&^\wedge\, \mathrm{d}(\phi)) + 2 \sin(\theta)^2 \, aa \cos(\theta) \, (\mathrm{d}(\theta) \,\&^\wedge\, \mathrm{d}(r)) , 0 \right]$$

$$\left[-\frac{\mathrm{d}(r) \,\&^\wedge\, \mathrm{d}(\phi)}{r^2} , -\frac{(\sin(\theta)^2 + \cos(\theta)^2) \, (\mathrm{d}(\theta) \,\&^\wedge\, \mathrm{d}(\phi))}{\sin(\theta)^2} , 0 , 0 \right]$$

$$[0 , 0 , 0 , 0]$$

Cartan matrix of curvature 2-forms based upon d[C] + [C]^[C]

first compute the [C][C] terms:

```
> OmegaCC11:=simpform(simplify(CGamma11&^CGamma11+CGamma12&^CGamma21+CGamma13&^CGa
  mma31+CGamma14&^CGamma41));OmegaCC12:=simpform(simplify(CGamma11&^CGamma12+CGamm
  a12&^CGamma22+CGamma13&^CGamma32+CGamma14&^CGamma42));OmegaCC13:=simpform(simpli
  fy(CGamma11&^CGamma13+CGamma12&^CGamma23+CGamma13&^CGamma33+CGamma14&^CGamma43))
  ;OmegaCC14:=simpform(simplify(CGamma11&^CGamma14+CGamma12&^CGamma24+CGamma13&^CG
  amma34+CGamma14&^CGamma44));
```

$OmegaCC11 :=$

$$(-\cos(\theta) \, aa + \cos(\theta)^3 \, aa) \, (\mathrm{d}(r) \,\&^\wedge\, \mathrm{d}(\phi)) + (r \, aa \, \sin(\theta) - 3 \, r \cos(\theta)^2 \, aa \, \sin(\theta)) \, (\mathrm{d}(\theta) \,\&^\wedge\, \mathrm{d}(\phi))$$

$OmegaCC12 :=$

$$(-3 \, r^2 \cos(\theta)^3 \, aa + 2 \, r^2 \cos(\theta) \, aa) \, (\mathrm{d}(\theta) \,\&^\wedge\, \mathrm{d}(\phi)) + 2 \sin(\theta) \, r \cos(\theta)^2 \, aa \, (\mathrm{d}(\phi) \,\&^\wedge\, \mathrm{d}(r)) - (\mathrm{d}(\theta) \,\&^\wedge\, \mathrm{d}(r))$$

$OmegaCC13 := 2 \, r \cos(\theta) \sin(\theta) \, (\mathrm{d}(\theta) \,\&^\wedge\, \mathrm{d}(\phi)) + (-1 + \cos(\theta)^2) \, (\mathrm{d}(\phi) \,\&^\wedge\, \mathrm{d}(r))$

$$+ (r \cos(\theta)^2 \, aa \, \sin(\theta) - r \, aa \, \sin(\theta)) \, (\mathrm{d}(\theta) \,\&^\wedge\, \mathrm{d}(r))$$

$$OmegaCC14 := 0$$

```
> OmegaCC21:=simpform(simplify(CGamma21&^CGamma11+CGamma22&^CGamma21+CGamma23&^CGa
  mma31+CGamma24&^CGamma41));OmegaCC22:=simpform(simplify(CGamma21&^CGamma12+CGamm
  a22&^CGamma22+CGamma23&^CGamma32+CGamma24&^CGamma42));OmegaCC23:=simpform(simpli
  fy(CGamma21&^CGamma13+CGamma22&^CGamma23+CGamma23&^CGamma33+CGamma24&^CGamma43))
  ;OmegaCC24:=simpform(simplify(CGamma21&^CGamma14+CGamma22&^CGamma24+CGamma23&^CG
  amma34+CGamma24&^CGamma44));
```

$$OmegaCC21 := \frac{\mathrm{d}(r) \,\&^\wedge\, \mathrm{d}(\theta)}{r^2} + (3 \cos(\theta) \, aa - 3 \cos(\theta)^3 \, aa) \, (\mathrm{d}(\theta) \,\&^\wedge\, \mathrm{d}(\phi))$$

$OmegaCC22 :=$

$$(\cos(\theta) \, aa - \cos(\theta)^3 \, aa) \, (\mathrm{d}(r) \,\&^\wedge\, \mathrm{d}(\phi)) + (-r \, aa \, \sin(\theta) + 3 \, r \cos(\theta)^2 \, aa \, \sin(\theta)) \, (\mathrm{d}(\theta) \,\&^\wedge\, \mathrm{d}(\phi))$$

$$OmegaCC23 := (-1 + 2 \cos(\theta)^2) \, (\mathrm{d}(\theta) \,\&^\wedge\, \mathrm{d}(\phi)) + (-2 \cos(\theta) \, aa + 2 \cos(\theta)^3 \, aa) \, (\mathrm{d}(\theta) \,\&^\wedge\, \mathrm{d}(r))$$

$$OmegaCC24 := 0$$

```
> OmegaCC31:=simpform(simplify(CGamma31&^CGamma11+CGamma32&^CGamma21+CGamma33&^CGa
  mma31+CGamma34&^CGamma41));OmegaCC32:=simpform(simplify(CGamma31&^CGamma12+CGamm
  a32&^CGamma22+CGamma33&^CGamma32+CGamma34&^CGamma42));OmegaCC33:=simpform(simpli
  fy(CGamma31&^CGamma13+CGamma32&^CGamma23+CGamma33&^CGamma33+CGamma34&^CGamma43))
  ;OmegaCC34:=simpform(simplify(CGamma31&^CGamma14+CGamma32&^CGamma24+CGamma33&^CG
  amma34+CGamma34&^CGamma44));
```

$$OmegaCC31 := \frac{\mathrm{d}(r) \,\&^\wedge\, \mathrm{d}(\phi)}{r^2}$$

$$OmegaCC32 := \frac{d(\phi)\; \&^\wedge\; d(\theta)}{-1 + \cos(\theta)^2}$$

$$OmegaCC33 := 0$$

$$OmegaCC34 := 0$$

```
> OmegaCC41:=simpform(simplify(CGamma41&^CGamma11+CGamma42&^CGamma21+CGamma43&^CGa
  mma31+CGamma44&^CGamma41));OmegaCC42:=simpform(simplify(CGamma41&^CGamma12+CGamm
  a42&^CGamma22+CGamma43&^CGamma32+CGamma44&^CGamma42));OmegaCC43:=simpform(simpli
  fy(CGamma41&^CGamma13+CGamma42&^CGamma23+CGamma43&^CGamma33+CGamma44&^CGamma43))
  ;OmegaCC44:=simpform(simplify(CGamma41&^CGamma14+CGamma42&^CGamma24+CGamma43&^CG
  amma34+CGamma44&^CGamma44));
```

$$OmegaCC41 := 0$$

$$OmegaCC42 := 0$$

$$OmegaCC43 := 0$$

$$OmegaCC44 := 0$$

```
> CartanC:=array([[CGamma11,CGamma12,CGamma13,CGamma14],[CGamma21,CGamma22,CGamma2
  3,CGamma24],[CGamma31,CGamma32,CGamma33,CGamma34],[CGamma41,CGamma42,CGamma43,CG
  amma44]]);
```

$CartanC :=$

$[(r\cos(\theta)\,aa - r\cos(\theta)^3\,aa)\,d(\phi)\,,\, -r\,d(\theta) + \sin(\theta)\,r^2\cos(\theta)^2\,aa\,d(\phi)\,,$

$(-r\cos(\theta)\,aa + r\cos(\theta)^3\,aa)\,d(r) - \sin(\theta)\,r^2\cos(\theta)^2\,aa\,d(\theta) + (-r + r\cos(\theta)^2)\,d(\phi)\,,\, 0]$

$\left[\dfrac{d(\theta)}{r} + (-\sin(\theta)\,aa + \sin(\theta)\,aa\cos(\theta)^2)\,d(\phi)\,,\, \dfrac{d(r)}{r} + (-r\cos(\theta)\,aa + r\cos(\theta)^3\,aa)\,d(\phi)\,,\right.$

$\left.(\sin(\theta)\,aa - \sin(\theta)\,aa\cos(\theta)^2)\,d(r) + (r\cos(\theta)\,aa - r\cos(\theta)^3\,aa)\,d(\theta) - \sin(\theta)\cos(\theta)\,d(\phi)\,,\, 0\right]$

$\left[\dfrac{d(\phi)}{r}\,,\, \dfrac{\cos(\theta)\,d(\phi)}{\sin(\theta)}\,,\, \dfrac{d(r)}{r} + \dfrac{\cos(\theta)\,d(\theta)}{\sin(\theta)}\,,\, 0\right]$

$[0\,,\, 0\,,\, 0\,,\, 0]$

```
> OmegaD:=simpform(d(CartanC));
> OMEGAC:=simpform(simplify(array([[OmegaD[1,1]+OmegaCC11,OmegaD[1,2]+OmegaCC12,Om
  egaD[1,3]+OmegaCC13,OmegaD[1,4]+OmegaCC14],[OmegaD[2,1]+OmegaCC21,OmegaD[2,2]+Om
  egaCC22,OmegaD[2,3]+OmegaCC23,OmegaD[2,4]+OmegaCC24],[OmegaD[3,1]+OmegaCC31,Omeg
  aD[3,2]+OmegaCC32,OmegaD[3,3]+OmegaCC33,OmegaD[3,4]+OmegaCC34],[OmegaD[4,1]+Omeg
  aCC41,OmegaD[4,2]+OmegaCC42,OmegaD[4,3]+OmegaCC43,OmegaD[4,4]+OmegaCC44]])));
```

$OmegaD :=$

$[(\cos(\theta)\,aa - \cos(\theta)^3\,aa)\,(d(r)\;\&^\wedge\; d(\phi)) + (-r\,aa\sin(\theta) + 3\,r\cos(\theta)^2\,aa\sin(\theta))\,(d(\theta)\;\&^\wedge\; d(\phi))\,,$

$-(d(r)\;\&^\wedge\; d(\theta)) + (r^2\cos(\theta)^3\,aa - 2\sin(\theta)^2\,r^2\cos(\theta)\,aa)\,(d(\theta)\;\&^\wedge\; d(\phi))$

$+\, 2\sin(\theta)\,r\cos(\theta)^2\,aa\,(d(r)\;\&^\wedge\; d(\phi))\,,\, (-1 + \cos(\theta)^2)\,(d(r)\;\&^\wedge\; d(\phi)) - 2\,r\cos(\theta)\sin(\theta)\,(d(\theta)\;\&^\wedge\; d(\phi))$

$+\, (r\,aa\sin(\theta) - r\cos(\theta)^2\,aa\sin(\theta))\,(d(\theta)\;\&^\wedge\; d(r))\,,\, 0]$

$\left[-\dfrac{d(r)\;\&^\wedge\; d(\theta)}{r^2} + (-\cos(\theta)\,aa + \cos(\theta)^3\,aa - 2\sin(\theta)^2\,aa\cos(\theta))\,(d(\theta)\;\&^\wedge\; d(\phi))\,,\right.$

$(-\cos(\theta)\,aa + \cos(\theta)^3\,aa)\,(d(r)\;\&^\wedge\; d(\phi)) + (r\,aa\sin(\theta) - 3\,r\cos(\theta)^2\,aa\sin(\theta))\,(d(\theta)\;\&^\wedge\; d(\phi))\,,$

$\left.(-\cos(\theta)^2 + \sin(\theta)^2)\,(d(\theta)\;\&^\wedge\; d(\phi)) + 2\sin(\theta)^2\,aa\cos(\theta)\,(d(\theta)\;\&^\wedge\; d(r))\,,\, 0\right]$

$$\left[-\frac{d(r) \,\&\wedge\, d(\phi)}{r^2} , -\frac{(\sin(\theta)^2 + \cos(\theta)^2)\,(d(\theta) \,\&\wedge\, d(\phi))}{\sin(\theta)^2} , 0 , 0 \right]$$

$$[0 , 0 , 0 , 0]$$

$$OMEGAC := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

OMEGA above IS THE MATRIX OF CURVATURE 2-FORMS.
IT IS A ZERO MATRIX FOR A CARTAN CONNECTION OF AN INVERTIBLE
FRAME FIELD.

```
> TRACEOMEGA_noTorsion_smallmass:=subs(aa=0,M^2=0,2*r+M=2*r,simpform(wcollect(OMEG
  AC[1,1]+OMEGAC[2,2]+OMEGAC[3,3]+OMEGAC[4,4])));JJ:=subs(d(TRACEOMEGA_noTorsion_s
  mallmass));
```

$$TRACEOMEGA\_noTorsion\_smallmass := 0$$

$$JJ := 0$$

```
> TRACEOMEGA_smallTorsion_nomass:=subs(aa^2=0,M=0,simpform(wcollect(OMEGAC[1,1]+OM
  EGAC[2,2]+OMEGAC[3,3]+OMEGAC[4,4])));JJ:=subs(d(TRACEOMEGA_smallTorsion_nomass))
  ;
```

$$TRACEOMEGA\_smallTorsion\_nomass := 0$$

$$JJ := 0$$

```
> TRACEOMEGA:=((subs(aa^2=0,M^2=0,2*r+M=2*r,simpform(wcollect(OMEGAC[1,1]+OMEGAC[2
  ,2]+OMEGAC[3,3]+OMEGAC[4,4]))))));JJ:=simpform(subs(M^2=0,d(TRACEOMEGA)));
```

$$TRACEOMEGA := 0$$

$$JJ := 0$$

THE TRACE OF THE CARTAN MATRIX OF 2-FORMS IS ZERO IN ALL APPROXIMATIONS.

```
>
> dtheta_dphi:=simplify(factor(subs(aa^2=0,(getcoeff(TRACEOMEGA&^d(r)&^d(tau))))))
  ;dr_dphi:=simplify(factor(subs(aa^2=0,(getcoeff(TRACEOMEGA&^d(theta)&^d(tau))))
  ));dr_dtheta:=simplify(factor(subs(aa^2=0,(getcoeff(TRACEOMEGA&^d(phi)&^d(tau)))
  )));simplify(factor(subs(aa^2=0,(getcoeff(TRACEOMEGA&^d(phi)&^d(theta))))));simpl
  ify(factor(subs(aa^2=0,(getcoeff(TRACEOMEGA&^d(phi)&^d(r))))));simplify(factor(s
  ubs(aa^2=0,(getcoeff(TRACEOMEGA&^d(theta)&^d(r))))));
```

$$dtheta\_dphi := 0$$

$$dr\_dphi := 0$$

$$dr\_dtheta := 0$$

$$0$$

$$0$$

$$0$$

The matrix of curvature 2-forms (based upon d[C]+[C]^[C])for the right
Cartan connection vanishes - and has zero trace.

```
>
```
****

NOW PERTURB THE METRIC ON THE FINAL STATE AND USE THE
COMPATIBILITY CONDITION (the metric is defined as a cotensor) TO

YIELD A PULLBACK METRIC ON THE INITIAL STATE.

The final state metric is perturbed as follows:

```
> Sch1:=(1+M/(2*r))^4;Sch2:=((1-0*M/(2*r))/(1+0*M/(2*r)))^2;
> finalmetric := array([[-Sch1, 0, 0,0], [0,-Sch1, 0,0], [0,
  0,-Sch1,0],[0,0,0,Sch2]]);
```

$$Sch1 := \left(1 + \frac{1}{2}\frac{M}{r}\right)^4$$

$$Sch2 := 1$$

$$finalmetric := \begin{bmatrix} -\left(1 + \frac{1}{2}\frac{M}{r}\right)^4 & 0 & 0 & 0 \\ 0 & -\left(1 + \frac{1}{2}\frac{M}{r}\right)^4 & 0 & 0 \\ 0 & 0 & -\left(1 + \frac{1}{2}\frac{M}{r}\right)^4 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

THE perturbed PULLBACK METRIC COEFFICIENTS ARE:

```
> pullbackmetric:=simplify(innerprod(transpose(FFI),finalmetric,FFI),trig):g11:=fa
  ctor(pullbackmetric[1,1]);g12:=factor(pullbackmetric[1,2]);g13:=factor(pullbackm
  etric[1,3]);g14:=factor(pullbackmetric[1,4]);g21:=factor(pullbackmetric[2,1]);g2
  2:=factor(pullbackmetric[2,2]);g23:=factor(pullbackmetric[2,3]);g24:=factor(pull
  backmetric[2,4]);g31:=factor(pullbackmetric[3,1]);g32:=factor(pullbackmetric[3,2
  ]);g33:=(factor(pullbackmetric[3,3]));g34:=factor(pullbackmetric[3,4]);g41:=fact
  or(pullbackmetric[4,1]);g42:=factor(pullbackmetric[4,2]);g43:=factor(pullbackmet
  ric[4,3]);g44:=factor(pullbackmetric[4,4]);
```

$$g11 := -\frac{1}{16}\frac{(2\,r + M)^4}{r^4}$$

$$g12 := 0$$

$$g13 := -\frac{1}{16}\frac{\cos(\theta)\,aa\,(2\,r + M)^4\,(\cos(\theta) - 1)\,(\cos(\theta) + 1)}{r^2}$$

$$g14 := 0$$

$$g21 := 0$$

$$g22 := -\frac{1}{16}\frac{(2\,r + M)^4}{r^2}$$

$$g23 := \frac{1}{16}\frac{\sin(\theta)\,aa\,(2\,r + M)^4\,(\cos(\theta) - 1)\,(\cos(\theta) + 1)}{r}$$

$$g24 := 0$$

$$g31 := -\frac{1}{16}\frac{\cos(\theta)\,aa\,(2\,r + M)^4\,(\cos(\theta) - 1)\,(\cos(\theta) + 1)}{r^2}$$

$$g32 := \frac{1}{16}\frac{\sin(\theta)\,aa\,(2\,r + M)^4\,(\cos(\theta) - 1)\,(\cos(\theta) + 1)}{r}$$

$$g33 := -\frac{1}{16}\frac{(2\,r + M)^4\,(\cos(\theta) - 1)\,(\cos(\theta) + 1)\,(r^2\cos(\theta)^2\,aa^2 - r^2\,aa^2 - 1)}{r^2}$$

$$g34 := 0$$

$$g41 := 0$$
$$g42 := 0$$
$$g43 := 0$$
$$g44 := 1$$

NEXT, obtain the Christoffel Connection coefficients from the induced perturbed pullback metric.

```
> metric:= simplify(subs(evalm(pullbackmetric)));
```

$metric :=$

$$\left[-\frac{1}{16}\frac{16\,r^4 + 32\,r^3 M + 24\,r^2 M^2 + 8\,r M^3 + M^4}{r^4},\, 0,\, -\frac{1}{16}\cos(\theta)\,aa\,(-16\,r^4 + 16\,r^4\cos(\theta)^2 - 32\,r^3 M \right.$$
$$\left. + 32\,r^3 M\cos(\theta)^2 - 24\,r^2 M^2 + 24\,r^2 M^2\cos(\theta)^2 - 8\,r M^3 + 8\,r M^3\cos(\theta)^2 - M^4 + M^4\cos(\theta)^2) \Big/ r^2,\, 0 \right]$$

$$\left[0,\, -\frac{1}{16}\frac{16\,r^4 + 32\,r^3 M + 24\,r^2 M^2 + 8\,r M^3 + M^4}{r^2},\, \frac{1}{16}\sin(\theta)\,aa\,(-16\,r^4 + 16\,r^4\cos(\theta)^2 - 32\,r^3 M \right.$$
$$\left. + 32\,r^3 M\cos(\theta)^2 - 24\,r^2 M^2 + 24\,r^2 M^2\cos(\theta)^2 - 8\,r M^3 + 8\,r M^3\cos(\theta)^2 - M^4 + M^4\cos(\theta)^2) / r,\, 0 \right]$$

$$\left[-\frac{1}{16}\cos(\theta)\,aa\,(-16\,r^4 + 16\,r^4\cos(\theta)^2 - 32\,r^3 M + 32\,r^3 M\cos(\theta)^2 - 24\,r^2 M^2 + 24\,r^2 M^2\cos(\theta)^2 - 8\,r M^3 \right.$$
$$+ 8\,r M^3\cos(\theta)^2 - M^4 + M^4\cos(\theta)^2) \Big/ r^2,\, \frac{1}{16}\sin(\theta)\,aa\,(-16\,r^4 + 16\,r^4\cos(\theta)^2 - 32\,r^3 M + 32\,r^3 M\cos(\theta)^2$$
$$- 24\,r^2 M^2 + 24\,r^2 M^2\cos(\theta)^2 - 8\,r M^3 + 8\,r M^3\cos(\theta)^2 - M^4 + M^4\cos(\theta)^2) / r,\, -\frac{1}{16}(24\,r^4 M^2\,aa^2\cos(\theta)^4 + 16\,r^4$$
$$+ 32\,r^3 M + 24\,r^2 M^2 + 8\,r M^3 + M^4 - 24\,r^2 M^2\cos(\theta)^2 - 16\,r^3 M^3\,aa^2\cos(\theta)^2 - 64\,r^5 M\,aa^2\cos(\theta)^2$$
$$+ 32\,r^5 M\,aa^2\cos(\theta)^4 - 48\,r^4 M^2\,aa^2\cos(\theta)^2 + 8\,r^3 M^3\,aa^2\cos(\theta)^4 + M^4 r^2\,aa^2\cos(\theta)^4 - 2\,M^4 r^2\,aa^2\cos(\theta)^2$$
$$+ 16\,r^6\,aa^2 - 16\,r^4\cos(\theta)^2 - 8\,r M^3\cos(\theta)^2 - 32\,r^3 M\cos(\theta)^2 - M^4\cos(\theta)^2 + 16\,r^6\,aa^2\cos(\theta)^4 - 32\,r^6\,aa^2\cos(\theta)^2$$
$$\left. + M^4 r^2\,aa^2 + 8\,r^3 M^3\,aa^2 + 24\,r^4 M^2\,aa^2 + 32\,r^5 M\,aa^2) \Big/ r^2,\, 0 \right]$$

$$[0,0,0,1]$$

```
> metricinverse:=inverse(metric):dim:=4;coord:=[r,theta,phi,tau];
```

$$dim := 4$$
$$coord := [r, \theta, \phi, \tau]$$

```
> for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do
  d1gun[i,j,k] := (diff(metric[i,j],coord[k])) od od od:
> #for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do if
  d1gun[i,j,k]=0 then  else print(`dgun`(i,j,k)=d1gun[i,j,k]) fi od od od;
> for i from 1 to dim do for j from i to dim do for k from 1 to dim do C1S[i,j,k]
  := 0  od od od; for i from 1 to dim do for j from 1 to dim do for k from 1 to
  dim do C1S[i,j,k] := 1/2*d1gun[i,k,j]+1/2*d1gun[j,k,i]-1/2*d1gun[i,j,k]  od od
  od;
>  for k from 1 to dim do for i from 1 to dim do for j from 1 to dim do ss := 0;
  for m to dim do ss := ss+metricinverse[k,m]*C1S[i,j,m] od; C2S[k,i,j] :=
  simplify(factor(ss),trig) od od od;
```

```
> for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do if
  C2S[i,j,k]=0 then  else
  print(`Christoffel_Gamma2`(i,j,k)=simplify(subs(-1+cos(theta)^2=-sin(theta)^2,(C
  2S[i,j,k])))) fi od od od;
```

The non zero Christoffel Connection coefficients 2nd kind for the perturbed frame and perturbed metric, on the initial space (domain) are:

Gamma2(i,j,k) index (1,-1,-1)

$$\text{Christoffel\_Gamma2}(1, 1, 1) = 2\,\frac{-2\,r^3\,aa^2\cos(\theta)^2 + 2\,r^3\,aa^2\cos(\theta)^4 - M}{r\,(2\,r+M)}$$

$$\text{Christoffel\_Gamma2}(1, 1, 2) = \cos(\theta)\,aa^2\,r^2\sin(\theta) - 2\cos(\theta)^3\,aa^2\,r^2\sin(\theta)$$

$$\text{Christoffel\_Gamma2}(1, 1, 3) = -\,r\,aa\,\cos(\theta)\,(-2\,r^3\,aa^2 + 4\,r^3\,aa^2\cos(\theta)^2 - 2\,r^3\,aa^2\cos(\theta)^4 - 5\,r^2\,M\,aa^2\cos(\theta)^4$$
$$+ 2\,r^2\,aa^2\cos(\theta)^6\,M - r^2\,M\,aa^2 + 4\,r^2\,M\,aa^2\cos(\theta)^2 + 2\,r\cos(\theta)^2 - 2\,r + M - M\cos(\theta)^2)\,/\,(2\,r+M)$$

$$\text{Christoffel\_Gamma2}(1, 2, 1) = \cos(\theta)\,aa^2\,r^2\sin(\theta) - 2\cos(\theta)^3\,aa^2\,r^2\sin(\theta)$$

$$\text{Christoffel\_Gamma2}(1, 2, 2) =$$
$$-\,\frac{r\,(-4\,r^3\,aa^2\cos(\theta)^2 + 4\,r^3\,aa^2\cos(\theta)^4 + 4\,r^2\,M\,aa^2\cos(\theta)^4 - 4\,r^2\,M\,aa^2\cos(\theta)^2 + 2\,r - M)}{2\,r+M}$$

$$\text{Christoffel\_Gamma2}(1, 2, 3) = r^2\sin(\theta)\,aa\,(2\,r^3\,aa^2\cos(\theta)^2 - 2\,r^3\,aa^2\cos(\theta)^4 + 3\,r^2\,M\,aa^2\cos(\theta)^2$$
$$- 5\,r^2\,M\,aa^2\cos(\theta)^4 + 2\,r^2\,aa^2\cos(\theta)^6\,M + 2\,r\cos(\theta)^2 - 2\,r + M - M\cos(\theta)^2)\,/\,(2\,r+M)$$

$$\text{Christoffel\_Gamma2}(1, 3, 1) = -\,r\,aa\,\cos(\theta)\,(-2\,r^3\,aa^2 + 4\,r^3\,aa^2\cos(\theta)^2 - 2\,r^3\,aa^2\cos(\theta)^4 - 5\,r^2\,M\,aa^2\cos(\theta)^4$$
$$+ 2\,r^2\,aa^2\cos(\theta)^6\,M - r^2\,M\,aa^2 + 4\,r^2\,M\,aa^2\cos(\theta)^2 + 2\,r\cos(\theta)^2 - 2\,r + M - M\cos(\theta)^2)\,/\,(2\,r+M)$$

$$\text{Christoffel\_Gamma2}(1, 3, 2) = r^2\sin(\theta)\,aa\,(2\,r^3\,aa^2\cos(\theta)^2 - 2\,r^3\,aa^2\cos(\theta)^4 + 3\,r^2\,M\,aa^2\cos(\theta)^2$$
$$- 5\,r^2\,M\,aa^2\cos(\theta)^4 + 2\,r^2\,aa^2\cos(\theta)^6\,M + 2\,r\cos(\theta)^2 - 2\,r + M - M\cos(\theta)^2)\,/\,(2\,r+M)$$

$$\text{Christoffel\_Gamma2}(1, 3, 3) = -\,r\,(-6\,r^4\cos(\theta)^6\,aa^4\,M + 2\,r^4\cos(\theta)^8\,aa^4\,M - 2\,r^4\cos(\theta)^2\,aa^4\,M$$
$$+ 6\,r^4\cos(\theta)^4\,aa^4\,M - 2\,r^2\,aa^2\cos(\theta)^6\,M + 4\,r^3\,aa^2 - M - 2\,r\cos(\theta)^2 + 2\,r - 2\,r^2\,M\,aa^2\cos(\theta)^2 + 4\,r^3\,aa^2\cos(\theta)^4$$
$$+ M\cos(\theta)^2 - 8\,r^3\,aa^2\cos(\theta)^2 + 4\,r^2\,M\,aa^2\cos(\theta)^4)\,/\,(2\,r+M)$$

$$\text{Christoffel\_Gamma2}(2, 1, 1) = -4\,\frac{(-1+\cos(\theta)^2)\,r\,aa^2\cos(\theta)\sin(\theta)}{2\,r+M}$$

$$\text{Christoffel\_Gamma2}(2, 1, 2) =$$
$$-\,\frac{-6\,r^3\,aa^2\cos(\theta)^2 + 2\,r^3\,aa^2 + 4\,r^3\,aa^2\cos(\theta)^4 - 3\,r^2\,M\,aa^2\cos(\theta)^2 + 2\,r^2\,M\,aa^2\cos(\theta)^4 + r^2\,M\,aa^2 - 2\,r + M}{r\,(2\,r+M)}$$

$$\text{Christoffel\_Gamma2}(2, 1, 3) = ((4\,r^3\,aa^2\cos(\theta)^2 - 2\,r^3\,aa^2 - 2\,r^3\,aa^2\cos(\theta)^4 + 2\,r^2\,aa^2\cos(\theta)^6\,M - r^2\,M\,aa^2$$
$$- 5\,r^2\,M\,aa^2\cos(\theta)^4 + 4\,r^2\,M\,aa^2\cos(\theta)^2 + 2\,r\cos(\theta)^2 + M\cos(\theta)^2)\sin(\theta)\,aa)\,/\,(2\,r+M)$$

$$\text{Christoffel\_Gamma2}(2, 2, 1) =$$
$$-\,\frac{-6\,r^3\,aa^2\cos(\theta)^2 + 2\,r^3\,aa^2 + 4\,r^3\,aa^2\cos(\theta)^4 - 3\,r^2\,M\,aa^2\cos(\theta)^2 + 2\,r^2\,M\,aa^2\cos(\theta)^4 + r^2\,M\,aa^2 - 2\,r + M}{r\,(2\,r+M)}$$

$$\text{Christoffel\_Gamma2}(2, 2, 2) = 4\,\frac{\cos(\theta)\,aa^2\,r^2\sin(\theta)\,(-r + r\cos(\theta)^2 + M\cos(\theta)^2 - M)}{2\,r+M}$$

$$\text{Christoffel\_Gamma2}(2, 2, 3) = r\,aa\,\cos(\theta)\,(-2\,r^3\,aa^2 + 4\,r^3\,aa^2\cos(\theta)^2 - 2\,r^3\,aa^2\cos(\theta)^4 - 3\,r^2\,M\,aa^2$$
$$+ 8\,r^2\,M\,aa^2\cos(\theta)^2 - 7\,r^2\,M\,aa^2\cos(\theta)^4 + 2\,r^2\,aa^2\cos(\theta)^6\,M - 2\,r + 2\,r\cos(\theta)^2 - M + M\cos(\theta)^2)\,/\,(2\,r+M)$$

Christoffel_Gamma2($2, 3, 1$) = (($4\, r^3\, aa^2 \cos(\theta)^2 - 2\, r^3\, aa^2 - 2\, r^3\, aa^2 \cos(\theta)^4 + 2\, r^2\, aa^2 \cos(\theta)^6\, M - r^2\, M\, aa^2$

$\quad - 5\, r^2\, M\, aa^2 \cos(\theta)^4 + 4\, r^2\, M\, aa^2 \cos(\theta)^2 + 2\, r \cos(\theta)^2 + M \cos(\theta)^2)\, \sin(\theta)\, aa$) / ($2\, r + M$)

Christoffel_Gamma2($2, 3, 2$) = $r\, aa \cos(\theta)\, (-2\, r^3\, aa^2 + 4\, r^3\, aa^2 \cos(\theta)^2 - 2\, r^3\, aa^2 \cos(\theta)^4 - 3\, r^2\, M\, aa^2$

$\quad + 8\, r^2\, M\, aa^2 \cos(\theta)^2 - 7\, r^2\, M\, aa^2 \cos(\theta)^4 + 2\, r^2\, aa^2 \cos(\theta)^6\, M - 2\, r + 2\, r \cos(\theta)^2 - M + M \cos(\theta)^2)$ / ($2\, r + M$)

Christoffel_Gamma2($2, 3, 3$) = (($2\, r^4 \cos(\theta)^6\, aa^4\, M - 2\, r^4\, aa^4\, M - 6\, r^4 \cos(\theta)^4\, aa^4\, M + 6\, r^4 \cos(\theta)^2\, aa^4\, M$

$\quad + 4\, r^3\, aa^2 \cos(\theta)^2 - 4\, r^3\, aa^2 - 4\, r^2\, M\, aa^2 - 2\, r^2\, M\, aa^2 \cos(\theta)^4 + 6\, r^2\, M\, aa^2 \cos(\theta)^2 - 2\, r - M) \cos(\theta) \sin(\theta)$) / (

$\quad 2\, r + M$)

$$\text{Christoffel\_Gamma2}(3, 1, 1) = -4\, \frac{\cos(\theta)\, aa}{2\, r + M}$$

$$\text{Christoffel\_Gamma2}(3, 1, 2) = -\frac{aa\, (-1 + 2 \cos(\theta)^2)}{\sin(\theta)}$$

Christoffel_Gamma2($3, 1, 3$) =

$$\frac{-2\, r^3\, aa^2 \cos(\theta)^2 - 3\, r^2\, M\, aa^2 \cos(\theta)^2 + 2\, r^2\, M\, aa^2 \cos(\theta)^4 + 2\, r^3\, aa^2 + r^2\, M\, aa^2 + 2\, r - M}{r\, (2\, r + M)}$$

$$\text{Christoffel\_Gamma2}(3, 2, 1) = -\frac{aa\, (-1 + 2 \cos(\theta)^2)}{\sin(\theta)}$$

$$\text{Christoffel\_Gamma2}(3, 2, 2) = 4\, \frac{(r + M) \cos(\theta)\, r\, aa}{2\, r + M}$$

Christoffel_Gamma2($3, 2, 3$) =

$$-\frac{\sin(\theta) \cos(\theta)\, (2\, r^3\, aa^2 - 2\, r^3\, aa^2 \cos(\theta)^2 + 2\, r^2\, M\, aa^2 \cos(\theta)^4 + 3\, r^2\, M\, aa^2 - 5\, r^2\, M\, aa^2 \cos(\theta)^2 + 2\, r + M)}{2\, r \cos(\theta)^2 - 2\, r - M + M \cos(\theta)^2}$$

Christoffel_Gamma2($3, 3, 1$) =

$$\frac{-2\, r^3\, aa^2 \cos(\theta)^2 - 3\, r^2\, M\, aa^2 \cos(\theta)^2 + 2\, r^2\, M\, aa^2 \cos(\theta)^4 + 2\, r^3\, aa^2 + r^2\, M\, aa^2 + 2\, r - M}{r\, (2\, r + M)}$$

Christoffel_Gamma2($3, 3, 2$) =

$$-\frac{\sin(\theta) \cos(\theta)\, (2\, r^3\, aa^2 - 2\, r^3\, aa^2 \cos(\theta)^2 + 2\, r^2\, M\, aa^2 \cos(\theta)^4 + 3\, r^2\, M\, aa^2 - 5\, r^2\, M\, aa^2 \cos(\theta)^2 + 2\, r + M)}{2\, r \cos(\theta)^2 - 2\, r - M + M \cos(\theta)^2}$$

$$\text{Christoffel\_Gamma2}(3, 3, 3) = 2\, \frac{\cos(\theta)\, r\, aa\, M\, (r^2 \cos(\theta)^4\, aa^2 - 2\, r^2 \cos(\theta)^2\, aa^2 + r^2\, aa^2 + 1 - \cos(\theta)^2)}{2\, r + M}$$

>

Now compute the matrix elements of the matrix of connection 1-forms based upon the Christoffel connection

```
> Gamma11:=C2S[1,1,1]*d(r)+C2S[1,1,2]*d(theta)+C2S[1,1,3]*d(phi)+C2S[1,1,4]*d(tau)
  ;
> Gamma12:=C2S[1,2,1]*d(r)+C2S[1,2,2]*d(theta)+C2S[1,2,3]*d(phi)+C2S[1,2,4]*d(tau)
  ;
> Gamma13:=((C2S[1,3,1]*d(r)+C2S[1,3,2]*d(theta)+C2S[1,3,3]*d(phi)+C2S[1,3,4]*d(ta
  u)));
> Gamma14:=C2S[1,4,1]*d(r)+C2S[1,4,2]*d(theta)+C2S[1,4,3]*d(phi)+C2S[1,4,4]*d(tau)
  ;
```

$\Gamma 11 := 2\, \dfrac{(-2\, r^3\, aa^2 \cos(\theta)^2 + 2\, r^3\, aa^2 \cos(\theta)^4 - M)\, \mathrm{d}(r)}{r\, (2\, r + M)} + (\cos(\theta)\, aa^2\, r^2 \sin(\theta) - 2 \cos(\theta)^3\, aa^2\, r^2 \sin(\theta))\, \mathrm{d}(\theta) -$

$\quad r\, aa \cos(\theta)\, (-2\, r^3\, aa^2 + 4\, r^3\, aa^2 \cos(\theta)^2 - 2\, r^3\, aa^2 \cos(\theta)^4 - 5\, r^2\, M\, aa^2 \cos(\theta)^4 + 2\, r^2\, aa^2 \cos(\theta)^6\, M - r^2\, M\, aa^2$

$+ 4\,r^2\,M\,aa^2\,\cos(\theta)^2 + 2\,r\cos(\theta)^2 - 2\,r + M - M\cos(\theta)^2)\,\mathrm{d}(\phi)\,/\,(2\,r+M)$

$\Gamma 12 := (\cos(\theta)\,aa^2\,r^2\,\sin(\theta) - 2\cos(\theta)^3\,aa^2\,r^2\,\sin(\theta))\,\mathrm{d}(r)$

$\quad - \dfrac{r\,(-4\,r^3\,aa^2\,\cos(\theta)^2 + 4\,r^3\,aa^2\,\cos(\theta)^4 + 4\,r^2\,M\,aa^2\,\cos(\theta)^4 - 4\,r^2\,M\,aa^2\,\cos(\theta)^2 + 2\,r - M)\,\mathrm{d}(\theta)}{2\,r+M} + r^2\,\sin(\theta)\,aa\,($

$\quad 2\,r^3\,aa^2\,\cos(\theta)^2 - 2\,r^3\,aa^2\,\cos(\theta)^4 + 3\,r^2\,M\,aa^2\,\cos(\theta)^2 - 5\,r^2\,M\,aa^2\,\cos(\theta)^4 + 2\,r^2\,aa^2\,\cos(\theta)^6\,M + 2\,r\cos(\theta)^2 - 2\,r$

$\quad + M - M\cos(\theta)^2)\,\mathrm{d}(\phi)\,/\,(2\,r+M)$

$\Gamma 13 := -\,r\,aa\,\cos(\theta)\,(-2\,r^3\,aa^2 + 4\,r^3\,aa^2\,\cos(\theta)^2 - 2\,r^3\,aa^2\,\cos(\theta)^4 - 5\,r^2\,M\,aa^2\,\cos(\theta)^4 + 2\,r^2\,aa^2\,\cos(\theta)^6\,M$

$\quad - r^2\,M\,aa^2 + 4\,r^2\,M\,aa^2\,\cos(\theta)^2 + 2\,r\cos(\theta)^2 - 2\,r + M - M\cos(\theta)^2)\,\mathrm{d}(r)\,/\,(2\,r+M) + r^2\,\sin(\theta)\,aa\,($

$\quad 2\,r^3\,aa^2\,\cos(\theta)^2 - 2\,r^3\,aa^2\,\cos(\theta)^4 + 3\,r^2\,M\,aa^2\,\cos(\theta)^2 - 5\,r^2\,M\,aa^2\,\cos(\theta)^4 + 2\,r^2\,aa^2\,\cos(\theta)^6\,M + 2\,r\cos(\theta)^2 - 2\,r$

$\quad + M - M\cos(\theta)^2)\,\mathrm{d}(\theta)\,/\,(2\,r+M) - r\,(-6\,r^4\,\cos(\theta)^6\,aa^4\,M + 2\,r^4\,\cos(\theta)^8\,aa^4\,M - 2\,r^4\,\cos(\theta)^2\,aa^4\,M$

$\quad + 6\,r^4\,\cos(\theta)^4\,aa^4\,M - 2\,r^2\,aa^2\,\cos(\theta)^6\,M + 4\,r^3\,aa^2 - M - 2\,r\cos(\theta)^2 + 2\,r - 2\,r^2\,M\,aa^2\,\cos(\theta)^2 + 4\,r^3\,aa^2\,\cos(\theta)^4$

$\quad + M\cos(\theta)^2 - 8\,r^3\,aa^2\,\cos(\theta)^2 + 4\,r^2\,M\,aa^2\,\cos(\theta)^4)\,\mathrm{d}(\phi)\,/\,(2\,r+M)$

$$\Gamma 14 := 0$$

```
> Gamma21:=C2S[2,1,1]*d(r)+C2S[2,1,2]*d(theta)+C2S[2,1,3]*d(phi)+C2S[2,1,4]*d(tau)
  ;
> Gamma22:=C2S[2,2,1]*d(r)+C2S[2,2,2]*d(theta)+C2S[2,2,3]*d(phi)+C2S[2,2,4]*d(tau)
  ;
> Gamma23:=C2S[2,3,1]*d(r)+C2S[2,3,2]*d(theta)+C2S[2,3,3]*d(phi)+C2S[2,3,4]*d(tau)
  ;
> Gamma24:=C2S[2,4,1]*d(r)+C2S[2,4,2]*d(theta)+C2S[2,4,3]*d(phi)+C2S[2,4,4]*d(tau)
  ;
```

$\Gamma 21 := -4\,\dfrac{(-1+\cos(\theta)^2)\,r\,aa^2\,\cos(\theta)\,\sin(\theta)\,\mathrm{d}(r)}{2\,r+M} - ($

$\quad (-6\,r^3\,aa^2\,\cos(\theta)^2 + 2\,r^3\,aa^2 + 4\,r^3\,aa^2\,\cos(\theta)^4 - 3\,r^2\,M\,aa^2\,\cos(\theta)^2 + 2\,r^2\,M\,aa^2\,\cos(\theta)^4 + r^2\,M\,aa^2 - 2\,r + M)\,\mathrm{d}(\theta)$

$\quad)\,/\,(r\,(2\,r+M)) + ((4\,r^3\,aa^2\,\cos(\theta)^2 - 2\,r^3\,aa^2 - 2\,r^3\,aa^2\,\cos(\theta)^4 + 2\,r^2\,aa^2\,\cos(\theta)^6\,M - r^2\,M\,aa^2$

$\quad - 5\,r^2\,M\,aa^2\,\cos(\theta)^4 + 4\,r^2\,M\,aa^2\,\cos(\theta)^2 + 2\,r\cos(\theta)^2 + M\cos(\theta)^2)\,\sin(\theta)\,aa\,\mathrm{d}(\phi))\,/\,(2\,r+M)$

$\Gamma 22 := -\,($

$\quad (-6\,r^3\,aa^2\,\cos(\theta)^2 + 2\,r^3\,aa^2 + 4\,r^3\,aa^2\,\cos(\theta)^4 - 3\,r^2\,M\,aa^2\,\cos(\theta)^2 + 2\,r^2\,M\,aa^2\,\cos(\theta)^4 + r^2\,M\,aa^2 - 2\,r + M)\,\mathrm{d}(r)$

$\quad)\,/\,(r\,(2\,r+M)) + 4\,\dfrac{\cos(\theta)\,aa^2\,r^2\,\sin(\theta)\,(-r + r\cos(\theta)^2 + M\cos(\theta)^2 - M)\,\mathrm{d}(\theta)}{2\,r+M} + r\,aa\,\cos(\theta)\,(-2\,r^3\,aa^2$

$\quad + 4\,r^3\,aa^2\,\cos(\theta)^2 - 2\,r^3\,aa^2\,\cos(\theta)^4 - 3\,r^2\,M\,aa^2 + 8\,r^2\,M\,aa^2\,\cos(\theta)^2 - 7\,r^2\,M\,aa^2\,\cos(\theta)^4 + 2\,r^2\,aa^2\,\cos(\theta)^6\,M$

$\quad - 2\,r + 2\,r\cos(\theta)^2 - M + M\cos(\theta)^2)\,\mathrm{d}(\phi)\,/\,(2\,r+M)$

$\Gamma 23 := ((4\,r^3\,aa^2\,\cos(\theta)^2 - 2\,r^3\,aa^2 - 2\,r^3\,aa^2\,\cos(\theta)^4 + 2\,r^2\,aa^2\,\cos(\theta)^6\,M - r^2\,M\,aa^2 - 5\,r^2\,M\,aa^2\,\cos(\theta)^4$

$\quad + 4\,r^2\,M\,aa^2\,\cos(\theta)^2 + 2\,r\cos(\theta)^2 + M\cos(\theta)^2)\,\sin(\theta)\,aa\,\mathrm{d}(r))\,/\,(2\,r+M) + r\,aa\,\cos(\theta)\,(-2\,r^3\,aa^2$

$\quad + 4\,r^3\,aa^2\,\cos(\theta)^2 - 2\,r^3\,aa^2\,\cos(\theta)^4 - 3\,r^2\,M\,aa^2 + 8\,r^2\,M\,aa^2\,\cos(\theta)^2 - 7\,r^2\,M\,aa^2\,\cos(\theta)^4 + 2\,r^2\,aa^2\,\cos(\theta)^6\,M$

$\quad - 2\,r + 2\,r\cos(\theta)^2 - M + M\cos(\theta)^2)\,\mathrm{d}(\theta)\,/\,(2\,r+M) + ((2\,r^4\,\cos(\theta)^6\,aa^4\,M - 2\,r^4\,aa^4\,M - 6\,r^4\,\cos(\theta)^4\,aa^4\,M$

$\quad + 6\,r^4\,\cos(\theta)^2\,aa^4\,M + 4\,r^3\,aa^2\,\cos(\theta)^2 - 4\,r^3\,aa^2 - 4\,r^2\,M\,aa^2 - 2\,r^2\,M\,aa^2\,\cos(\theta)^4 + 6\,r^2\,M\,aa^2\,\cos(\theta)^2 - 2\,r - M)$

$\quad \cos(\theta)\,\sin(\theta)\,\mathrm{d}(\phi))\,/\,(2\,r+M)$

$$\Gamma 24 := 0$$

```
> Gamma31:=C2S[3,1,1]*d(r)+C2S[3,1,2]*d(theta)+C2S[3,1,3]*d(phi)+C2S[3,1,4]*d(tau)
  ;
> Gamma32:=simpform(simplify(C2S[3,2,1]*d(r)+C2S[3,2,2]*d(theta)+C2S[3,2,3]*d(phi)
  +C2S[3,2,4]*d(tau)));
```

```
> Gamma33:=simpform((wcollect(simplify(C2S[3,3,1]*d(r)+simplify(C2S[3,3,2])*d(thet
  a)+simplify(C2S[3,3,3])*d(phi)+C2S[3,3,4]*d(tau))))));
> Gamma34:=C2S[3,4,1]*d(r)+C2S[3,4,2]*d(theta)+C2S[3,4,3]*d(phi)+C2S[3,4,4]*d(tau)
  ;subs(-1+cos(theta)^2=-sin(theta),C2S[3,2,3]);
>
```

$$\Gamma 31 := -4\,\frac{\cos(\theta)\,aa\,\mathrm{d}(r)}{2\,r+M} + \frac{\sin(\theta)\,aa\,(-1+2\,\cos(\theta)^2)\,\mathrm{d}(\theta)}{-1+\cos(\theta)^2}$$

$$+\frac{(-2\,r^3\,aa^2\,\cos(\theta)^2 - 3\,r^2\,M\,aa^2\,\cos(\theta)^2 + 2\,r^2\,M\,aa^2\,\cos(\theta)^4 + 2\,r^3\,aa^2 + r^2\,M\,aa^2 + 2\,r - M)\,\mathrm{d}(\phi)}{r\,(2\,r+M)}$$

$$\Gamma 32 := \frac{\sin(\theta)\,aa\,(-1+2\,\cos(\theta)^2)\,\mathrm{d}(r)}{-1+\cos(\theta)^2} + 4\,\frac{(r+M)\,\cos(\theta)\,r\,aa\,\mathrm{d}(\theta)}{2\,r+M}$$

$$-\frac{\sin(\theta)\,\cos(\theta)\,(2\,r^3\,aa^2 - 2\,r^3\,aa^2\,\cos(\theta)^2 + 2\,r^2\,M\,aa^2\,\cos(\theta)^4 + 3\,r^2\,M\,aa^2 - 5\,r^2\,M\,aa^2\,\cos(\theta)^2 + 2\,r + M)\,\mathrm{d}(\phi)}{2\,r\,\cos(\theta)^2 - 2\,r - M + M\,\cos(\theta)^2}$$

$$\Gamma 33 := \frac{(-2\,r^3\,aa^2\,\cos(\theta)^2 - 3\,r^2\,M\,aa^2\,\cos(\theta)^2 + 2\,r^2\,M\,aa^2\,\cos(\theta)^4 + 2\,r^3\,aa^2 + r^2\,M\,aa^2 + 2\,r - M)\,\mathrm{d}(r)}{r\,(2\,r+M)}$$

$$-\frac{\sin(\theta)\,\cos(\theta)\,(2\,r^3\,aa^2 - 2\,r^3\,aa^2\,\cos(\theta)^2 + 2\,r^2\,M\,aa^2\,\cos(\theta)^4 + 3\,r^2\,M\,aa^2 - 5\,r^2\,M\,aa^2\,\cos(\theta)^2 + 2\,r + M)\,\mathrm{d}(\theta)}{2\,r\,\cos(\theta)^2 - 2\,r - M + M\,\cos(\theta)^2}$$

$$+2\,\frac{\cos(\theta)\,r\,aa\,M\,(r^2\,\cos(\theta)^4\,aa^2 - 2\,r^2\,\cos(\theta)^2\,aa^2 + r^2\,aa^2 + 1 - \cos(\theta)^2)\,\mathrm{d}(\phi)}{2\,r+M}$$

$$\Gamma 34 := 0$$

$$-\frac{\sin(\theta)\,\cos(\theta)\,(2\,r^3\,aa^2 - 2\,r^3\,aa^2\,\cos(\theta)^2 + 2\,r^2\,M\,aa^2\,\cos(\theta)^4 + 3\,r^2\,M\,aa^2 - 5\,r^2\,M\,aa^2\,\cos(\theta)^2 + 2\,r + M)}{2\,r\,\cos(\theta)^2 - 2\,r - M + M\,\cos(\theta)^2}$$

```
> Gamma41:=C2S[4,1,1]*d(r)+C2S[4,1,2]*d(theta)+C2S[4,1,3]*d(phi)+C2S[4,1,4]*d(tau)
  ;
> Gamma42:=C2S[4,2,1]*d(r)+C2S[4,2,2]*d(theta)+C2S[4,2,3]*d(phi)+C2S[4,2,4]*d(tau)
  ;
> Gamma43:=C2S[4,3,1]*d(r)+C2S[4,3,2]*d(theta)+C2S[4,3,3]*d(phi)+C2S[4,3,4]*d(tau)
  ;
> Gamma44:=C2S[4,4,1]*d(r)+C2S[4,4,2]*d(theta)+C2S[4,4,3]*d(phi)+C2S[4,4,4]*d(tau)
  ;
```

$$\Gamma 41 := 0$$
$$\Gamma 42 := 0$$
$$\Gamma 43 := 0$$
$$\Gamma 44 := 0$$

Compute the "Christoffel curvature" based upon the formula d[Gamma] + [Gamma]^[Gamma]

```
>
> OmegaCC11:=simpform(simplify(Gamma11&^Gamma11+Gamma12&^Gamma21+Gamma13&^Gamma31+
  Gamma14&^Gamma41));OmegaCC12:=simpform(simplify(Gamma11&^Gamma12+Gamma12&^Gamma2
  2+Gamma13&^Gamma32+Gamma14&^Gamma42));OmegaCC13:=simpform(simplify(Gamma11&^Gamm
  a13+Gamma12&^Gamma23+Gamma13&^Gamma33+Gamma14&^Gamma43));OmegaCC14:=simpform(sim
  plify(Gamma11&^Gamma14+Gamma12&^Gamma24+Gamma13&^Gamma34+Gamma14&^Gamma44));
```

$OmegaCC11 := -2 \dfrac{r\sin(\theta)\,aa^2\cos(\theta)\,(-2\,r - 2\,M\cos(\theta)^2 + M + 4\,r\cos(\theta)^2)\,(\mathrm{d}(r)\,\&\^{}\,\mathrm{d}(\theta))}{2\,r + M} - r\sin(\theta)\,aa\,($

$6\,r\cos(\theta)^2 + 6\,r^2 M\,aa^2\cos(\theta)^2 - 3\,M\cos(\theta)^2 + 10\,r^2 aa^2\cos(\theta)^6 M - r^2 M\,aa^2 - 15\,r^2 M\,aa^2\cos(\theta)^4 - 2\,r^3 aa^2$

$-\,2\,r + M + 4\,r^3 aa^2\cos(\theta)^2 - 2\,r^3 aa^2\cos(\theta)^4)\,(\mathrm{d}(\theta)\,\&\^{}\,\mathrm{d}(\phi))\,/\,(2\,r + M) + \cos(\theta)\,aa\,(-13\,r^2 M\,aa^2\cos(\theta)^4$

$+\,9\,r^2 M\,aa^2\cos(\theta)^2 + 6\,r^2 aa^2\cos(\theta)^6 M + 2\,r^3 aa^2\cos(\theta)^4 - 2\,r^3 aa^2\cos(\theta)^2 + 2\,r\cos(\theta)^2 - 2\,r - 2\,r^2 M\,aa^2$

$-\,M\cos(\theta)^2 + M)\,(\mathrm{d}(r)\,\&\^{}\,\mathrm{d}(\phi))\,/\,(2\,r + M)$

$OmegaCC12 :=$

$-\dfrac{(4\,r^3 aa^2\cos(\theta)^4 - 4\,r^3 aa^2\cos(\theta)^2 + M + 8\,r^2 M\,aa^2\cos(\theta)^2 - 8\,r^2 M\,aa^2\cos(\theta)^4 - 2\,r)\,(\mathrm{d}(r)\,\&\^{}\,\mathrm{d}(\theta))}{2\,r + M} - r^2 aa$

$\cos(\theta)\,(17\,r^2 M\,aa^2\cos(\theta)^2 - 3\,M\cos(\theta)^2 + 3\,M - 23\,r^2 M\,aa^2\cos(\theta)^4 - 4\,r^2 M\,aa^2 + 10\,r^2 aa^2\cos(\theta)^6 M$

$-\,2\,r^3 aa^2\cos(\theta)^4 + 6\,r^3 aa^2\cos(\theta)^2 - 4\,r^3 aa^2 + 6\,r\cos(\theta)^2 - 6\,r)\,(\mathrm{d}(\theta)\,\&\^{}\,\mathrm{d}(\phi))\,/\,(2\,r + M) - r\sin(\theta)\,aa\,($

$-2\,M\cos(\theta)^2 - 18\,r^2 M\,aa^2\cos(\theta)^4 + 8\,r^2 aa^2\cos(\theta)^6 M + 11\,r^2 M\,aa^2\cos(\theta)^2 - r^2 M\,aa^2 + 4\,r\cos(\theta)^2 + 2\,M$

$+\,2\,r^3 aa^2\cos(\theta)^2 - 2\,r^3 aa^2 - 4\,r)\,(\mathrm{d}(r)\,\&\^{}\,\mathrm{d}(\phi))\,/\,(2\,r + M)$

$OmegaCC13 := r\sin(\theta)\,aa\,(-M - M\cos(\theta)^2 + 2\,r\cos(\theta)^2 - 5\,r^2 M\,aa^2\cos(\theta)^2 + 2\,r^2 aa^2\cos(\theta)^6 M$

$+\,3\,r^2 M\,aa^2\cos(\theta)^4 + 2\,r^3 aa^2\cos(\theta)^2 - 2\,r^3 aa^2\cos(\theta)^4 + 2\,r)\,(\mathrm{d}(r)\,\&\^{}\,\mathrm{d}(\theta))\,/\,(2\,r + M) - 2\,r\cos(\theta)\sin(\theta)\,(M$

$+\,r^2 M\,aa^2 - 4\,r^2 M\,aa^2\cos(\theta)^4 + 3\,r^2 M\,aa^2\cos(\theta)^2 - 4\,r^3 aa^2 - 13\,r^4\cos(\theta)^4 aa^4 M + 8\,r^4\cos(\theta)^2 aa^4 M$

$+\,4\,r^3 aa^2\cos(\theta)^2 - r^4 aa^4 M + 6\,r^4\cos(\theta)^6 aa^4 M - 2\,r)\,(\mathrm{d}(\theta)\,\&\^{}\,\mathrm{d}(\phi))\,/\,(2\,r + M) + ((-12\,r^4\cos(\theta)^2 aa^4 M - M$

$+\,M\cos(\theta)^2 - 6\,r^2 aa^2\cos(\theta)^6 M + 10\,r^2 M\,aa^2\cos(\theta)^4 - 3\,r^2 M\,aa^2\cos(\theta)^2 + 2\,r + 6\,r^3 aa^2 - 30\,r^4\cos(\theta)^6 aa^4 M$

$+\,10\,r^4\cos(\theta)^8 aa^4 M - 2\,r\cos(\theta)^2 + 31\,r^4\cos(\theta)^4 aa^4 M + 2\,r^5 aa^4 - r^2 M\,aa^2 + 4\,r^3 aa^2\cos(\theta)^4$

$-\,10\,r^3 aa^2\cos(\theta)^2 + r^4 aa^4 M + 2\,r^5 aa^4\cos(\theta)^4 - 4\,r^5 aa^4\cos(\theta)^2)\,(\mathrm{d}(r)\,\&\^{}\,\mathrm{d}(\phi)))\,/\,(2\,r + M)$

$OmegaCC14 := 0$

> **OmegaCC21:=simpform(simplify(Gamma21&^Gamma11+Gamma22&^Gamma21+Gamma23&^Gamma31+ Gamma24&^Gamma41));OmegaCC22:=simpform(simplify(Gamma21&^Gamma12+Gamma22&^Gamma2 2+Gamma23&^Gamma32+Gamma24&^Gamma42));OmegaCC23:=simpform(simplify(Gamma21&^Gamm a13+Gamma22&^Gamma23+Gamma23&^Gamma33+Gamma24&^Gamma43));OmegaCC24:=simpform(sim plify(Gamma21&^Gamma14+Gamma22&^Gamma24+Gamma23&^Gamma34+Gamma24&^Gamma44));**

$OmegaCC21 := -\,((M^2 - r^2 M^2 aa^2\cos(\theta)^2 + 8\,r^3 aa^2\cos(\theta)^4 M - 8\,r^3 aa^2\cos(\theta)^2 M + 2\,r^2 M^2 aa^2\cos(\theta)^4$

$+\,4\,r^3 aa^2 M + 24\,r^4 aa^2\cos(\theta)^4 - 28\,r^4 aa^2\cos(\theta)^2 + 8\,r^4 aa^2 - 4\,r^2)\,(\mathrm{d}(r)\,\&\^{}\,\mathrm{d}(\theta)))\,\Big/\,((4\,r^2 + 4\,r M + M^2)\,r^2)$

$-\,aa\cos(\theta)\,(-4\,r^3 aa^2 M + 11\,r^2 M^2 aa^2\cos(\theta)^2 + 6\,r^2 M^2 aa^2\cos(\theta)^6 - 15\,r^2 M^2 aa^2\cos(\theta)^4 - 2\,r^2 M^2 aa^2 - 2\,M^2$

$+\,12\,r^2\cos(\theta)^2 + 3\,M^2\cos(\theta)^2 - 4\,r^4 aa^2\cos(\theta)^4 + 20\cos(\theta)^6 r^3 aa^2 M - 8\,r^2 - 48\,r^3 aa^2\cos(\theta)^4 M + 4\,r\cos(\theta)^2 M$

$+\,32\,r^3 aa^2\cos(\theta)^2 M + 4\,r^4 aa^2\cos(\theta)^2)\,(\mathrm{d}(\theta)\,\&\^{}\,\mathrm{d}(\phi))\,\Big/\,(4\,r^2 + 4\,r M + M^2)$

$-\,4\,\dfrac{aa^3\sin(\theta)\,r^2\,(-3\cos(\theta)^4 M + 2\cos(\theta)^6 M + 2\,r\cos(\theta)^4 - 4\,r\cos(\theta)^2 + 2\,r + M)\,(\mathrm{d}(r)\,\&\^{}\,\mathrm{d}(\phi))}{4\,r^2 + 4\,r M + M^2}$

$OmegaCC22 := 2\,\dfrac{aa^2\sin(\theta)\,r\cos(\theta)\,(-4\,r^2 + 4\,r M + 8\,r^2\cos(\theta)^2 + M^2)\,(\mathrm{d}(r)\,\&\^{}\,\mathrm{d}(\theta))}{4\,r^2 + 4\,r M + M^2} + aa\sin(\theta)\,r\,($

$12\,r^2\cos(\theta)^2 - 4\,r^2 + 24\,r^3 aa^2\cos(\theta)^2 M + 4\,r\cos(\theta)^2 M + 7\,r^2 M^2 aa^2\cos(\theta)^2 + 6\,r^2 M^2 aa^2\cos(\theta)^6$

$-\,4\,r^4 aa^2\cos(\theta)^4 + 3\,M^2\cos(\theta)^2 - 13\,r^2 M^2 aa^2\cos(\theta)^4 + 4\,r^4 aa^2\cos(\theta)^2 - 44\,r^3 aa^2\cos(\theta)^4 M$

$+\,20\cos(\theta)^6 r^3 aa^2 M - M^2 + 4\,r M)\,(\mathrm{d}(\theta)\,\&\^{}\,\mathrm{d}(\phi))\,\Big/\,(4\,r^2 + 4\,r M + M^2) - aa\cos(\theta)\,(-4\,r^2 + 4\,r^2\cos(\theta)^2 - M^2$

$+\,M^2\cos(\theta)^2 - 36\,r^3 aa^2\cos(\theta)^4 M + 4\,r^4 aa^2 + 12\cos(\theta)^6 r^3 aa^2 M + 4\,r^4 aa^2\cos(\theta)^4 + 2\,r^2 M^2 aa^2\cos(\theta)^6$

$-\,8\,r^4 aa^2\cos(\theta)^2 - 7\,r^2 M^2 aa^2\cos(\theta)^4 - 12\,r^3 aa^2 M + 36\,r^3 aa^2\cos(\theta)^2 M + 8\,r^2 M^2 aa^2\cos(\theta)^2 - 3\,r^2 M^2 aa^2$

$-\,4\,r M + 4\,r\cos(\theta)^2 M)\,(\mathrm{d}(r)\,\&\^{}\,\mathrm{d}(\phi))\,\Big/\,(4\,r^2 + 4\,r M + M^2)$

$OmegaCC23 := aa\cos(\theta)\,(4\,r M + 3\,r^2 M^2 aa^2\cos(\theta)^2 + 8\,r^2\cos(\theta)^2 + 12\,r^4 aa^2\cos(\theta)^2 + 8\cos(\theta)^6 r^3 aa^2 M$

$$- 12\, r^3\, aa^2 \cos(\theta)^4\, M + 8\, r^3\, aa^2\, M + 2\, M^2 \cos(\theta)^2 - 8\, r^4\, aa^2 \cos(\theta)^4 - 4\, r^3\, aa^2 \cos(\theta)^2\, M - M^2$$
$$+ 4\, r^2\, M^2\, aa^2 \cos(\theta)^6 + r^2\, M^2\, aa^2 - 4\, r^2 - 8\, r^2\, M^2\, aa^2 \cos(\theta)^4 - 4\, r^4\, aa^2)\, (\mathrm{d}(r)\, \&^\wedge\, \mathrm{d}(\theta))\, \Big/\, (4\, r^2 + 4\, r\, M + M^2) - ((\, $$
$$M^2 - 20\, r^4\, aa^2 \cos(\theta)^2 - 9\, r^2\, M^2\, aa^2 \cos(\theta)^2 + 12\, r^2\, M^2\, aa^2 \cos(\theta)^4 + r^2\, M^2\, aa^2 - 4\, r^3\, aa^2\, M - 4\, r^2\, M^2\, aa^2 \cos(\theta)^6$$
$$- 20\, r^3\, aa^2 \cos(\theta)^2\, M + 16\, r^4\, aa^2 \cos(\theta)^4 - r^4\, M^2\, aa^4 - 16\, r^5\, aa^4 \cos(\theta)^2\, M + 68\, r^5\, aa^4 \cos(\theta)^4\, M$$
$$- 72\, r^5\, aa^4 \cos(\theta)^6\, M + 40\, r^3\, aa^2 \cos(\theta)^4\, M - 16 \cos(\theta)^6\, r^3\, aa^2\, M - 6\, r^4\, M^2\, aa^4 \cos(\theta)^2 - 24\, r^4\, M^2\, aa^4 \cos(\theta)^6$$
$$+ 23\, r^4\, M^2\, aa^4 \cos(\theta)^4 + 8\, r^4\, M^2\, aa^4 \cos(\theta)^8 - 4\, r^6\, aa^4 \cos(\theta)^4 + 8\, r^6\, aa^4 \cos(\theta)^2 + 4\, r^4\, aa^2 - 4\, r\, M + 4\, r^2$$
$$- 4\, r^5\, M\, aa^4 - 4\, r^6\, aa^4 + 24\, r^5\, aa^4 \cos(\theta)^8\, M - 8\, r^2 \cos(\theta)^2 - 2\, M^2 \cos(\theta)^2)\, (\mathrm{d}(\theta)\, \&^\wedge\, \mathrm{d}(\phi)))\, \Big/\, (4\, r^2 + 4\, r\, M + M^2$$
$$) - 2\, r\, aa^2 \sin(\theta) \cos(\theta)\, M\, (-6\, r^2\, M\, aa^2 \cos(\theta)^4 - M + M \cos(\theta)^2 + 6\, r^2\, M\, aa^2 \cos(\theta)^2 + 10\, r \cos(\theta)^2$$
$$+ 2\, r^2\, aa^2 \cos(\theta)^6\, M - 6\, r - 2\, r^2\, M\, aa^2 + 24\, r^3\, aa^2 \cos(\theta)^2 + 8\, r^3\, aa^2 \cos(\theta)^6 - 24\, r^3\, aa^2 \cos(\theta)^4 - 4\, r \cos(\theta)^4$$
$$- 8\, r^3\, aa^2)\, (\mathrm{d}(r)\, \&^\wedge\, \mathrm{d}(\phi))\, \Big/\, (4\, r^2 + 4\, r\, M + M^2)$$

$$OmegaCC24 := 0$$

```
> OmegaCC31:=simpform(simplify(Gamma31&^Gamma11+Gamma32&^Gamma21+Gamma33&^Gamma31+
  Gamma34&^Gamma41));OmegaCC32:=simpform(simplify(Gamma31&^Gamma12+Gamma32&^Gamma2
  2+Gamma33&^Gamma32+Gamma34&^Gamma42));OmegaCC33:=simpform(simplify(Gamma31&^Gamm
  a13+Gamma32&^Gamma23+Gamma33&^Gamma33+Gamma34&^Gamma43));OmegaCC34:=simpform(sim
  plify(Gamma31&^Gamma14+Gamma32&^Gamma24+Gamma33&^Gamma34+Gamma34&^Gamma44));
```

$$OmegaCC31 := 4\, \frac{\sin(\theta)\, aa\, (\mathrm{d}(r)\, \&^\wedge\, \mathrm{d}(\theta))}{2\, r + M} + 4\, \frac{r \sin(\theta)\, aa^2 \cos(\theta)\, (M \cos(\theta)^2 + r)\, (\mathrm{d}(\theta)\, \&^\wedge\, \mathrm{d}(\phi))}{2\, r + M}$$
$$- \frac{(2\, r^2\, M\, aa^2 \cos(\theta)^4 + 6\, r^3\, aa^2 \cos(\theta)^2 - r^2\, M\, aa^2 \cos(\theta)^2 - 4\, r^3\, aa^2 - 2\, r + M)\, (\mathrm{d}(r)\, \&^\wedge\, \mathrm{d}(\phi))}{(2\, r + M)\, r^2}$$

$$OmegaCC32 := - \frac{\cos(\theta)\, aa\, (2\, M^2 \cos(\theta)^2 + 12\, r \cos(\theta)^2\, M + 4\, r^2 - M^2 - 8\, r\, M)\, (\mathrm{d}(r)\, \&^\wedge\, \mathrm{d}(\theta))}{4\, r^2 \cos(\theta)^2 + 4\, r \cos(\theta)^2\, M + M^2 \cos(\theta)^2 - 4\, r^2 - 4\, r\, M - M^2} + ((\, $$
$$8 \cos(\theta)^6\, r^3\, aa^2\, M + 8\, r^4\, aa^2 \cos(\theta)^4 - 8\, r^3\, aa^2 \cos(\theta)^4\, M + 2\, r^2\, M^2\, aa^2 \cos(\theta)^4 - 8\, r^4\, aa^2 \cos(\theta)^2$$
$$- 2\, r^2\, M^2\, aa^2 \cos(\theta)^2 - 8\, r \cos(\theta)^2\, M + 4\, r\, M - 4\, r^2 - M^2)\, (\mathrm{d}(\theta)\, \&^\wedge\, \mathrm{d}(\phi)))\, \Big/\, (\, $$
$$4\, r^2 \cos(\theta)^2 + 4\, r \cos(\theta)^2\, M + M^2 \cos(\theta)^2 - 4\, r^2 - 4\, r\, M - M^2)$$
$$+ 4\, \frac{(2\, M \cos(\theta)^2 + 2\, r - M)\, r^2 \sin(\theta)\, aa^2 \cos(\theta)\, (\mathrm{d}(r)\, \&^\wedge\, \mathrm{d}(\phi))}{4\, r^2 + 4\, r\, M + M^2}$$

$$OmegaCC33 := -4\, \frac{aa^2 \sin(\theta)\, r \cos(\theta)\, M\, (2\, r + M \cos(\theta)^2)\, (\mathrm{d}(r)\, \&^\wedge\, \mathrm{d}(\theta))}{4\, r^2 + 4\, r\, M + M^2} + aa \sin(\theta)\, r\, (-8\, r^3\, aa^2 \cos(\theta)^2\, M$$
$$- 4\, r \cos(\theta)^2\, M - r^2\, M^2\, aa^2 \cos(\theta)^2 + 4\, r^2\, M^2\, aa^2 \cos(\theta)^6 - 6\, M^2 \cos(\theta)^2 - 2\, r^2\, M^2\, aa^2 \cos(\theta)^4 - r^2\, M^2\, aa^2$$
$$+ 4\, r^4\, aa^2 \cos(\theta)^2 - 4\, r^3\, aa^2\, M + 12\, r^3\, aa^2 \cos(\theta)^4\, M - 4\, r^4\, aa^2 + 2\, M^2 - 4\, r\, M)\, (\mathrm{d}(\theta)\, \&^\wedge\, \mathrm{d}(\phi))\, \Big/\, (\, $$
$$4\, r^2 + 4\, r\, M + M^2) - \cos(\theta)\, aa\, (2\, M^2 - 2\, M^2 \cos(\theta)^2 + 12\, r^3\, aa^2 \cos(\theta)^4\, M - 4\, r^4\, aa^2 + 4\, r^2\, M^2\, aa^2 \cos(\theta)^6$$
$$+ 4\, r^4\, aa^2 \cos(\theta)^2 - 6\, r^2\, M^2\, aa^2 \cos(\theta)^4 + 8\, r^3\, aa^2\, M - 20\, r^3\, aa^2 \cos(\theta)^2\, M + r^2\, M^2\, aa^2 \cos(\theta)^2 + r^2\, M^2\, aa^2 + 4\, r\, M$$
$$- 4\, r \cos(\theta)^2\, M)\, (\mathrm{d}(r)\, \&^\wedge\, \mathrm{d}(\phi))\, \Big/\, (4\, r^2 + 4\, r\, M + M^2)$$

$$OmegaCC34 := 0$$

```
> OmegaCC41:=simpform(simplify(Gamma41&^Gamma11+Gamma42&^Gamma21+Gamma43&^Gamma31+
  Gamma44&^Gamma41));OmegaCC42:=simpform(simplify(Gamma41&^Gamma12+Gamma42&^Gamma2
  2+Gamma43&^Gamma32+Gamma44&^Gamma42));OmegaCC43:=simpform(simplify(Gamma41&^Gamm
  a13+Gamma42&^Gamma23+Gamma43&^Gamma33+Gamma44&^Gamma43));OmegaCC44:=simpform(sim
  plify(Gamma41&^Gamma14+Gamma42&^Gamma24+Gamma43&^Gamma34+Gamma44&^Gamma44));
```

$$OmegaCC41 := 0$$
$$OmegaCC42 := 0$$
$$OmegaCC43 := 0$$

```
> Christ:=array([[Gamma11,Gamma12,Gamma13,Gamma14],[Gamma21,Gamma22,Gamma23,Gamma2
  4],[Gamma31,Gamma32,Gamma33,Gamma34],[Gamma41,Gamma42,Gamma43,Gamma44]]);CartanC
  :=array([[CGamma11,CGamma12,CGamma13,CGamma14],[CGamma21,CGamma22,CGamma23,CGamm
  a24],[CGamma31,CGamma32,CGamma33,CGamma34],[CGamma41,CGamma42,CGamma43,CGamma44]
  ]);
```

$Christ :=$

$$\left[2\frac{(-2\,r^3\,aa^2\cos(\theta)^2 + 2\,r^3\,aa^2\cos(\theta)^4 - M)\,\mathrm{d}(r)}{r\,(2\,r+M)} + (\cos(\theta)\,aa^2\,r^2\sin(\theta) - 2\cos(\theta)^3\,aa^2\,r^2\sin(\theta))\,\mathrm{d}(\theta) - r\,aa \right.$$

$$\cos(\theta)\,(-2\,r^3\,aa^2 + 4\,r^3\,aa^2\cos(\theta)^2 - 2\,r^3\,aa^2\cos(\theta)^4 - 5\,r^2\,M\,aa^2\cos(\theta)^4 + 2\,r^2\,aa^2\cos(\theta)^6\,M - r^2\,M\,aa^2$$

$$+ 4\,r^2\,M\,aa^2\cos(\theta)^2 + 2\,r\cos(\theta)^2 - 2\,r + M - M\cos(\theta)^2)\,\mathrm{d}(\phi)\,/\,(2\,r+M),$$

$$(\cos(\theta)\,aa^2\,r^2\sin(\theta) - 2\cos(\theta)^3\,aa^2\,r^2\sin(\theta))\,\mathrm{d}(r)$$

$$- \frac{r\,(-4\,r^3\,aa^2\cos(\theta)^2 + 4\,r^3\,aa^2\cos(\theta)^4 + 4\,r^2\,M\,aa^2\cos(\theta)^4 - 4\,r^2\,M\,aa^2\cos(\theta)^2 + 2\,r - M)\,\mathrm{d}(\theta)}{2\,r+M} + r^2\sin(\theta)\,aa\,($$

$$2\,r^3\,aa^2\cos(\theta)^2 - 2\,r^3\,aa^2\cos(\theta)^4 + 3\,r^2\,M\,aa^2\cos(\theta)^2 - 5\,r^2\,M\,aa^2\cos(\theta)^4 + 2\,r^2\,aa^2\cos(\theta)^6\,M + 2\,r\cos(\theta)^2 - 2\,r$$

$$+ M - M\cos(\theta)^2)\,\mathrm{d}(\phi)\,/\,(2\,r+M),\; - r\,aa\cos(\theta)\,(-2\,r^3\,aa^2 + 4\,r^3\,aa^2\cos(\theta)^2 - 2\,r^3\,aa^2\cos(\theta)^4$$

$$- 5\,r^2\,M\,aa^2\cos(\theta)^4 + 2\,r^2\,aa^2\cos(\theta)^6\,M - r^2\,M\,aa^2 + 4\,r^2\,M\,aa^2\cos(\theta)^2 + 2\,r\cos(\theta)^2 - 2\,r + M - M\cos(\theta)^2)$$

$$\mathrm{d}(r)\,/\,(2\,r+M) + r^2\sin(\theta)\,aa\,(2\,r^3\,aa^2\cos(\theta)^2 - 2\,r^3\,aa^2\cos(\theta)^4 + 3\,r^2\,M\,aa^2\cos(\theta)^2 - 5\,r^2\,M\,aa^2\cos(\theta)^4$$

$$+ 2\,r^2\,aa^2\cos(\theta)^6\,M + 2\,r\cos(\theta)^2 - 2\,r + M - M\cos(\theta)^2)\,\mathrm{d}(\theta)\,/\,(2\,r+M) - r\,(-6\,r^4\cos(\theta)^6\,aa^4\,M$$

$$+ 2\,r^4\cos(\theta)^8\,aa^4\,M - 2\,r^4\cos(\theta)^2\,aa^4\,M + 6\,r^4\cos(\theta)^4\,aa^4\,M - 2\,r^2\,aa^2\cos(\theta)^6\,M + 4\,r^3\,aa^2 - M - 2\,r\cos(\theta)^2$$

$$\left. + 2\,r - 2\,r^2\,M\,aa^2\cos(\theta)^2 + 4\,r^3\,aa^2\cos(\theta)^4 + M\cos(\theta)^2 - 8\,r^3\,aa^2\cos(\theta)^2 + 4\,r^2\,M\,aa^2\cos(\theta)^4)\,\mathrm{d}(\phi)\,/\,(2\,r+M), \right.$$

$$\left. 0 \right]$$

$$\left[-4\frac{(-1+\cos(\theta)^2)\,r\,aa^2\cos(\theta)\sin(\theta)\,\mathrm{d}(r)}{2\,r+M} - (\right.$$

$$(-6\,r^3\,aa^2\cos(\theta)^2 + 2\,r^3\,aa^2 + 4\,r^3\,aa^2\cos(\theta)^4 - 3\,r^2\,M\,aa^2\cos(\theta)^2 + 2\,r^2\,M\,aa^2\cos(\theta)^4 + r^2\,M\,aa^2 - 2\,r + M)\,\mathrm{d}(\theta)$$

$$)\,/\,(r\,(2\,r+M)) + ((4\,r^3\,aa^2\cos(\theta)^2 - 2\,r^3\,aa^2 - 2\,r^3\,aa^2\cos(\theta)^4 + 2\,r^2\,aa^2\cos(\theta)^6\,M - r^2\,M\,aa^2$$

$$- 5\,r^2\,M\,aa^2\cos(\theta)^4 + 4\,r^2\,M\,aa^2\cos(\theta)^2 + 2\,r\cos(\theta)^2 + M\cos(\theta)^2)\sin(\theta)\,aa\,\mathrm{d}(\phi))\,/\,(2\,r+M),\; - ($$

$$(-6\,r^3\,aa^2\cos(\theta)^2 + 2\,r^3\,aa^2 + 4\,r^3\,aa^2\cos(\theta)^4 - 3\,r^2\,M\,aa^2\cos(\theta)^2 + 2\,r^2\,M\,aa^2\cos(\theta)^4 + r^2\,M\,aa^2 - 2\,r + M)\,\mathrm{d}(r)$$

$$)\,/\,(r\,(2\,r+M)) + 4\frac{\cos(\theta)\,aa^2\,r^2\sin(\theta)\,(-r + r\cos(\theta)^2 + M\cos(\theta)^2 - M)\,\mathrm{d}(\theta)}{2\,r+M} + r\,aa\cos(\theta)\,(-2\,r^3\,aa^2$$

$$+ 4\,r^3\,aa^2\cos(\theta)^2 - 2\,r^3\,aa^2\cos(\theta)^4 - 3\,r^2\,M\,aa^2 + 8\,r^2\,M\,aa^2\cos(\theta)^2 - 7\,r^2\,M\,aa^2\cos(\theta)^4 + 2\,r^2\,aa^2\cos(\theta)^6\,M$$

$$- 2\,r + 2\,r\cos(\theta)^2 - M + M\cos(\theta)^2)\,\mathrm{d}(\phi)\,/\,(2\,r+M),\; ((4\,r^3\,aa^2\cos(\theta)^2 - 2\,r^3\,aa^2 - 2\,r^3\,aa^2\cos(\theta)^4$$

$$+ 2\,r^2\,aa^2\cos(\theta)^6\,M - r^2\,M\,aa^2 - 5\,r^2\,M\,aa^2\cos(\theta)^4 + 4\,r^2\,M\,aa^2\cos(\theta)^2 + 2\,r\cos(\theta)^2 + M\cos(\theta)^2)\sin(\theta)\,aa$$

$$\mathrm{d}(r))\,/\,(2\,r+M) + r\,aa\cos(\theta)\,(-2\,r^3\,aa^2 + 4\,r^3\,aa^2\cos(\theta)^2 - 2\,r^3\,aa^2\cos(\theta)^4 - 3\,r^2\,M\,aa^2 + 8\,r^2\,M\,aa^2\cos(\theta)^2$$

$$- 7\,r^2\,M\,aa^2\cos(\theta)^4 + 2\,r^2\,aa^2\cos(\theta)^6\,M - 2\,r + 2\,r\cos(\theta)^2 - M + M\cos(\theta)^2)\,\mathrm{d}(\theta)\,/\,(2\,r+M) + (($$

$$2\,r^4\cos(\theta)^6\,aa^4\,M - 2\,r^4\,aa^4\,M - 6\,r^4\cos(\theta)^4\,aa^4\,M + 6\,r^4\cos(\theta)^2\,aa^4\,M + 4\,r^3\,aa^2\cos(\theta)^2 - 4\,r^3\,aa^2 - 4\,r^2\,M\,aa^2$$

$$\left. - 2\,r^2\,M\,aa^2\cos(\theta)^4 + 6\,r^2\,M\,aa^2\cos(\theta)^2 - 2\,r - M)\cos(\theta)\sin(\theta)\,\mathrm{d}(\phi))\,/\,(2\,r+M),\; 0 \right]$$

$$\left[-4\frac{\cos(\theta)\,aa\,\mathrm{d}(r)}{2\,r+M} + \frac{\sin(\theta)\,aa\,(-1 + 2\cos(\theta)^2)\,\mathrm{d}(\theta)}{-1 + \cos(\theta)^2} \right.$$

$$+ \frac{(-2\,r^3\,aa^2\cos(\theta)^2 - 3\,r^2\,M\,aa^2\cos(\theta)^2 + 2\,r^2\,M\,aa^2\cos(\theta)^4 + 2\,r^3\,aa^2 + r^2\,M\,aa^2 + 2\,r - M)\,\mathrm{d}(\phi)}{r\,(2\,r + M)}\,,$$

$$\frac{\sin(\theta)\,aa\,(-1 + 2\cos(\theta)^2)\,\mathrm{d}(r)}{-1 + \cos(\theta)^2} + 4\,\frac{(r + M)\cos(\theta)\,r\,aa\,\mathrm{d}(\theta)}{2\,r + M}$$

$$-\frac{\sin(\theta)\cos(\theta)\,(2\,r^3\,aa^2 - 2\,r^3\,aa^2\cos(\theta)^2 + 2\,r^2\,M\,aa^2\cos(\theta)^4 + 3\,r^2\,M\,aa^2 - 5\,r^2\,M\,aa^2\cos(\theta)^2 + 2\,r + M)\,\mathrm{d}(\phi)}{2\,r\cos(\theta)^2 - 2\,r - M + M\cos(\theta)^2}$$

$$,\frac{(-2\,r^3\,aa^2\cos(\theta)^2 - 3\,r^2\,M\,aa^2\cos(\theta)^2 + 2\,r^2\,M\,aa^2\cos(\theta)^4 + 2\,r^3\,aa^2 + r^2\,M\,aa^2 + 2\,r - M)\,\mathrm{d}(r)}{r\,(2\,r + M)}$$

$$-\frac{\sin(\theta)\cos(\theta)\,(2\,r^3\,aa^2 - 2\,r^3\,aa^2\cos(\theta)^2 + 2\,r^2\,M\,aa^2\cos(\theta)^4 + 3\,r^2\,M\,aa^2 - 5\,r^2\,M\,aa^2\cos(\theta)^2 + 2\,r + M)\,\mathrm{d}(\theta)}{2\,r\cos(\theta)^2 - 2\,r - M + M\cos(\theta)^2}$$

$$+ 2\,\frac{\cos(\theta)\,r\,aa\,M\,(r^2\cos(\theta)^4\,aa^2 - 2\,r^2\cos(\theta)^2\,aa^2 + r^2\,aa^2 + 1 - \cos(\theta)^2)\,\mathrm{d}(\phi)}{2\,r + M}\,,\,0\Bigg]$$

$$[\,0\,,\,0\,,\,0\,,\,0\,]$$

CartanC :=

$$[\,(r\cos(\theta)\,aa - r\cos(\theta)^3\,aa)\,\mathrm{d}(\phi)\,,\,-r\,\mathrm{d}(\theta) + \sin(\theta)\,r^2\cos(\theta)^2\,aa\,\mathrm{d}(\phi)\,,$$

$$(-r\cos(\theta)\,aa + r\cos(\theta)^3\,aa)\,\mathrm{d}(r) - \sin(\theta)\,r^2\cos(\theta)^2\,aa\,\mathrm{d}(\theta) + (-r + r\cos(\theta)^2)\,\mathrm{d}(\phi)\,,\,0\,]$$

$$\left[\frac{\mathrm{d}(\theta)}{r} + (-\sin(\theta)\,aa + \sin(\theta)\,aa\cos(\theta)^2)\,\mathrm{d}(\phi)\,,\,\frac{\mathrm{d}(r)}{r} + (-r\cos(\theta)\,aa + r\cos(\theta)^3\,aa)\,\mathrm{d}(\phi)\,,\right.$$

$$\left.(\sin(\theta)\,aa - \sin(\theta)\,aa\cos(\theta)^2)\,\mathrm{d}(r) + (r\cos(\theta)\,aa - r\cos(\theta)^3\,aa)\,\mathrm{d}(\theta) - \sin(\theta)\cos(\theta)\,\mathrm{d}(\phi)\,,\,0\right]$$

$$\left[\frac{\mathrm{d}(\phi)}{r}\,,\,\frac{\cos(\theta)\,\mathrm{d}(\phi)}{\sin(\theta)}\,,\,\frac{\mathrm{d}(r)}{r} + \frac{\cos(\theta)\,\mathrm{d}(\theta)}{\sin(\theta)}\,,\,0\right]$$

$$[\,0\,,\,0\,,\,0\,,\,0\,]$$

```
> OmegaD:=simpform(d(Christ));
> OMEGAG:=simpform(simplify(array([[OmegaD[1,1]+OmegaCC11,OmegaD[1,2]+OmegaCC12,Om
  egaD[1,3]+OmegaCC13,OmegaD[1,4]+OmegaCC14],[OmegaD[2,1]+OmegaCC21,OmegaD[2,2]+Om
  egaCC22,OmegaD[2,3]+OmegaCC23,OmegaD[2,4]+OmegaCC24],[OmegaD[3,1]+OmegaCC31,Omeg
  aD[3,2]+OmegaCC32,OmegaD[3,3]+OmegaCC33,OmegaD[3,4]+OmegaCC34],[OmegaD[4,1]+Omeg
  aCC41,OmegaD[4,2]+OmegaCC42,OmegaD[4,3]+OmegaCC43,OmegaD[4,4]+OmegaCC44]])));
```

OmegaD :=

$$\Bigg[aa\sin(\theta)\,r\,(-2\,r^3\,aa^2 + 12\,r^3\,aa^2\cos(\theta)^2 - 10\,r^3\,aa^2\cos(\theta)^4 - 25\,r^2\,M\,aa^2\cos(\theta)^4 + 14\,r^2\,aa^2\cos(\theta)^6\,M$$

$$- r^2\,M\,aa^2 + 12\,r^2\,M\,aa^2\cos(\theta)^2 + 6\,r\cos(\theta)^2 - 2\,r + M - 3\,M\cos(\theta)^2)\,(\mathrm{d}(\theta)\,\&^{\wedge}\,\mathrm{d}(\phi))\,/\,(2\,r + M) - aa\cos(\theta)\,($$

$$-12\,r^4\,aa^2\cos(\theta)^4 + 24\,r^4\,aa^2\cos(\theta)^2 + M^2 - 3\,r^2\,M^2\,aa^2 - 12\,r^3\,aa^2\,M - M^2\cos(\theta)^2 - 28\,r^3\,aa^2\cos(\theta)^4\,M$$

$$+ 32\,r^3\,aa^2\cos(\theta)^2\,M + 6\,r^2\,M^2\,aa^2\cos(\theta)^6 + 8\cos(\theta)^6\,r^3\,aa^2\,M - 15\,r^2\,M^2\,aa^2\cos(\theta)^4 + 12\,r^2\,M^2\,aa^2\cos(\theta)^2$$

$$- 12\,r^4\,aa^2 - 4\,r\,M - 4\,r^2 + 4\,r^2\cos(\theta)^2 + 4\,r\cos(\theta)^2\,M)\,(\mathrm{d}(r)\,\&^{\wedge}\,\mathrm{d}(\phi))\,\Big/\,(2\,r + M)^2$$

$$- 2\,\frac{\cos(\theta)\,r\sin(\theta)\,aa^2\,(-2\,r - 2\,M\cos(\theta)^2 + M + 4\,r\cos(\theta)^2)\,(\mathrm{d}(\theta)\,\&^{\wedge}\,\mathrm{d}(r))}{2\,r + M}\,,\,r^2\,aa\cos(\theta)\,(2\,r^3\,aa^2\cos(\theta)^2$$

$$- 2\,r^3\,aa^2\cos(\theta)^4 + 3\,r^2\,M\,aa^2\cos(\theta)^2 - 5\,r^2\,M\,aa^2\cos(\theta)^4 + 2\,r^2\,aa^2\cos(\theta)^6\,M + 2\,r\cos(\theta)^2 - 2\,r + M$$

$$- M\cos(\theta)^2 - 4\sin(\theta)^2\,aa^2\,r^3 + 8\sin(\theta)^2\,aa^2\,r^3\cos(\theta)^2 - 6\sin(\theta)^2\,aa^2\,r^2\,M + 20\sin(\theta)^2\,aa^2\,r^2\cos(\theta)^2\,M$$

$$- 12\sin(\theta)^2\,aa^2\,r^2\cos(\theta)^4\,M - 4\sin(\theta)^2\,r + 2\sin(\theta)^2\,M)\,(\mathrm{d}(\theta)\,\&^{\wedge}\,\mathrm{d}(\phi))\,/\,(2\,r + M) + 2\,aa\sin(\theta)\,r\,($$

$$-8\,r^4\,aa^2\cos(\theta)^4 + 8\,r^4\,aa^2\cos(\theta)^2 + M^2 - M^2\cos(\theta)^2 - 20\,r^3\,aa^2\cos(\theta)^4\,M + 14\,r^3\,aa^2\cos(\theta)^2\,M$$

$$+ 4\,r^2\,M^2\,aa^2\cos(\theta)^6 + 6\cos(\theta)^6\,r^3\,aa^2\,M - 10\,r^2\,M^2\,aa^2\cos(\theta)^4 + 6\,r^2\,M^2\,aa^2\cos(\theta)^2 - 2\,r\,M - 4\,r^2 + 4\,r^2\cos(\theta)^2$$
$$+ 2\,r\cos(\theta)^2\,M)\,(d(r)\,\&^\wedge\,d(\phi))\Big/(2\,r+M)^2 + ((-4\,r^4\sin(\theta)^2\,aa^2 - 4\,r^3\sin(\theta)^2\,aa^2\,M - r^2\sin(\theta)^2\,aa^2\,M^2$$
$$- 20\,r^4\,aa^2\cos(\theta)^2 - 28\,r^3\,aa^2\cos(\theta)^2\,M - 11\,r^2\,M^2\,aa^2\cos(\theta)^2 + 24\,r^4\sin(\theta)^2\,aa^2\cos(\theta)^2$$
$$+ 24\,r^3\sin(\theta)^2\,aa^2\cos(\theta)^2\,M + 6\,r^2\sin(\theta)^2\,aa^2\cos(\theta)^2\,M^2 + 16\,r^4\,aa^2\cos(\theta)^4 + 24\,r^3\,aa^2\cos(\theta)^4\,M$$
$$+ 10\,r^2\,M^2\,aa^2\cos(\theta)^4 + 4\,r^2 - M^2 + 4\,r\,M)\,(d(\theta)\,\&^\wedge\,d(r)))\Big/(2\,r+M)^2,\ 2\,r\cos(\theta)\sin(\theta)\,($$
$$-18\,r^4\cos(\theta)^4\,aa^4\,M + 8\,r^4\cos(\theta)^6\,aa^4\,M - 2\,r^4\,aa^4\,M + 12\,r^4\cos(\theta)^2\,aa^4\,M - 6\,r^2\,M\,aa^2\cos(\theta)^4 - 2\,r - 2\,r^2\,M\,aa^2$$
$$+ 8\,r^3\,aa^2\cos(\theta)^2 + M - 8\,r^3\,aa^2 + 8\,r^2\,M\,aa^2\cos(\theta)^2)\,(d(\theta)\,\&^\wedge\,d(\phi))/(2\,r+M) - ((24\,r^4\,aa^2\cos(\theta)^4$$
$$- 48\,r^4\,aa^2\cos(\theta)^2 - M^2 + 16\,r^3\,aa^2\,M + M^2\cos(\theta)^2 + 32\,r^3\,aa^2\cos(\theta)^4\,M - 40\,r^3\,aa^2\cos(\theta)^2\,M$$
$$- 6\,r^2\,M^2\,aa^2\cos(\theta)^6 - 8\cos(\theta)^6\,r^3\,aa^2\,M + 12\,r^2\,M^2\,aa^2\cos(\theta)^4 - 6\,r^2\,M^2\,aa^2\cos(\theta)^2 + 24\,r^4\,aa^2$$
$$+ 30\,r^4\,M^2\,aa^4\cos(\theta)^4 + 48\,r^5\,aa^4\cos(\theta)^4\,M - 10\,r^4\,M^2\,aa^4\cos(\theta)^2 - 16\,r^5\,aa^4\cos(\theta)^2\,M + 10\,r^4\,M^2\,aa^4\cos(\theta)^8$$
$$+ 16\,r^5\,aa^4\cos(\theta)^8\,M - 30\,r^4\,M^2\,aa^4\cos(\theta)^6 - 48\,r^5\,aa^4\cos(\theta)^6\,M + 4\,r\,M + 4\,r^2 - 4\,r^2\cos(\theta)^2 - 4\,r\cos(\theta)^2\,M)$$
$$(d(r)\,\&^\wedge\,d(\phi)))\Big/(2\,r+M)^2 + aa\sin(\theta)\,r\,(-4\,r^4\,aa^2\cos(\theta)^4 + 8\,r^4\,aa^2\cos(\theta)^2 - M^2 - r^2\,M^2\,aa^2 - 4\,r^3\,aa^2\,M$$
$$- M^2\cos(\theta)^2 - 20\,r^3\,aa^2\cos(\theta)^4\,M + 8\,r^3\,aa^2\cos(\theta)^2\,M + 6\,r^2\,M^2\,aa^2\cos(\theta)^6 + 16\cos(\theta)^6\,r^3\,aa^2\,M$$
$$- 5\,r^2\,M^2\,aa^2\cos(\theta)^4 - 4\,r^4\,aa^2 + 4\,r\,M + 4\,r^2 + 4\,r^2\cos(\theta)^2 - 4\,r\cos(\theta)^2\,M)\,(d(\theta)\,\&^\wedge\,d(r))\Big/(2\,r+M)^2,\ 0\ \Bigg]$$

$$\Bigg[\,aa\cos(\theta)\,(-8\sin(\theta)^2\,aa^2\,r^3 + 8\sin(\theta)^2\,aa^2\,r^3\cos(\theta)^2 - 12\sin(\theta)^2\,aa^2\,r^2\cos(\theta)^4\,M + 20\sin(\theta)^2\,aa^2\,r^2\cos(\theta)^2\,M$$
$$- 8\sin(\theta)^2\,aa^2\,r^2\,M - 4\sin(\theta)^2\,r - 2\sin(\theta)^2\,M + 4\,r^3\,aa^2\cos(\theta)^2 - 2\,r^3\,aa^2 - 2\,r^3\,aa^2\cos(\theta)^4 + 2\,r^2\,aa^2\cos(\theta)^6\,M$$
$$- r^2\,M\,aa^2 - 5\,r^2\,M\,aa^2\cos(\theta)^4 + 4\,r^2\,M\,aa^2\cos(\theta)^2 + 2\,r\cos(\theta)^2 + M\cos(\theta)^2)\,(d(\theta)\,\&^\wedge\,d(\phi))/(2\,r+M) + 2$$
$$aa^3\sin(\theta)\,r\,(8\,r^2\cos(\theta)^2 + 10\,r\cos(\theta)^2\,M - 4\,r^2 - 4\,r\,M - 4\cos(\theta)^4\,r^2 - 8\cos(\theta)^4\,r\,M + 2\,r\cos(\theta)^6\,M$$
$$+ 2\cos(\theta)^6\,M^2 - M^2 - 5\cos(\theta)^4\,M^2 + 4\,M^2\cos(\theta)^2)\,(d(r)\,\&^\wedge\,d(\phi))\Big/(2\,r+M)^2 + ((24\,r^4\sin(\theta)^2\,aa^2\cos(\theta)^2$$
$$+ 12\,r^3\sin(\theta)^2\,aa^2\cos(\theta)^2\,M - 8\,r^4\sin(\theta)^2\,aa^2 - 4\,r^3\sin(\theta)^2\,aa^2\,M - 4\,r^4\,aa^2\cos(\theta)^2 - 8\,r^3\,aa^2\cos(\theta)^2\,M$$
$$+ 4\,r^3\,aa^2\cos(\theta)^4\,M - M^2 + r^2\,M^2\,aa^2 + 4\,r^3\,aa^2\,M + 2\,r^2\,M^2\,aa^2\cos(\theta)^4 - 3\,r^2\,M^2\,aa^2\cos(\theta)^2 + 4\,r^4\,aa^2 - 4\,r\,M$$
$$+ 4\,r^2)\,(d(\theta)\,\&^\wedge\,d(r)))\Big/(r^2\,(2\,r+M)^2),\ - aa\sin(\theta)\,r\,(-2\,r^3\,aa^2 + 12\,r^3\,aa^2\cos(\theta)^2 - 10\,r^3\,aa^2\cos(\theta)^4$$
$$- 3\,r^2\,M\,aa^2 + 24\,r^2\,M\,aa^2\cos(\theta)^2 - 35\,r^2\,M\,aa^2\cos(\theta)^4 + 14\,r^2\,aa^2\cos(\theta)^6\,M - 2\,r + 6\,r\cos(\theta)^2 - M$$
$$+ 3\,M\cos(\theta)^2)\,(d(\theta)\,\&^\wedge\,d(\phi))/(2\,r+M) + \cos(\theta)\,aa\,(-M^2 - 9\,r^2\,M^2\,aa^2 - 20\,r^3\,aa^2\,M + M^2\cos(\theta)^2$$
$$+ 8\cos(\theta)^6\,r^3\,aa^2\,M + 24\,r^4\,aa^2\cos(\theta)^2 - 12\,r^4\,aa^2\cos(\theta)^4 + 6\,r^2\,M^2\,aa^2\cos(\theta)^6 - 12\,r^4\,aa^2 - 4\,r\,M - 4\,r^2$$
$$- 21\,r^2\,M^2\,aa^2\cos(\theta)^4 + 24\,r^2\,M^2\,aa^2\cos(\theta)^2 + 4\,r^2\cos(\theta)^2 - 36\,r^3\,aa^2\cos(\theta)^4\,M + 4\,r\cos(\theta)^2\,M$$
$$+ 48\,r^3\,aa^2\cos(\theta)^2\,M)\,(d(r)\,\&^\wedge\,d(\phi))\Big/(2\,r+M)^2$$
$$+ 2\,\frac{\cos(\theta)\,r\sin(\theta)\,aa^2\,(-4\,r^2 - 2\,r\,M + M^2 + 8\,r^2\cos(\theta)^2 + 6\,r\cos(\theta)^2\,M)\,(d(\theta)\,\&^\wedge\,d(r))}{(2\,r+M)^2},\ ((2\,r^4\cos(\theta)^8\,aa^4\,M$$
$$- 14\sin(\theta)^2\,r^4\,aa^4\cos(\theta)^6\,M - 18\sin(\theta)^2\,r^4\,aa^4\cos(\theta)^2\,M + 2\sin(\theta)^2\,r^4\,aa^4\,M + 30\sin(\theta)^2\,r^4\,aa^4\cos(\theta)^4\,M$$
$$+ 10\sin(\theta)^2\,aa^2\,r^2\cos(\theta)^4\,M - 18\sin(\theta)^2\,aa^2\,r^2\cos(\theta)^2\,M + 4\sin(\theta)^2\,aa^2\,r^3 + 4\sin(\theta)^2\,aa^2\,r^2\,M$$
$$- 12\sin(\theta)^2\,aa^2\,r^3\cos(\theta)^2 + \sin(\theta)^2\,M + 2\sin(\theta)^2\,r - 2\,r^4\cos(\theta)^2\,aa^4\,M - 6\,r^4\cos(\theta)^6\,aa^4\,M + 6\,r^4\cos(\theta)^4\,aa^4\,M$$
$$- 2\,r\cos(\theta)^2 - 4\,r^3\,aa^2\cos(\theta)^2 - M\cos(\theta)^2 - 4\,r^2\,M\,aa^2\cos(\theta)^2 + 6\,r^2\,M\,aa^2\cos(\theta)^4 + 4\,r^3\,aa^2\cos(\theta)^4$$
$$- 2\,r^2\,aa^2\cos(\theta)^6\,M)\,(d(\theta)\,\&^\wedge\,d(\phi)))/(2\,r+M) + 4\cos(\theta)\sin(\theta)\,r\,aa^2\,(-3\,r^3\,aa^2\,M - 4\,r^2 - 2\,r^2\,M^2\,aa^2$$
$$+ 6\,r^2\,M^2\,aa^2\cos(\theta)^2 + 2\,r^2\,M^2\,aa^2\cos(\theta)^6 - 6\,r^2\,M^2\,aa^2\cos(\theta)^4 - 5\,r\,M - 2\,M^2 + 9\,r^3\,aa^2\cos(\theta)^2\,M$$
$$+ 3\cos(\theta)^6\,r^3\,aa^2\,M - 9\,r^3\,aa^2\cos(\theta)^4\,M - \cos(\theta)^4\,M^2 + 3\,M^2\cos(\theta)^2 + 4\,r^2\cos(\theta)^2 + 6\,r\cos(\theta)^2\,M - \cos(\theta)^4\,r\,M$$
$$)\,(d(r)\,\&^\wedge\,d(\phi))\Big/(2\,r+M)^2 - aa\cos(\theta)\,(-M^2 + 16\,r^4\,aa^2\cos(\theta)^2 + 20\,r^2\,M^2\,aa^2\cos(\theta)^2 - 16\,r^2\,M^2\,aa^2\cos(\theta)^4$$
$$- 8\,r^2\,M^2\,aa^2 - 16\,r^3\,aa^2\,M + 4\,r^2\,M^2\,aa^2\cos(\theta)^6 + 36\,r^3\,aa^2\cos(\theta)^2\,M - 8\,r^4\,aa^2\cos(\theta)^4 - 24\,r^3\,aa^2\cos(\theta)^4\,M$$
$$+ 4\cos(\theta)^6\,r^3\,aa^2\,M + 12\,r^2\sin(\theta)^2\,aa^2\cos(\theta)^4\,M^2 - 20\,r^2\sin(\theta)^2\,aa^2\cos(\theta)^2\,M^2 + 8\,r^2\sin(\theta)^2\,aa^2\,M^2$$

$$-16 \, r^4 \sin(\theta)^2 \, aa^2 \cos(\theta)^2 + 8 \sin(\theta)^2 \, r \, M - 8 \, r^4 \, aa^2 - 4 \, r \, M - 4 \, r^2 + 2 \sin(\theta)^2 \, M^2 + 8 \sin(\theta)^2 \, r^2$$

$$+ \, 16 \, r^4 \sin(\theta)^2 \, aa^2 - 48 \, r^3 \sin(\theta)^2 \, aa^2 \cos(\theta)^2 \, M + 24 \, r^3 \sin(\theta)^2 \, aa^2 \cos(\theta)^4 \, M + 24 \, r^3 \sin(\theta)^2 \, aa^2 \, M)$$

$$(\mathrm{d}(\theta) \, \&^{\wedge} \, \mathrm{d}(r)) \Big/ (2 \, r + M)^2 \, , \, 0 \Big]$$

$$\Bigg[4 \, \frac{aa \sin(\theta) \, (\mathrm{d}(\theta) \, \&^{\wedge} \, \mathrm{d}(r))}{2 \, r + M} - 2 \, \frac{r \, aa^2 \cos(\theta) \sin(\theta) \, (-2 \, r - 3 \, M + 4 \, M \cos(\theta)^2) \, (\mathrm{d}(\theta) \, \&^{\wedge} \, \mathrm{d}(\phi))}{2 \, r + M} + ((($$

$$-4 \, r^4 \, aa^2 \cos(\theta)^2 + M^2 + r^2 \, M^2 \, aa^2 + 4 \, r^3 \, aa^2 \, M - 4 \, r^3 \, aa^2 \cos(\theta)^2 \, M + 2 \, r^2 \, M^2 \, aa^2 \cos(\theta)^4 - 3 \, r^2 \, M^2 \, aa^2 \cos(\theta)^2$$

$$+ \, 4 \, r^4 \, aa^2 + 4 \, r \, M - 4 \, r^2) \, (\mathrm{d}(r) \, \&^{\wedge} \, \mathrm{d}(\phi))) \Big/ (r^2 \, (2 \, r + M)^2) \, , \, - ((15 \sin(\theta)^2 \, aa^2 \, r^2 \cos(\theta)^4 \, M$$

$$- \, 12 \sin(\theta)^2 \, aa^2 \, r^2 \cos(\theta)^2 \, M + 3 \sin(\theta)^2 \, aa^2 \, r^2 \, M + 2 \sin(\theta)^2 \, aa^2 \, r^3 - 4 \sin(\theta)^2 \, aa^2 \, r^3 \cos(\theta)^2 + \sin(\theta)^2 \, M$$

$$+ \, 2 \sin(\theta)^2 \, r - 2 \, r \cos(\theta)^2 + \cos(\theta)^4 \, M - 2 \, r^3 \, aa^2 \cos(\theta)^2 - M \cos(\theta)^2 - 3 \, r^2 \, M \, aa^2 \cos(\theta)^2 + 8 \, r^2 \, M \, aa^2 \cos(\theta)^4$$

$$- \, 6 \sin(\theta)^2 \, r^2 \, aa^2 \cos(\theta)^6 \, M + 2 \sin(\theta)^2 \, aa^2 \, r^3 \cos(\theta)^4 - 2 \, r^3 \, aa^2 \cos(\theta)^6 + 2 \cos(\theta)^8 \, r^2 \, aa^2 \, M + 4 \, r^3 \, aa^2 \cos(\theta)^4$$

$$+ \, 2 \cos(\theta)^2 \sin(\theta)^2 \, r + \cos(\theta)^2 \sin(\theta)^2 \, M - 7 \, r^2 \, aa^2 \cos(\theta)^6 \, M + 2 \, r \cos(\theta)^4) \, (\mathrm{d}(\theta) \, \&^{\wedge} \, \mathrm{d}(\phi))) \Big/ ($$

$$(-1 + \cos(\theta)^2) \, (2 \, r \cos(\theta)^2 - 2 \, r - M + M \cos(\theta)^2))$$

$$- \, 2 \, \frac{(2 \, r \cos(\theta)^2 \, M + 2 \, M^2 \cos(\theta)^2 - 6 \, r \, M - 4 \, r^2 - 3 \, M^2) \cos(\theta) \, r \sin(\theta) \, aa^2 \, (\mathrm{d}(r) \, \&^{\wedge} \, \mathrm{d}(\phi))}{(2 \, r + M)^2} - aa \cos(\theta) \, (4 \, r^2$$

$$+ \, 4 \, r \, M + 3 \, M^2 - 4 \, r^2 \cos(\theta)^2 - 4 \, r \cos(\theta)^2 \, M - 5 \, M^2 \cos(\theta)^2 + 2 \cos(\theta)^4 \, M^2 - 8 \sin(\theta)^2 \, r^2 - 8 \sin(\theta)^2 \, r \, M$$

$$- \, 2 \sin(\theta)^2 \, M^2) \, (\mathrm{d}(\theta) \, \&^{\wedge} \, \mathrm{d}(r)) \Big/ ((-1 + \cos(\theta)^2)^2 \, (2 \, r + M)^2) \, ,$$

$$-2 \, \frac{r \, aa \, M \sin(\theta) \, (5 \, r^2 \cos(\theta)^4 \, aa^2 - 6 \, r^2 \cos(\theta)^2 \, aa^2 + r^2 \, aa^2 + 1 - 3 \cos(\theta)^2) \, (\mathrm{d}(\theta) \, \&^{\wedge} \, \mathrm{d}(\phi))}{2 \, r + M} + 2 \, M \, aa \cos(\theta) \, ($$

$$4 \, r^3 \, aa^2 \cos(\theta)^4 + 3 \, r^2 \, M \, aa^2 \cos(\theta)^4 - 8 \, r^3 \, aa^2 \cos(\theta)^2 - 6 \, r^2 \, M \, aa^2 \cos(\theta)^2 + 4 \, r^3 \, aa^2 + 3 \, r^2 \, M \, aa^2 + M - M \cos(\theta)^2$$

$$) \, (\mathrm{d}(r) \, \&^{\wedge} \, \mathrm{d}(\phi)) \Big/ (2 \, r + M)^2 - 4 \, \frac{\cos(\theta) \, r \sin(\theta) \, aa^2 \, M \, (-r + 3 \, r \cos(\theta)^2 + M \cos(\theta)^2) \, (\mathrm{d}(\theta) \, \&^{\wedge} \, \mathrm{d}(r))}{(2 \, r + M)^2} \, , \, 0 \Bigg]$$

$$[0 \, , \, 0 \, , \, 0 \, , \, 0]$$

$OMEGAG :=$

$$\Bigg[2 \, \frac{(4 \, r - 4 \, r \cos(\theta)^2 - 5 \, M \cos(\theta)^2 + 3 \, M + 2 \cos(\theta)^4 \, M) \, aa^3 \sin(\theta) \, r^3 \cos(\theta)^2 \, (\mathrm{d}(\theta) \, \&^{\wedge} \, \mathrm{d}(\phi))}{2 \, r + M} + \cos(\theta) \, r \, aa \, ($$

$$4 \, r^2 \, M \, aa^2 \cos(\theta)^4 + 12 \, r^3 \, aa^2 + 4 \, r^2 \, aa^2 \cos(\theta)^6 \, M + 16 \, r^3 \, aa^2 \cos(\theta)^4 - 28 \, r^3 \, aa^2 \cos(\theta)^2 + 2 \, aa^2 \, r \cos(\theta)^4 \, M^2$$

$$+ \, 8 \, r^2 \, M \, aa^2 - 16 \, r^2 \, M \, aa^2 \cos(\theta)^2 - 3 \, aa^2 \, r \cos(\theta)^2 \, M^2 + aa^2 \, r \, M^2 + 4 \, M - 4 \, M \cos(\theta)^2) \, (\mathrm{d}(r) \, \&^{\wedge} \, \mathrm{d}(\phi)) \Big/ ($$

$$4 \, r^2 + 4 \, r \, M + M^2) \, ,$$

$$2 \, \frac{(-4 \, r \cos(\theta)^4 + 4 \, r \cos(\theta)^2 + 6 \, M \cos(\theta)^2 - M + 2 \cos(\theta)^6 \, M - 7 \cos(\theta)^4 \, M) \, aa^3 \, r^4 \cos(\theta) \, (\mathrm{d}(\theta) \, \&^{\wedge} \, \mathrm{d}(\phi))}{2 \, r + M} - aa$$

$$\sin(\theta) \, r^2 \, (4 \, M + 4 \, r^2 \, aa^2 \cos(\theta)^6 \, M + 4 \, r^2 \, M \, aa^2 \cos(\theta)^4 - 12 \, r^3 \, aa^2 \cos(\theta)^2 + 16 \, r^3 \, aa^2 \cos(\theta)^4 - 4 \, r^3 \, aa^2$$

$$- \, 4 \, r^2 \, M \, aa^2 - 4 \, M \cos(\theta)^2 - 4 \, r^2 \, M \, aa^2 \cos(\theta)^2 - aa^2 \, r \, M^2 + 2 \, aa^2 \, r \cos(\theta)^4 \, M^2 - aa^2 \, r \cos(\theta)^2 \, M^2)$$

$$(\mathrm{d}(r) \, \&^{\wedge} \, \mathrm{d}(\phi)) \Big/ (4 \, r^2 + 4 \, r \, M + M^2) - r \, (4 \, aa^2 \, r \cos(\theta)^4 \, M^2 - 4 \, M + 12 \, r^2 \, M \, aa^2 \cos(\theta)^4 - 12 \, r^2 \, M \, aa^2 \cos(\theta)^2$$

$$- \, 4 \, aa^2 \, r \cos(\theta)^2 \, M^2 + 4 \, r^3 \, aa^2 + aa^2 \, r \, M^2 + 4 \, r^2 \, M \, aa^2) \, (\mathrm{d}(\theta) \, \&^{\wedge} \, \mathrm{d}(r)) \Big/ (4 \, r^2 + 4 \, r \, M + M^2) \, , \, 2 \, (($$

$$4 \, r^2 \, M \, aa^2 \cos(\theta)^2 - 5 \, r^2 \, M \, aa^2 \cos(\theta)^4 + 2 \, r^2 \, aa^2 \cos(\theta)^6 \, M - r^2 \, M \, aa^2 - 4 \, r + 4 \, r \cos(\theta)^2 + 5 \, M \cos(\theta)^2$$

$$- \, 2 \cos(\theta)^4 \, M - 3 \, M) \, aa^2 \sin(\theta) \, r^3 \cos(\theta) \, (\mathrm{d}(\theta) \, \&^{\wedge} \, \mathrm{d}(\phi))) / (2 \, r + M) + r \, (3 \, aa^2 \, r \cos(\theta)^2 \, M^2$$

$$- \, 2 \, r^3 \, M^2 \, aa^4 \cos(\theta)^2 + r^3 \, M^2 \, aa^4 \cos(\theta)^4 - 2 \, aa^2 \, r \cos(\theta)^4 \, M^2 - aa^2 \, r \, M^2 - 12 \, r^2 \, M \, aa^2 - 12 \, r^3 \, aa^2 + 4 \, r^5 \, aa^4$$

$$- \, 8 \, r^5 \, aa^4 \cos(\theta)^2 + 16 \, r^4 \cos(\theta)^4 \, aa^4 \, M + 4 \, r^4 \cos(\theta)^8 \, aa^4 \, M - 12 \, r^4 \cos(\theta)^6 \, aa^4 \, M + 24 \, r^2 \, M \, aa^2 \cos(\theta)^2 - 4 \, M$$

$$- \, 4 \, r^2 \, aa^2 \cos(\theta)^6 \, M - 8 \, r^2 \, M \, aa^2 \cos(\theta)^4 - 16 \, r^3 \, aa^2 \cos(\theta)^4 - 12 \, r^4 \cos(\theta)^2 \, aa^4 \, M + 28 \, r^3 \, aa^2 \cos(\theta)^2 + 4 \, r^4 \, aa^4 \, M$$

$$+ \, 4 \, r^5 \, aa^4 \cos(\theta)^4 + 4 \, M \cos(\theta)^2 + r^3 \, M^2 \, aa^4) \, (\mathrm{d}(r) \, \&^{\wedge} \, \mathrm{d}(\phi)) \Big/ (4 \, r^2 + 4 \, r \, M + M^2) + aa \sin(\theta) \, r^2 \, ($$

$4 r^3 aa^2 \cos(\theta)^2 - aa^2 r M^2 - 24 r^2 M aa^2 \cos(\theta)^4 + 16 r^2 M aa^2 \cos(\theta)^2 + 4 aa^2 r \cos(\theta)^6 M^2 - 8 aa^2 r \cos(\theta)^4 M^2$

$+ 12 r^2 aa^2 \cos(\theta)^6 M - 4 M \cos(\theta)^2 + 5 aa^2 r \cos(\theta)^2 M^2 - 4 r^3 aa^2 - 4 r^2 M aa^2 + 4 M) (d(\theta) \&^\wedge d(r)) \Big/ ($

$4 r^2 + 4 r M + M^2), 0\Big]$

$\Big[\cos(\theta) r aa (-24 r^2 M aa^2 + 21 aa^2 r \cos(\theta)^2 M^2 + 8 aa^2 r \cos(\theta)^6 M^2 - 22 aa^2 r \cos(\theta)^4 M^2 - 7 aa^2 r M^2$

$- 20 r^3 aa^2 - 16 r^3 aa^2 \cos(\theta)^4 + 8 r^2 aa^2 \cos(\theta)^6 M - 36 r^2 M aa^2 \cos(\theta)^4 + 8 M \cos(\theta)^2 + 52 r^2 M aa^2 \cos(\theta)^2$

$+ 36 r^3 aa^2 \cos(\theta)^2 - 8 M) (d(\theta) \&^\wedge d(\phi)) \Big/ (4 r^2 + 4 r M + M^2) - 2 r aa^3 \sin(\theta) (-2 \cos(\theta)^6 M^2 + 2 \cos(\theta)^4 r M$

$+ 2 r \cos(\theta)^6 M + 8 \cos(\theta)^4 r^2 - 16 r^2 \cos(\theta)^2 + 8 r^2 + 6 r M + M^2 - 10 r \cos(\theta)^2 M + 5 \cos(\theta)^4 M^2 - 4 M^2 \cos(\theta)^2)$

$(d(r) \&^\wedge d(\phi)) \Big/ (4 r^2 + 4 r M + M^2)$

$+ \dfrac{(-4 M + 4 r^3 aa^2 + 4 r^2 M aa^2 + 4 aa^2 r \cos(\theta)^4 M^2 - 4 aa^2 r \cos(\theta)^2 M^2 + aa^2 r M^2) (d(\theta) \&^\wedge d(r))}{r (4 r^2 + 4 r M + M^2)},$

$-12 \dfrac{aa^2 \sin(\theta) r^2 \cos(\theta) M (-1 + \cos(\theta)^2) (d(r) \&^\wedge d(\theta))}{4 r^2 + 4 r M + M^2} - aa \sin(\theta) r^2 (36 r^2 M aa^2 \cos(\theta)^2 + 8 M \cos(\theta)^2$

$+ 17 aa^2 r \cos(\theta)^2 M^2 + 8 aa^2 r \cos(\theta)^6 M^2 - 16 r^3 aa^2 \cos(\theta)^4 - 22 aa^2 r \cos(\theta)^4 M^2 - 3 aa^2 r M^2$

$+ 20 r^3 aa^2 \cos(\theta)^2 - 8 r^2 M aa^2 - 36 r^2 M aa^2 \cos(\theta)^4 + 8 r^2 aa^2 \cos(\theta)^6 M - 4 r^3 aa^2 - 8 M) (d(\theta) \&^\wedge d(\phi)) \Big/ ($

$4 r^2 + 4 r M + M^2) - 2 aa^3 r^2 \cos(\theta) (8 r^2 + 2 r \cos(\theta)^6 M + 8 \cos(\theta)^4 r^2 - 2 \cos(\theta)^6 M^2 - 16 r^2 \cos(\theta)^2$

$+ 7 \cos(\theta)^4 M^2 + 4 r M - 6 r \cos(\theta)^2 M - 8 M^2 \cos(\theta)^2 + 3 M^2) (d(r) \&^\wedge d(\phi)) \Big/ (4 r^2 + 4 r M + M^2), r ($

$4 r^5 aa^4 \cos(\theta)^4 - 8 r^5 aa^4 \cos(\theta)^2 - 8 r^2 aa^2 \cos(\theta)^6 M + 4 r^5 aa^4 - 28 r^4 \cos(\theta)^2 aa^4 M + 40 r^4 \cos(\theta)^4 aa^4 M$

$- 28 r^4 \cos(\theta)^6 aa^4 M + 8 r^4 \cos(\theta)^8 aa^4 M + 8 r^4 aa^4 M + 22 aa^2 r \cos(\theta)^4 M^2 - 8 aa^2 r \cos(\theta)^6 M^2 + 3 aa^2 r M^2$

$- 17 aa^2 r \cos(\theta)^2 M^2 + 4 r^3 aa^2 - 8 M \cos(\theta)^2 + 8 M - 16 r^3 M^2 aa^4 \cos(\theta)^2 - 26 r^3 M^2 aa^4 \cos(\theta)^6$

$- 20 r^3 aa^2 \cos(\theta)^2 + 16 r^3 aa^2 \cos(\theta)^4 + 31 r^3 M^2 aa^4 \cos(\theta)^4 + 8 r^3 M^2 aa^4 \cos(\theta)^8 + 16 r^2 M aa^2$

$- 52 r^2 M aa^2 \cos(\theta)^2 + 44 r^2 M aa^2 \cos(\theta)^4 + 3 r^3 M^2 aa^4) (d(\theta) \&^\wedge d(\phi)) \Big/ (4 r^2 + 4 r M + M^2) - 2 r aa^2 \sin(\theta)$

$\cos(\theta) (6 r^2 M^2 aa^2 \cos(\theta)^4 + 3 M^2 - 5 M^2 \cos(\theta)^2 - 6 r^2 M^2 aa^2 \cos(\theta)^2 - 2 r \cos(\theta)^2 M - 2 r^2 M^2 aa^2 \cos(\theta)^6$

$+ 4 r M + 2 r^2 M^2 aa^2 + 6 r^3 aa^2 \cos(\theta)^2 M + 2 \cos(\theta)^6 r^3 aa^2 M + 8 r^2 - 8 r^2 \cos(\theta)^2 - 6 r^3 aa^2 \cos(\theta)^4 M$

$- 2 \cos(\theta)^4 r M - 2 r^3 aa^2 M + 2 \cos(\theta)^4 M^2) (d(r) \&^\wedge d(\phi)) \Big/ (4 r^2 + 4 r M + M^2) + \cos(\theta) r aa (-8 M$

$+ 5 aa^2 r \cos(\theta)^2 M^2 + 4 aa^2 r \cos(\theta)^6 M^2 + 4 r^3 aa^2 \cos(\theta)^2 + 12 r^2 aa^2 \cos(\theta)^6 M - 36 r^2 M aa^2 \cos(\theta)^4$

$- 16 r^2 M aa^2 - 4 r^3 aa^2 + 40 r^2 M aa^2 \cos(\theta)^2 - 8 aa^2 r \cos(\theta)^4 M^2 - aa^2 r M^2 + 8 M \cos(\theta)^2) (d(\theta) \&^\wedge d(r)) \Big/ ($

$4 r^2 + 4 r M + M^2), 0\Big]$

$\Big[-2 \dfrac{(-4 r - 3 M + 2 M \cos(\theta)^2) r \cos(\theta) \sin(\theta) aa^2 (d(\theta) \&^\wedge d(\phi))}{2 r + M} - ((8 r^2 M aa^2 \cos(\theta)^2 - 12 r^3 aa^2 - 4 M$

$+ 2 aa^2 r \cos(\theta)^2 M^2 + 16 r^3 aa^2 \cos(\theta)^2 - aa^2 r M^2 - 8 r^2 M aa^2 + 4 r^2 M aa^2 \cos(\theta)^4) (d(r) \&^\wedge d(\phi))) \Big/ ($

$r (4 r^2 + 4 r M + M^2)), - ((8 aa^2 r \cos(\theta)^4 M^2 + 8 r^2 M aa^2 \cos(\theta)^4 - 14 aa^2 r \cos(\theta)^2 M^2 - 28 r^2 M aa^2 \cos(\theta)^2$

$- 16 r^3 aa^2 \cos(\theta)^2 + 8 M + 4 r^3 aa^2 + 8 r^2 M aa^2 + 3 aa^2 r M^2) r (d(\theta) \&^\wedge d(\phi))) \Big/ (4 r^2 + 4 r M + M^2)$

$+ 2 \dfrac{(-2 M^2 \cos(\theta)^2 + 2 r \cos(\theta)^2 M + 4 r M + 8 r^2 + 3 M^2) r \cos(\theta) \sin(\theta) aa^2 (d(r) \&^\wedge d(\phi))}{4 r^2 + 4 r M + M^2}$

$+ 12 \dfrac{r aa \cos(\theta) (d(\theta) \&^\wedge d(r)) M}{4 r^2 + 4 r M + M^2}, 12 \dfrac{aa^2 \sin(\theta) r^2 \cos(\theta) M (-1 + \cos(\theta)^2) (d(r) \&^\wedge d(\theta))}{4 r^2 + 4 r M + M^2} + aa \sin(\theta) r^2 ($

$16 r^2 M aa^2 \cos(\theta)^2 + 8 M \cos(\theta)^2 + 11 aa^2 r \cos(\theta)^2 M^2 + 4 aa^2 r \cos(\theta)^6 M^2 - 12 aa^2 r \cos(\theta)^4 M^2 - 3 aa^2 r M^2$

$+ 4 r^3 aa^2 \cos(\theta)^2 - 8 r^2 M aa^2 - 8 r^2 M aa^2 \cos(\theta)^4 - 4 r^3 aa^2 - 8 M) (d(\theta) \&^\wedge d(\phi)) \Big/ (4 r^2 + 4 r M + M^2) -$

$$\cos(\theta)\, r\, aa\, (4\, r^2\, M\, aa^2\, \cos(\theta)^4 - 4\, r^3\, aa^2 + 4\, aa^2\, r\, \cos(\theta)^6\, M^2 + 4\, r^3\, aa^2\, \cos(\theta)^2 - 12\, aa^2\, r\, \cos(\theta)^4\, M^2$$
$$- 4\, r^2\, M\, aa^2\, \cos(\theta)^2 + 13\, aa^2\, r\, \cos(\theta)^2\, M^2 - 5\, aa^2\, r\, M^2 + 4\, M - 4\, M\, \cos(\theta)^2)\, (\mathrm{d}(r)\, \&^{\wedge}\, \mathrm{d}(\phi))\, \Big/ \, ($$
$$4\, r^2 + 4\, r\, M + M^2)\, , 0 \Bigg]$$

$$[\, 0\, , 0\, , 0\, , 0\,]$$

`> Christoffel_curvature_no_torsion:=(subs(M=M,aa=0,evalm(OMEGAG)));`

Christoffel_curvature_no_torsion :=

$$\begin{bmatrix} 0 & 4\,\dfrac{(\mathrm{d}(\theta)\,\&^{\wedge}\,\mathrm{d}(r))\, r\, M}{4\, r^2 + 4\, r\, M + M^2} & \dfrac{r\,(-4\, M + 4\, M\, \cos(\theta)^2)\,(\mathrm{d}(r)\,\&^{\wedge}\,\mathrm{d}(\phi))}{4\, r^2 + 4\, r\, M + M^2} & 0 \\[4mm] -4\,\dfrac{(\mathrm{d}(\theta)\,\&^{\wedge}\,\mathrm{d}(r))\, M}{r\,(4\, r^2 + 4\, r\, M + M^2)} & 0 & \dfrac{r\,(-8\, M\, \cos(\theta)^2 + 8\, M)\,(\mathrm{d}(\theta)\,\&^{\wedge}\,\mathrm{d}(\phi))}{4\, r^2 + 4\, r\, M + M^2} & 0 \\[4mm] 4\,\dfrac{M\,(\mathrm{d}(r)\,\&^{\wedge}\,\mathrm{d}(\phi))}{r\,(4\, r^2 + 4\, r\, M + M^2)} & -8\,\dfrac{r\,(\mathrm{d}(\theta)\,\&^{\wedge}\,\mathrm{d}(\phi))\, M}{4\, r^2 + 4\, r\, M + M^2} & 0 & 0 \\[4mm] 0 & 0 & 0 & 0 \end{bmatrix}$$

`>`
`> Christoffel_curvature_no_mass:=(subs(M=0,evalm(OMEGAG)));`

Christoffel_curvature_no_mass :=

$$\Bigg[(4\, r - 4\, r\, \cos(\theta)^2)\, aa^3\, \sin(\theta)\, r^2\, \cos(\theta)^2\, (\mathrm{d}(\theta)\,\&^{\wedge}\,\mathrm{d}(\phi))$$

$$+ \frac{1}{4}\,\frac{\cos(\theta)\, aa\, (12\, r^3\, aa^2 + 16\, r^3\, aa^2\, \cos(\theta)^4 - 28\, r^3\, aa^2\, \cos(\theta)^2)\,(\mathrm{d}(r)\,\&^{\wedge}\,\mathrm{d}(\phi))}{r}\, ,$$

$$(-4\, r\, \cos(\theta)^4 + 4\, r\, \cos(\theta)^2)\, aa^3\, r^3\, \cos(\theta)\,(\mathrm{d}(\theta)\,\&^{\wedge}\,\mathrm{d}(\phi))$$

$$- \frac{1}{4}\, aa\, \sin(\theta)\,(-12\, r^3\, aa^2\, \cos(\theta)^2 + 16\, r^3\, aa^2\, \cos(\theta)^4 - 4\, r^3\, aa^2)\,(\mathrm{d}(r)\,\&^{\wedge}\,\mathrm{d}(\phi)) - r^2\, aa^2\,(\mathrm{d}(\theta)\,\&^{\wedge}\,\mathrm{d}(r))\, ,$$

$$(-4\, r + 4\, r\, \cos(\theta)^2)\, aa^2\, \sin(\theta)\, r^2\, \cos(\theta)\,(\mathrm{d}(\theta)\,\&^{\wedge}\,\mathrm{d}(\phi)) + \frac{1}{4}$$

$$\frac{(-12\, r^3\, aa^2 + 4\, r^5\, aa^4 - 8\, r^5\, aa^4\, \cos(\theta)^2 - 16\, r^3\, aa^2\, \cos(\theta)^4 + 28\, r^3\, aa^2\, \cos(\theta)^2 + 4\, r^5\, aa^4\, \cos(\theta)^4)\,(\mathrm{d}(r)\,\&^{\wedge}\,\mathrm{d}(\phi))}{r}$$

$$+ \frac{1}{4}\, aa\, \sin(\theta)\,(4\, r^3\, aa^2\, \cos(\theta)^2 - 4\, r^3\, aa^2)\,(\mathrm{d}(\theta)\,\&^{\wedge}\,\mathrm{d}(r))\, , 0 \Bigg]$$

$$\Bigg[\frac{1}{4}\,\frac{\cos(\theta)\, aa\, (-20\, r^3\, aa^2 - 16\, r^3\, aa^2\, \cos(\theta)^4 + 36\, r^3\, aa^2\, \cos(\theta)^2)\,(\mathrm{d}(\theta)\,\&^{\wedge}\,\mathrm{d}(\phi))}{r}$$

$$- \frac{1}{2}\,\frac{aa^3\, \sin(\theta)\,(8\, r^2 + 8\, \cos(\theta)^4\, r^2 - 16\, r^2\, \cos(\theta)^2)\,(\mathrm{d}(r)\,\&^{\wedge}\,\mathrm{d}(\phi))}{r} + aa^2\,(\mathrm{d}(\theta)\,\&^{\wedge}\,\mathrm{d}(r))\, ,$$

$$- \frac{1}{4}\, aa\, \sin(\theta)\,(-16\, r^3\, aa^2\, \cos(\theta)^4 + 20\, r^3\, aa^2\, \cos(\theta)^2 - 4\, r^3\, aa^2)\,(\mathrm{d}(\theta)\,\&^{\wedge}\,\mathrm{d}(\phi))$$

$$- \frac{1}{2}\, aa^3\, \cos(\theta)\,(8\, r^2 + 8\, \cos(\theta)^4\, r^2 - 16\, r^2\, \cos(\theta)^2)\,(\mathrm{d}(r)\,\&^{\wedge}\,\mathrm{d}(\phi))\, ,$$

$$\frac{1}{4}\,\frac{(4\, r^5\, aa^4\, \cos(\theta)^4 - 8\, r^5\, aa^4\, \cos(\theta)^2 + 4\, r^5\, aa^4 + 4\, r^3\, aa^2 - 20\, r^3\, aa^2\, \cos(\theta)^2 + 16\, r^3\, aa^2\, \cos(\theta)^4)\,(\mathrm{d}(\theta)\,\&^{\wedge}\,\mathrm{d}(\phi))}{r}$$

$$- \frac{1}{2}\,\frac{aa^2\, \sin(\theta)\, \cos(\theta)\,(8\, r^2 - 8\, r^2\, \cos(\theta)^2)\,(\mathrm{d}(r)\,\&^{\wedge}\,\mathrm{d}(\phi))}{r}$$

$$+ \frac{1}{4} \frac{\cos(\theta)\, aa\, (4\, r^3\, aa^2 \cos(\theta)^2 - 4\, r^3\, aa^2)\, (d(\theta)\, \&{}^\wedge\, d(r))}{r}\,,\, 0 \Bigg]$$

$$\Bigg[4\, r \cos(\theta) \sin(\theta)\, aa^2\, (d(\theta)\, \&{}^\wedge\, d(\phi)) - \frac{1}{4} \frac{(-12\, r^3\, aa^2 + 16\, r^3\, aa^2 \cos(\theta)^2)\, (d(r)\, \&{}^\wedge\, d(\phi))}{r^3}\,,$$

$$-\frac{1}{4} \frac{(-16\, r^3\, aa^2 \cos(\theta)^2 + 4\, r^3\, aa^2)\, (d(\theta)\, \&{}^\wedge\, d(\phi))}{r} + 4 \sin(\theta) \cos(\theta)\, (d(r)\, \&{}^\wedge\, d(\phi))\, r\, aa^2\,,$$

$$\frac{1}{4} aa \sin(\theta)\, (4\, r^3\, aa^2 \cos(\theta)^2 - 4\, r^3\, aa^2)\, (d(\theta)\, \&{}^\wedge\, d(\phi))$$

$$-\frac{1}{4} \frac{\cos(\theta)\, aa\, (4\, r^3\, aa^2 \cos(\theta)^2 - 4\, r^3\, aa^2)\, (d(r)\, \&{}^\wedge\, d(\phi))}{r}\,,\, 0 \Bigg]$$

$$[0,0,0,0]$$

```
>
> Christoffel_curvature_no_torsion_noMass:=(subs(M=0,aa=0,aa^2=0,aa^3=0,aa^4=0,eva
  lm(OMEGAG)));
>
```

$$Christoffel\_curvature\_no\_torsion\_noMass := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

THE matrix of 2-forms (Christoffel curvature = d[Gamma] + [Gamma]^[Gamma]) is non-zero for the perturbed cases. The matrix depends upon Mass(the metric perturbation), but is of second order in aa (the frame perturbation). But as shown below the trace is ZERO

```
> TRACEOMEGA_noTorsion_smallmass:=subs(aa=0,M^2=0,2*r+M=2*r,simpform(wcollect(OMEG
  AG[1,1]+OMEGAG[2,2]+OMEGAG[3,3]+OMEGAG[4,4])));JJ:=subs(d(TRACEOMEGA_noTorsion_s
  mallmass));
```

$$TRACEOMEGA\_noTorsion\_smallmass := 0$$

$$JJ := 0$$

```
> TRACEOMEGA_smallTorsion_nomass:=subs(aa^2=0,M=0,simpform(wcollect(OMEGAG[1,1]+OM
  EGAG[2,2]+OMEGAG[3,3]+OMEGAG[4,4])));JJ:=subs(d(TRACEOMEGA_smallTorsion_nomass))
  ;
```

$$TRACEOMEGA\_smallTorsion\_nomass := 0$$

$$JJ := 0$$

```
> TRACEOMEGA:=((subs(aa^2=0,M^2=0,2*r+M=2*r,simpform(wcollect(OMEGAG[1,1]+OMEGAG[2
  ,2]+OMEGAG[3,3]+OMEGAG[4,4])))));JJ:=simpform(subs(M^2=0,d(TRACEOMEGA)));
```

$$TRACEOMEGA := 0$$

$$JJ := 0$$

SURPRISE - BUT IN AGREEMENT WITH BILL PAGE's RESULT. THE FORMULA FOR THE CURVATURE 2-FORMS GIVEN BY d[Gamma] + [Gamma]^[Gamma] for the symmetric (perturbed metric) gives a non-zero curvature matrix , but the trace is ZERO.

```
>
>
```

```
> dtheta_dphi:=simplify(factor(subs(aa^2=0,(getcoeff(TRACEOMEGA&^d(r)&^d(tau))))))
  ;dr_dphi:=simplify(factor(subs(aa^2=0,(getcoeff(TRACEOMEGA&^d(theta)&^d(tau)))))
  );dr_dtheta:=simplify(factor(subs(aa^2=0,(getcoeff(TRACEOMEGA&^d(phi)&^d(tau))))
  ));simplify(factor(subs(aa^2=0,(getcoeff(TRACEOMEGA&^d(phi)&^d(theta))))));simpl
  ify(factor(subs(aa^2=0,(getcoeff(TRACEOMEGA&^d(phi)&^d(r))))));simplify(factor(s
  ubs(aa^2=0,(getcoeff(TRACEOMEGA&^d(theta)&^d(r))))));
```

$$dtheta\_dphi := 0$$
$$dr\_dphi := 0$$
$$dr\_dtheta := 0$$
$$0$$
$$0$$
$$0$$

****

BUT NOw compute curvature with the formula d[Gamma] + [C]^[Gamma] which is the definition of " metric curvature "

```
> OmegaCC11:=simpform(simplify(CGamma11&^Gamma11+CGamma12&^Gamma21+CGamma13&^Gamma
  31+CGamma14&^Gamma41));OmegaCC12:=simpform(simplify(CGamma11&^Gamma12+CGamma12&^
  Gamma22+CGamma13&^Gamma32+CGamma14&^Gamma42));OmegaCC13:=simpform(simplify(CGamm
  a11&^Gamma13+CGamma12&^Gamma23+CGamma13&^Gamma33+CGamma14&^Gamma43));OmegaCC14:=
  simpform(simplify(CGamma11&^Gamma14+CGamma12&^Gamma24+CGamma13&^Gamma34+CGamma14
  &^Gamma44));
```

$$OmegaCC11 := -\frac{r\sin(\theta)\,aa^2\cos(\theta)\,(-2\,r - 2\,M\cos(\theta)^2 + M + 4\,r\cos(\theta)^2)\,(\mathrm{d}(r)\,\&^\wedge\,\mathrm{d}(\theta))}{2\,r + M} - r\sin(\theta)\,aa\,($$
$$10\,r\cos(\theta)^2 + 5\,r^2\,M\,aa^2\cos(\theta)^2 + 4\,r^2\,aa^2\cos(\theta)^6\,M + 6\,r^3\,aa^2\cos(\theta)^2 - 8\,r^2\,M\,aa^2\cos(\theta)^4 - 4\,r^3\,aa^2\cos(\theta)^4$$
$$+ M\cos(\theta)^2 - r^2\,M\,aa^2 - 2\,r^3\,aa^2 - 2\,r - M)\,(\mathrm{d}(\theta)\,\&^\wedge\,\mathrm{d}(\phi))\,/\,(2\,r + M) + \cos(\theta)\,aa\,(-5\,r^2\,M\,aa^2\cos(\theta)^4$$
$$+ 4\,r^2\,M\,aa^2\cos(\theta)^2 - 2\,r^3\,aa^2 + 2\,r^2\,aa^2\cos(\theta)^6\,M - 2\,r^3\,aa^2\cos(\theta)^4 + 4\,r^3\,aa^2\cos(\theta)^2 + 6\,r\cos(\theta)^2 - 6\,r$$
$$- r^2\,M\,aa^2 - 3\,M\cos(\theta)^2 + 3\,M)\,(\mathrm{d}(r)\,\&^\wedge\,\mathrm{d}(\phi))\,/\,(2\,r + M)$$

$$OmegaCC12 := r^2\,aa\cos(\theta)\,(-4\,r^3\,aa^2\cos(\theta)^4 + 11\,r^2\,M\,aa^2\cos(\theta)^2 - 12\,r^2\,M\,aa^2\cos(\theta)^4 + 6\,r^3\,aa^2\cos(\theta)^2$$
$$+ 5\,M\cos(\theta)^2 - 8\,r + 10\,r\cos(\theta)^2 - 4\,M - 3\,r^2\,M\,aa^2 - 2\,r^3\,aa^2 + 4\,r^2\,aa^2\cos(\theta)^6\,M)\,(\mathrm{d}(\phi)\,\&^\wedge\,\mathrm{d}(\theta))\,/\,(2\,r + M)$$
$$+ \sin(\theta)\,r\,aa\,(8\,r\cos(\theta)^2 + 2\,M\cos(\theta)^2 - 2\,r^3\,aa^2\cos(\theta)^2 - 5\,r^2\,M\,aa^2\cos(\theta)^4 + 2\,r^2\,aa^2\cos(\theta)^6\,M$$
$$+ 3\,r^2\,M\,aa^2\cos(\theta)^2 - M + 2\,r^3\,aa^2\cos(\theta)^2 - 2\,r)\,(\mathrm{d}(\phi)\,\&^\wedge\,\mathrm{d}(r))\,/\,(2\,r + M)$$
$$+ \frac{(2\,r^3\,aa^2 + M - 2\,r - 4\,r^3\,aa^2\cos(\theta)^2 + r^2\,M\,aa^2 + 4\,r^3\,aa^2\cos(\theta)^4)\,(\mathrm{d}(\theta)\,\&^\wedge\,\mathrm{d}(r))}{2\,r + M}$$

$$OmegaCC13 := -2\,r\cos(\theta)\sin(\theta)\,(-M - 2\,r^2\,M\,aa^2\cos(\theta)^4 + 2\,r^4\cos(\theta)^6\,aa^4\,M + 4\,r^4\cos(\theta)^2\,aa^4\,M - r^4\,aa^4\,M$$
$$+ 4\,r^3\,aa^2\cos(\theta)^2 - 5\,r^4\cos(\theta)^4\,aa^4\,M - 4\,r^3\,aa^2 + 5\,r^2\,M\,aa^2\cos(\theta)^2 - 3\,r^2\,M\,aa^2 - 2\,r)\,(\mathrm{d}(\theta)\,\&^\wedge\,\mathrm{d}(\phi))\,/\,($$
$$2\,r + M) - ((6\,r^2\,M\,aa^2\cos(\theta)^4 - 2\,r^4\cos(\theta)^2\,aa^4\,M - M + 4\,r^3\,aa^2\cos(\theta)^4 + 2\,r - 6\,r^3\,aa^2\cos(\theta)^2 - 2\,r\cos(\theta)^2$$
$$+ 2\,r^3\,aa^2 + M\cos(\theta)^2 + r^2\,M\,aa^2 + 6\,r^4\cos(\theta)^4\,aa^4\,M + 2\,r^4\cos(\theta)^8\,aa^4\,M - 6\,r^4\cos(\theta)^6\,aa^4\,M$$
$$- 5\,r^2\,M\,aa^2\cos(\theta)^2 - 2\,r^2\,aa^2\cos(\theta)^6\,M)\,(\mathrm{d}(\phi)\,\&^\wedge\,\mathrm{d}(r)))\,/\,(2\,r + M) - aa\sin(\theta)\,r\,(-2\,r^3\,aa^2\cos(\theta)^4 - r^2\,M\,aa^2$$
$$+ 4\,r^3\,aa^2\cos(\theta)^2 + 2\,r^2\,M\,aa^2\cos(\theta)^2 - 3\,r^2\,M\,aa^2\cos(\theta)^4 + 2\,r\cos(\theta)^2 - M\cos(\theta)^2 + 2\,r^2\,aa^2\cos(\theta)^6\,M$$
$$- 2\,r^3\,aa^2)\,(\mathrm{d}(\theta)\,\&^\wedge\,\mathrm{d}(r))\,/\,(2\,r + M)$$

$$OmegaCC14 := 0$$

```
> OmegaCC21:=simpform(simplify(CGamma21&^Gamma11+CGamma22&^Gamma21+CGamma23&^Gamma
  31+CGamma24&^Gamma41));OmegaCC22:=simpform(simplify(CGamma21&^Gamma12+CGamma22&^
```

```
Gamma22+CGamma23&^Gamma32+CGamma24&^Gamma42));OmegaCC23:=simpform(simplify(CGamm
a21&^Gamma13+CGamma22&^Gamma23+CGamma23&^Gamma33+CGamma24&^Gamma43));OmegaCC24:=
simpform(simplify(CGamma21&^Gamma14+CGamma22&^Gamma24+CGamma23&^Gamma34+CGamma24
&^Gamma44));
```

$OmegaCC21 := \cos(\theta)\, aa\, (2\,M + 4\,r^2\,aa^2\cos(\theta)^6\,M - 4\,r^3\,aa^2\cos(\theta)^4 + 8\,r^2\,M\,aa^2\cos(\theta)^2 - M\cos(\theta)^2$

$\quad - 10\,r^2\,M\,aa^2\cos(\theta)^4 - 2\,r^2\,M\,aa^2 - 4\,r^3\,aa^2 + 8\,r^3\,aa^2\cos(\theta)^2 - 8\,r + 10\,r\cos(\theta)^2)\,(\mathrm{d}(\phi)\,\&^{\wedge}\,\mathrm{d}(\theta))\,/\,(2\,r + M)$

$\quad + \dfrac{\sin(\theta)\,aa\,(-4\,M\cos(\theta)^2 + 3\,M - 2\,r + 4\,r\cos(\theta)^2)\,(\mathrm{d}(\phi)\,\&^{\wedge}\,\mathrm{d}(r))}{r\,(2\,r + M)}$

$\quad + \dfrac{(-M - 2\,r - 8\,r^3\,aa^2\cos(\theta)^2 + 8\,r^3\,aa^2\cos(\theta)^4)\,(\mathrm{d}(\theta)\,\&^{\wedge}\,\mathrm{d}(r))}{r^2\,(2\,r + M)}$

$OmegaCC22 := 2\cos(\theta)\,aa^2\,r\sin(\theta)\,(-1 + 2\cos(\theta)^2)\,(\mathrm{d}(r)\,\&^{\wedge}\,\mathrm{d}(\theta)) + r\sin(\theta)\,aa\,(10\,r\cos(\theta)^2$

$\quad + 6\,r^2\,M\,aa^2\cos(\theta)^2 + 4\,r^2\,aa^2\cos(\theta)^6\,M + 4\,r^3\,aa^2\cos(\theta)^2 - 10\,r^2\,M\,aa^2\cos(\theta)^4 - 4\,r^3\,aa^2\cos(\theta)^4 + 3\,M\cos(\theta)^2$

$\quad - 4\,r + 2\,M)\,(\mathrm{d}(\theta)\,\&^{\wedge}\,\mathrm{d}(\phi))\,/\,(2\,r + M) - \dfrac{\cos(\theta)\,aa\,(6\,r\cos(\theta)^2 - 4\,r + M\cos(\theta)^2)\,(\mathrm{d}(r)\,\&^{\wedge}\,\mathrm{d}(\phi))}{2\,r + M}$

$OmegaCC23 := 2\,aa\cos(\theta)\,(-2\,r^3\,aa^2 + 4\,r^3\,aa^2\cos(\theta)^2 - 2\,r^3\,aa^2\cos(\theta)^4 - 5\,r^2\,M\,aa^2\cos(\theta)^4$

$\quad + 2\,r^2\,aa^2\cos(\theta)^6\,M - r^2\,M\,aa^2 + 4\,r^2\,M\,aa^2\cos(\theta)^2 + 2\,r\cos(\theta)^2 - 2\,r + M - M\cos(\theta)^2)\,(\mathrm{d}(r)\,\&^{\wedge}\,\mathrm{d}(\theta))\,/\,($

$\quad 2\,r + M) + ((-4\,r^4\cos(\theta)^2\,aa^4\,M - 14\,r^3\,aa^2\cos(\theta)^2 + 12\,r^4\cos(\theta)^4\,aa^4\,M + 6\,r^3\,aa^2 - 4\,r\cos(\theta)^2$

$\quad + 8\,r^3\,aa^2\cos(\theta)^4 + 4\,r^4\cos(\theta)^8\,aa^4\,M + 8\,r^2\,M\,aa^2\cos(\theta)^4 - 12\,r^4\cos(\theta)^6\,aa^4\,M - r^2\,M\,aa^2 - 3\,r^2\,M\,aa^2\cos(\theta)^2$

$\quad + 2\,r - 4\,r^2\,aa^2\cos(\theta)^6\,M - M)\,(\mathrm{d}(\phi)\,\&^{\wedge}\,\mathrm{d}(\theta)))\,/\,(2\,r + M)$

$\quad + 2\,\dfrac{\cos(\theta)\sin(\theta)\,M\,(r^2\cos(\theta)^2\,aa^2 - r^2\,aa^2 - 1)\,(\mathrm{d}(r)\,\&^{\wedge}\,\mathrm{d}(\phi))}{r\,(2\,r + M)}$

$$OmegaCC24 := 0$$

```
> OmegaCC31:=simpform(simplify(CGamma31&^Gamma11+CGamma32&^Gamma21+CGamma33&^Gamma
31+CGamma34&^Gamma41));OmegaCC32:=simpform(simplify(CGamma31&^Gamma12+CGamma32&^
Gamma22+CGamma33&^Gamma32+CGamma34&^Gamma42));OmegaCC33:=simpform(simplify(CGamm
a31&^Gamma13+CGamma32&^Gamma23+CGamma33&^Gamma33+CGamma34&^Gamma43));OmegaCC34:=
simpform(simplify(CGamma31&^Gamma14+CGamma32&^Gamma24+CGamma33&^Gamma34+CGamma34
&^Gamma44));
```

$OmegaCC31 := -\dfrac{aa\,(-2\,r - M + 2\,M\cos(\theta)^2)\,(\mathrm{d}(r)\,\&^{\wedge}\,\mathrm{d}(\theta))}{\sin(\theta)\,r\,(2\,r + M)}$

$\quad + \dfrac{\cos(\theta)\,r\,aa^2\,(2\cos(\theta)^4\,M - 2\,r\cos(\theta)^2 - 3\,M\cos(\theta)^2 + M + 2\,r)\,(\mathrm{d}(\theta)\,\&^{\wedge}\,\mathrm{d}(\phi))}{\sin(\theta)\,(2\,r + M)}$

$\quad + \dfrac{(2\,r^2\,M\,aa^2\cos(\theta)^4 - 2\,r^3\,aa^2\cos(\theta)^2 - 3\,r^2\,M\,aa^2\cos(\theta)^2 + M + 2\,r + 2\,r^3\,aa^2 + r^2\,M\,aa^2)\,(\mathrm{d}(r)\,\&^{\wedge}\,\mathrm{d}(\phi))}{r^2\,(2\,r + M)}$

$OmegaCC32 := \dfrac{\cos(\theta)\,aa\,(-2\,r + 2\,M\cos(\theta)^2 - 3\,M)\,(\mathrm{d}(r)\,\&^{\wedge}\,\mathrm{d}(\theta))}{2\,r\cos(\theta)^2 - 2\,r - M + M\cos(\theta)^2} - ((2\,r^2\,aa^2\cos(\theta)^6\,M - 2\,r^3\,aa^2\cos(\theta)^4$

$\quad - 5\,r^2\,M\,aa^2\cos(\theta)^4 + 2\,r^3\,aa^2\cos(\theta)^2 + 3\,r^2\,M\,aa^2\cos(\theta)^2 + 2\,M\cos(\theta)^2 + 2\,r - M)\,(\mathrm{d}(\theta)\,\&^{\wedge}\,\mathrm{d}(\phi)))\,/\,($

$\quad 2\,r\cos(\theta)^2 - 2\,r - M + M\cos(\theta)^2) - \cos(\theta)\sin(\theta)$

$\quad (2\,r^2\,M\,aa^2\cos(\theta)^4 - 2\,r^3\,aa^2\cos(\theta)^2 - 5\,r^2\,M\,aa^2\cos(\theta)^2 + 2\,r^3\,aa^2 + 3\,r^2\,M\,aa^2 + 2\,M)\,(\mathrm{d}(r)\,\&^{\wedge}\,\mathrm{d}(\phi))\,/\,($

$\quad r\,(2\,r\cos(\theta)^2 - 2\,r - M + M\cos(\theta)^2))$

$OmegaCC33 := -2\,\dfrac{\cos(\theta)\,M\,(r^2\cos(\theta)^2\,aa^2 - r^2\,aa^2 - 1)\,(\mathrm{d}(r)\,\&^{\wedge}\,\mathrm{d}(\theta))}{\sin(\theta)\,r\,(2\,r + M)} + r\,aa$

$\quad (2\,r^2\,aa^2\cos(\theta)^6\,M - 4\cos(\theta)^4\,M - 4\,r^2\,M\,aa^2\cos(\theta)^4 - 2\,r\cos(\theta)^2 + 2\,r^2\,M\,aa^2\cos(\theta)^2 + 5\,M\cos(\theta)^2 - M + 2\,r)$

$$(d(\theta) \,\&^\wedge d(\phi)) / (\sin(\theta) (2 r + M))$$

$$+ \frac{\cos(\theta) aa (2 r^2 M aa^2 \cos(\theta)^4 - 4 M \cos(\theta)^2 - 4 r^2 M aa^2 \cos(\theta)^2 + 3 M - 2 r + 2 r^2 M aa^2) (d(r) \,\&^\wedge d(\phi))}{2 r + M}$$

$$OmegaCC34 := 0$$

```
> OmegaCC41:=simpform(simplify(CGamma41&^Gamma11+CGamma42&^Gamma21+CGamma43&^Gamma
  31+CGamma44&^Gamma41));OmegaCC42:=simpform(simplify(CGamma41&^Gamma12+CGamma42&^
  Gamma22+CGamma43&^Gamma32+CGamma44&^Gamma42));OmegaCC43:=simpform(simplify(CGamm
  a41&^Gamma13+CGamma42&^Gamma23+CGamma43&^Gamma33+CGamma44&^Gamma43));OmegaCC44:=
  simpform(simplify(CGamma41&^Gamma14+CGamma42&^Gamma24+CGamma43&^Gamma34+CGamma44
  &^Gamma44));
```

$$OmegaCC41 := 0$$
$$OmegaCC42 := 0$$
$$OmegaCC43 := 0$$
$$OmegaCC44 := 0$$

```
> Christ:=array([[Gamma11,Gamma12,Gamma13,Gamma14],[Gamma21,Gamma22,Gamma23,Gamma2
  4],[Gamma31,Gamma32,Gamma33,Gamma34],[Gamma41,Gamma42,Gamma43,Gamma44]]);
```

$Christ :=$

$$\Bigg[2 \frac{(-2 r^3 aa^2 \cos(\theta)^2 + 2 r^3 aa^2 \cos(\theta)^4 - M) d(r)}{r (2 r + M)} + (\cos(\theta) aa^2 r^2 \sin(\theta) - 2 \cos(\theta)^3 aa^2 r^2 \sin(\theta)) d(\theta) - r aa$$

$$\cos(\theta) (-2 r^3 aa^2 + 4 r^3 aa^2 \cos(\theta)^2 - 2 r^3 aa^2 \cos(\theta)^4 - 5 r^2 M aa^2 \cos(\theta)^4 + 2 r^2 aa^2 \cos(\theta)^6 M - r^2 M aa^2$$

$$+ 4 r^2 M aa^2 \cos(\theta)^2 + 2 r \cos(\theta)^2 - 2 r + M - M \cos(\theta)^2) d(\phi) / (2 r + M) ,$$

$$(\cos(\theta) aa^2 r^2 \sin(\theta) - 2 \cos(\theta)^3 aa^2 r^2 \sin(\theta)) d(r)$$

$$- \frac{r (-4 r^3 aa^2 \cos(\theta)^2 + 4 r^3 aa^2 \cos(\theta)^4 + 4 r^2 M aa^2 \cos(\theta)^4 - 4 r^2 M aa^2 \cos(\theta)^2 + 2 r - M) d(\theta)}{2 r + M} + r^2 \sin(\theta) aa ($$

$$2 r^3 aa^2 \cos(\theta)^2 - 2 r^3 aa^2 \cos(\theta)^4 + 3 r^2 M aa^2 \cos(\theta)^2 - 5 r^2 M aa^2 \cos(\theta)^4 + 2 r^2 aa^2 \cos(\theta)^6 M + 2 r \cos(\theta)^2 - 2 r$$

$$+ M - M \cos(\theta)^2) d(\phi) / (2 r + M) , - r aa \cos(\theta) (-2 r^3 aa^2 + 4 r^3 aa^2 \cos(\theta)^2 - 2 r^3 aa^2 \cos(\theta)^4$$

$$- 5 r^2 M aa^2 \cos(\theta)^4 + 2 r^2 aa^2 \cos(\theta)^6 M - r^2 M aa^2 + 4 r^2 M aa^2 \cos(\theta)^2 + 2 r \cos(\theta)^2 - 2 r + M - M \cos(\theta)^2)$$

$$d(r) / (2 r + M) + r^2 \sin(\theta) aa (2 r^3 aa^2 \cos(\theta)^2 - 2 r^3 aa^2 \cos(\theta)^4 + 3 r^2 M aa^2 \cos(\theta)^2 - 5 r^2 M aa^2 \cos(\theta)^4$$

$$+ 2 r^2 aa^2 \cos(\theta)^6 M + 2 r \cos(\theta)^2 - 2 r + M - M \cos(\theta)^2) d(\theta) / (2 r + M) - r (-6 r^4 \cos(\theta)^6 aa^4 M$$

$$+ 2 r^4 \cos(\theta)^8 aa^4 M - 2 r^4 \cos(\theta)^2 aa^4 M + 6 r^4 \cos(\theta)^4 aa^4 M - 2 r^2 aa^2 \cos(\theta)^6 M + 4 r^3 aa^2 - M - 2 r \cos(\theta)^2$$

$$+ 2 r - 2 r^2 M aa^2 \cos(\theta)^2 + 4 r^3 aa^2 \cos(\theta)^4 + M \cos(\theta)^2 - 8 r^3 aa^2 \cos(\theta)^2 + 4 r^2 M aa^2 \cos(\theta)^4) d(\phi) / (2 r + M) ,$$

$$0 \Bigg]$$

$$\Bigg[-4 \frac{(-1 + \cos(\theta)^2) r aa^2 \cos(\theta) \sin(\theta) d(r)}{2 r + M} - ($$

$$(-6 r^3 aa^2 \cos(\theta)^2 + 2 r^3 aa^2 + 4 r^3 aa^2 \cos(\theta)^4 - 3 r^2 M aa^2 \cos(\theta)^2 + 2 r^2 M aa^2 \cos(\theta)^4 + r^2 M aa^2 - 2 r + M) d(\theta)$$

$$) / (r (2 r + M)) + ((4 r^3 aa^2 \cos(\theta)^2 - 2 r^3 aa^2 - 2 r^3 aa^2 \cos(\theta)^4 + 2 r^2 aa^2 \cos(\theta)^6 M - r^2 M aa^2$$

$$- 5 r^2 M aa^2 \cos(\theta)^4 + 4 r^2 M aa^2 \cos(\theta)^2 + 2 r \cos(\theta)^2 + M \cos(\theta)^2) \sin(\theta) aa d(\phi)) / (2 r + M) , - ($$

$$(-6 r^3 aa^2 \cos(\theta)^2 + 2 r^3 aa^2 + 4 r^3 aa^2 \cos(\theta)^4 - 3 r^2 M aa^2 \cos(\theta)^2 + 2 r^2 M aa^2 \cos(\theta)^4 + r^2 M aa^2 - 2 r + M) d(r)$$

$$) / (r (2 r + M)) + 4 \frac{\cos(\theta) aa^2 r^2 \sin(\theta) (-r + r \cos(\theta)^2 + M \cos(\theta)^2 - M) d(\theta)}{2 r + M} + r aa \cos(\theta) (-2 r^3 aa^2$$

$$+ 4 r^3 aa^2 \cos(\theta)^2 - 2 r^3 aa^2 \cos(\theta)^4 - 3 r^2 M aa^2 + 8 r^2 M aa^2 \cos(\theta)^2 - 7 r^2 M aa^2 \cos(\theta)^4 + 2 r^2 aa^2 \cos(\theta)^6 M$$

$$- 2 r + 2 r \cos(\theta)^2 - M + M \cos(\theta)^2) d(\phi) / (2 r + M) , ((4 r^3 aa^2 \cos(\theta)^2 - 2 r^3 aa^2 - 2 r^3 aa^2 \cos(\theta)^4$$

$+ 2\,r^2\,aa^2\cos(\theta)^6\,M - r^2\,M\,aa^2 - 5\,r^2\,M\,aa^2\cos(\theta)^4 + 4\,r^2\,M\,aa^2\cos(\theta)^2 + 2\,r\cos(\theta)^2 + M\cos(\theta)^2)\sin(\theta)\,aa$

$\mathrm{d}(r))\,/\,(2\,r + M) + r\,aa\cos(\theta)\,(-2\,r^3\,aa^2 + 4\,r^3\,aa^2\cos(\theta)^2 - 2\,r^3\,aa^2\cos(\theta)^4 - 3\,r^2\,M\,aa^2 + 8\,r^2\,M\,aa^2\cos(\theta)^2$

$- 7\,r^2\,M\,aa^2\cos(\theta)^4 + 2\,r^2\,aa^2\cos(\theta)^6\,M - 2\,r + 2\,r\cos(\theta)^2 - M + M\cos(\theta)^2)\,\mathrm{d}(\theta)\,/\,(2\,r + M) + ((\ $

$2\,r^4\cos(\theta)^6\,aa^4\,M - 2\,r^4\,aa^4\,M - 6\,r^4\cos(\theta)^4\,aa^4\,M + 6\,r^4\cos(\theta)^2\,aa^4\,M + 4\,r^3\,aa^2\cos(\theta)^2 - 4\,r^3\,aa^2 - 4\,r^2\,M\,aa^2$

$- 2\,r^2\,M\,aa^2\cos(\theta)^4 + 6\,r^2\,M\,aa^2\cos(\theta)^2 - 2\,r - M)\cos(\theta)\sin(\theta)\,\mathrm{d}(\phi))\,/\,(2\,r + M)\,,\,0\,\Big]$

$$\Big[-4\,\frac{\cos(\theta)\,aa\,\mathrm{d}(r)}{2\,r + M} + \frac{\sin(\theta)\,aa\,(-1 + 2\cos(\theta)^2)\,\mathrm{d}(\theta)}{-1 + \cos(\theta)^2}$$

$$+ \frac{(-2\,r^3\,aa^2\cos(\theta)^2 - 3\,r^2\,M\,aa^2\cos(\theta)^2 + 2\,r^2\,M\,aa^2\cos(\theta)^4 + 2\,r^3\,aa^2 + r^2\,M\,aa^2 + 2\,r - M)\,\mathrm{d}(\phi)}{r\,(2\,r + M)}\,,$$

$$\frac{\sin(\theta)\,aa\,(-1 + 2\cos(\theta)^2)\,\mathrm{d}(r)}{-1 + \cos(\theta)^2} + 4\,\frac{(r + M)\cos(\theta)\,r\,aa\,\mathrm{d}(\theta)}{2\,r + M}$$

$$- \frac{\sin(\theta)\cos(\theta)\,(2\,r^3\,aa^2 - 2\,r^3\,aa^2\cos(\theta)^2 + 2\,r^2\,M\,aa^2\cos(\theta)^4 + 3\,r^2\,M\,aa^2 - 5\,r^2\,M\,aa^2\cos(\theta)^2 + 2\,r + M)\,\mathrm{d}(\phi)}{2\,r\cos(\theta)^2 - 2\,r - M + M\cos(\theta)^2}$$

$$\frac{(-2\,r^3\,aa^2\cos(\theta)^2 - 3\,r^2\,M\,aa^2\cos(\theta)^2 + 2\,r^2\,M\,aa^2\cos(\theta)^4 + 2\,r^3\,aa^2 + r^2\,M\,aa^2 + 2\,r - M)\,\mathrm{d}(r)}{r\,(2\,r + M)}\,,$$

$$- \frac{\sin(\theta)\cos(\theta)\,(2\,r^3\,aa^2 - 2\,r^3\,aa^2\cos(\theta)^2 + 2\,r^2\,M\,aa^2\cos(\theta)^4 + 3\,r^2\,M\,aa^2 - 5\,r^2\,M\,aa^2\cos(\theta)^2 + 2\,r + M)\,\mathrm{d}(\theta)}{2\,r\cos(\theta)^2 - 2\,r - M + M\cos(\theta)^2}$$

$$+ 2\,\frac{\cos(\theta)\,r\,aa\,M\,(r^2\cos(\theta)^4\,aa^2 - 2\,r^2\cos(\theta)^2\,aa^2 + r^2\,aa^2 + 1 - \cos(\theta)^2)\,\mathrm{d}(\phi)}{2\,r + M}\,,\,0\,\Big]$$

$$[\,0\,,\,0\,,\,0\,,\,0\,]$$

```
> OmegaD:=simpform(d(Christ));OmegaD[1,3];
> OMEGAM:=simpform(simplify(array([[OmegaD[1,1]+OmegaCC11,OmegaD[1,2]+OmegaCC12,Om
  egaD[1,3]+OmegaCC13,OmegaD[1,4]+OmegaCC14],[OmegaD[2,1]+OmegaCC21,OmegaD[2,2]+Om
  egaCC22,OmegaD[2,3]+OmegaCC23,OmegaD[2,4]+OmegaCC24],[OmegaD[3,1]+OmegaCC31,Omeg
  aD[3,2]+OmegaCC32,OmegaD[3,3]+OmegaCC33,OmegaD[3,4]+OmegaCC34],[OmegaD[4,1]+Omeg
  aCC41,OmegaD[4,2]+OmegaCC42,OmegaD[4,3]+OmegaCC43,OmegaD[4,4]+OmegaCC44]])));
```

$OmegaD :=$

$\Big[aa\sin(\theta)\,r\,(-2\,r^3\,aa^2 + 12\,r^3\,aa^2\cos(\theta)^2 - 10\,r^3\,aa^2\cos(\theta)^4 - 25\,r^2\,M\,aa^2\cos(\theta)^4 + 14\,r^2\,aa^2\cos(\theta)^6\,M$

$- r^2\,M\,aa^2 + 12\,r^2\,M\,aa^2\cos(\theta)^2 + 6\,r\cos(\theta)^2 - 2\,r + M - 3\,M\cos(\theta)^2)\,(\mathrm{d}(\theta)\,\&^\wedge\,\mathrm{d}(\phi))\,/\,(2\,r + M) - aa\cos(\theta)\,(\ $

$-12\,r^4\,aa^2\cos(\theta)^4 + 24\,r^4\,aa^2\cos(\theta)^2 + M^2 - 3\,r^2\,M^2\,aa^2 - 12\,r^3\,aa^2\,M - M^2\cos(\theta)^2 - 28\,r^3\,aa^2\cos(\theta)^4\,M$

$+ 32\,r^3\,aa^2\cos(\theta)^2\,M + 6\,r^2\,M^2\,aa^2\cos(\theta)^6 + 8\cos(\theta)^6\,r^3\,aa^2\,M - 15\,r^2\,M^2\,aa^2\cos(\theta)^4 + 12\,r^2\,M^2\,aa^2\cos(\theta)^2$

$- 12\,r^4\,aa^2 - 4\,r\,M - 4\,r^2 + 4\,r^2\cos(\theta)^2 + 4\,r\cos(\theta)^2\,M)\,(\mathrm{d}(r)\,\&^\wedge\,\mathrm{d}(\phi))\,\Big/\,(2\,r + M)^2$

$- 2\,\frac{\cos(\theta)\,r\sin(\theta)\,aa^2\,(-2\,r - 2\,M\cos(\theta)^2 + M + 4\,r\cos(\theta)^2)\,(\mathrm{d}(\theta)\,\&^\wedge\,\mathrm{d}(r))}{2\,r + M}\,,\,r^2\,aa\cos(\theta)\,(2\,r^3\,aa^2\cos(\theta)^2$

$- 2\,r^3\,aa^2\cos(\theta)^4 + 3\,r^2\,M\,aa^2\cos(\theta)^2 - 5\,r^2\,M\,aa^2\cos(\theta)^4 + 2\,r^2\,aa^2\cos(\theta)^6\,M + 2\,r\cos(\theta)^2 - 2\,r + M$

$- M\cos(\theta)^2 - 4\sin(\theta)^2\,aa^2\,r^3 + 8\sin(\theta)^2\,aa^2\,r^3\cos(\theta)^2 - 6\sin(\theta)^2\,aa^2\,r^2\,M + 20\sin(\theta)^2\,aa^2\,r^2\cos(\theta)^2\,M$

$- 12\sin(\theta)^2\,aa^2\,r^2\cos(\theta)^4\,M - 4\sin(\theta)^2\,r + 2\sin(\theta)^2\,M)\,(\mathrm{d}(\theta)\,\&^\wedge\,\mathrm{d}(\phi))\,/\,(2\,r + M) + 2\,aa\sin(\theta)\,r\,(\ $

$-8\,r^4\,aa^2\cos(\theta)^4 + 8\,r^4\,aa^2\cos(\theta)^2 + M^2 - M^2\cos(\theta)^2 - 20\,r^3\,aa^2\cos(\theta)^4\,M + 14\,r^3\,aa^2\cos(\theta)^2\,M$

$+ 4\,r^2\,M^2\,aa^2\cos(\theta)^6 + 6\cos(\theta)^6\,r^3\,aa^2\,M - 10\,r^2\,M^2\,aa^2\cos(\theta)^4 + 6\,r^2\,M^2\,aa^2\cos(\theta)^2 - 2\,r\,M - 4\,r^2 + 4\,r^2\cos(\theta)^2$

$+ 2\,r\cos(\theta)^2\,M)\,(\mathrm{d}(r)\,\&^\wedge\,\mathrm{d}(\phi))\,\Big/\,(2\,r + M)^2 + ((-4\,r^4\sin(\theta)^2\,aa^2 - 4\,r^3\sin(\theta)^2\,aa^2\,M - r^2\sin(\theta)^2\,aa^2\,M^2$

$$- 20\ r^4\ aa^2 \cos(\theta)^2 - 28\ r^3\ aa^2 \cos(\theta)^2\ M - 11\ r^2\ M^2\ aa^2 \cos(\theta)^2 + 24\ r^4 \sin(\theta)^2\ aa^2 \cos(\theta)^2$$
$$+ 24\ r^3 \sin(\theta)^2\ aa^2 \cos(\theta)^2\ M + 6\ r^2 \sin(\theta)^2\ aa^2 \cos(\theta)^2\ M^2 + 16\ r^4\ aa^2 \cos(\theta)^4 + 24\ r^3\ aa^2 \cos(\theta)^4\ M$$
$$+ 10\ r^2\ M^2\ aa^2 \cos(\theta)^4 + 4\ r^2 - M^2 + 4\ r\ M)\ (\mathrm{d}(\theta)\ \&^\wedge\ \mathrm{d}(r))) \Big/ (2\ r + M)^2,\ 2\ r \cos(\theta) \sin(\theta)\ ($$
$$-18\ r^4 \cos(\theta)^4\ aa^4\ M + 8\ r^4 \cos(\theta)^6\ aa^4\ M - 2\ r^4\ aa^4\ M + 12\ r^4 \cos(\theta)^2\ aa^4\ M - 6\ r^2\ M\ aa^2 \cos(\theta)^4 - 2\ r - 2\ r^2\ M\ aa^2$$
$$+ 8\ r^3\ aa^2 \cos(\theta)^2 + M - 8\ r^3\ aa^2 + 8\ r^2\ M\ aa^2 \cos(\theta)^2)\ (\mathrm{d}(\theta)\ \&^\wedge\ \mathrm{d}(\phi)) / (2\ r + M) - ((24\ r^4\ aa^2 \cos(\theta)^4$$
$$- 48\ r^4\ aa^2 \cos(\theta)^2 - M^2 + 16\ r^3\ aa^2\ M + M^2 \cos(\theta)^2 + 32\ r^3\ aa^2 \cos(\theta)^4\ M - 40\ r^3\ aa^2 \cos(\theta)^2\ M$$
$$- 6\ r^2\ M^2\ aa^2 \cos(\theta)^6 - 8 \cos(\theta)^6\ r^3\ aa^2\ M + 12\ r^2\ M^2\ aa^2 \cos(\theta)^4 - 6\ r^2\ M^2\ aa^2 \cos(\theta)^2 + 24\ r^4\ aa^2$$
$$+ 30\ r^4\ M^2\ aa^4 \cos(\theta)^4 + 48\ r^5\ aa^4 \cos(\theta)^4\ M - 10\ r^4\ M^2\ aa^4 \cos(\theta)^2 - 16\ r^5\ aa^4 \cos(\theta)^2\ M + 10\ r^4\ M^2\ aa^4 \cos(\theta)^8$$
$$+ 16\ r^5\ aa^4 \cos(\theta)^8\ M - 30\ r^4\ M^2\ aa^4 \cos(\theta)^6 - 48\ r^5\ aa^4 \cos(\theta)^6\ M + 4\ r\ M + 4\ r^2 - 4\ r^2 \cos(\theta)^2 - 4\ r \cos(\theta)^2\ M)$$
$$(\mathrm{d}(r)\ \&^\wedge\ \mathrm{d}(\phi))) \Big/ (2\ r + M)^2 + aa \sin(\theta)\ r\ (-4\ r^4\ aa^2 \cos(\theta)^4 + 8\ r^4\ aa^2 \cos(\theta)^2 - M^2 - r^2\ M^2\ aa^2 - 4\ r^3\ aa^2\ M$$
$$- M^2 \cos(\theta)^2 - 20\ r^3\ aa^2 \cos(\theta)^4\ M + 8\ r^3\ aa^2 \cos(\theta)^2\ M + 6\ r^2\ M^2\ aa^2 \cos(\theta)^6 + 16 \cos(\theta)^6\ r^3\ aa^2\ M$$
$$- 5\ r^2\ M^2\ aa^2 \cos(\theta)^4 - 4\ r^4\ aa^2 + 4\ r\ M + 4\ r^2 + 4\ r^2 \cos(\theta)^2 - 4\ r \cos(\theta)^2\ M)\ (\mathrm{d}(\theta)\ \&^\wedge\ \mathrm{d}(r)) \Big/ (2\ r + M)^2,\ 0 \Bigg]$$

$$\Bigg[aa \cos(\theta)\ (-8 \sin(\theta)^2\ aa^2\ r^3 + 8 \sin(\theta)^2\ aa^2\ r^3 \cos(\theta)^2 - 12 \sin(\theta)^2\ aa^2\ r^2 \cos(\theta)^4\ M + 20 \sin(\theta)^2\ aa^2\ r^2 \cos(\theta)^2\ M$$
$$- 8 \sin(\theta)^2\ aa^2\ r^2\ M - 4 \sin(\theta)^2\ r - 2 \sin(\theta)^2\ M + 4\ r^3\ aa^2 \cos(\theta)^2 - 2\ r^3\ aa^2 - 2\ r^3\ aa^2 \cos(\theta)^4 + 2\ r^2\ aa^2 \cos(\theta)^6\ M$$
$$- r^2\ M\ aa^2 - 5\ r^2\ M\ aa^2 \cos(\theta)^4 + 4\ r^2\ M\ aa^2 \cos(\theta)^2 + 2\ r \cos(\theta)^2 + M \cos(\theta)^2)\ (\mathrm{d}(\theta)\ \&^\wedge\ \mathrm{d}(\phi)) / (2\ r + M) + 2$$
$$aa^3 \sin(\theta)\ r\ (8\ r^2 \cos(\theta)^2 + 10\ r \cos(\theta)^2\ M - 4\ r^2 - 4\ r\ M - 4 \cos(\theta)^4\ r^2 - 8 \cos(\theta)^4\ r\ M + 2\ r \cos(\theta)^6\ M$$
$$+ 2 \cos(\theta)^6\ M^2 - M^2 - 5 \cos(\theta)^4\ M^2 + 4\ M^2 \cos(\theta)^2)\ (\mathrm{d}(r)\ \&^\wedge\ \mathrm{d}(\phi)) \Big/ (2\ r + M)^2 + ((24\ r^4 \sin(\theta)^2\ aa^2 \cos(\theta)^2$$
$$+ 12\ r^3 \sin(\theta)^2\ aa^2 \cos(\theta)^2\ M - 8\ r^4 \sin(\theta)^2\ aa^2 - 4\ r^3 \sin(\theta)^2\ aa^2\ M - 4\ r^4\ aa^2 \cos(\theta)^2 - 8\ r^3\ aa^2 \cos(\theta)^2\ M$$
$$+ 4\ r^3\ aa^2 \cos(\theta)^4\ M - M^2 + r^2\ M^2\ aa^2 + 4\ r^3\ aa^2\ M + 2\ r^2\ M^2\ aa^2 \cos(\theta)^4 - 3\ r^2\ M^2\ aa^2 \cos(\theta)^2 + 4\ r^4\ aa^2 - 4\ r\ M$$
$$+ 4\ r^2)\ (\mathrm{d}(\theta)\ \&^\wedge\ \mathrm{d}(r))) \Big/ (r^2\ (2\ r + M)^2),\ - aa \sin(\theta)\ r\ (-2\ r^3\ aa^2 + 12\ r^3\ aa^2 \cos(\theta)^2 - 10\ r^3\ aa^2 \cos(\theta)^4$$
$$- 3\ r^2\ M\ aa^2 + 24\ r^2\ M\ aa^2 \cos(\theta)^2 - 35\ r^2\ M\ aa^2 \cos(\theta)^4 + 14\ r^2\ aa^2 \cos(\theta)^6\ M - 2\ r + 6\ r \cos(\theta)^2 - M$$
$$+ 3\ M \cos(\theta)^2)\ (\mathrm{d}(\theta)\ \&^\wedge\ \mathrm{d}(\phi)) / (2\ r + M) + \cos(\theta)\ aa\ (-M^2 - 9\ r^2\ M^2\ aa^2 - 20\ r^3\ aa^2\ M + M^2 \cos(\theta)^2$$
$$+ 8 \cos(\theta)^6\ r^3\ aa^2\ M + 24\ r^4\ aa^2 \cos(\theta)^2 - 12\ r^4\ aa^2 \cos(\theta)^4 + 6\ r^2\ M^2\ aa^2 \cos(\theta)^6 - 12\ r^4\ aa^2 - 4\ r\ M - 4\ r^2$$
$$- 21\ r^2\ M^2\ aa^2 \cos(\theta)^4 + 24\ r^2\ M^2\ aa^2 \cos(\theta)^2 + 4\ r^2 \cos(\theta)^2 - 36\ r^3\ aa^2 \cos(\theta)^4\ M + 4\ r \cos(\theta)^2\ M$$
$$+ 48\ r^3\ aa^2 \cos(\theta)^2\ M)\ (\mathrm{d}(r)\ \&^\wedge\ \mathrm{d}(\phi)) \Big/ (2\ r + M)^2$$
$$+ 2\ \frac{\cos(\theta)\ r \sin(\theta)\ aa^2\ (-4\ r^2 - 2\ r\ M + M^2 + 8\ r^2 \cos(\theta)^2 + 6\ r \cos(\theta)^2\ M)\ (\mathrm{d}(\theta)\ \&^\wedge\ \mathrm{d}(r))}{(2\ r + M)^2},\ ((2\ r^4 \cos(\theta)^8\ aa^4\ M$$
$$- 14 \sin(\theta)^2\ r^4\ aa^4 \cos(\theta)^6\ M - 18 \sin(\theta)^2\ r^4\ aa^4 \cos(\theta)^2\ M + 2 \sin(\theta)^2\ r^4\ aa^4\ M + 30 \sin(\theta)^2\ r^4\ aa^4 \cos(\theta)^4\ M$$
$$+ 10 \sin(\theta)^2\ aa^2\ r^2 \cos(\theta)^4\ M - 18 \sin(\theta)^2\ aa^2\ r^2 \cos(\theta)^2\ M + 4 \sin(\theta)^2\ aa^2\ r^3 + 4 \sin(\theta)^2\ aa^2\ r^2\ M$$
$$- 12 \sin(\theta)^2\ aa^2\ r^3 \cos(\theta)^2 + \sin(\theta)^2\ M + 2 \sin(\theta)^2\ r - 2\ r^4 \cos(\theta)^2\ aa^4\ M - 6\ r^4 \cos(\theta)^6\ aa^4\ M + 6\ r^4 \cos(\theta)^4\ aa^4\ M$$
$$- 2\ r \cos(\theta)^2 - 4\ r^3\ aa^2 \cos(\theta)^2 - M \cos(\theta)^2 - 4\ r^2\ M\ aa^2 \cos(\theta)^2 + 6\ r^2\ M\ aa^2 \cos(\theta)^4 + 4\ r^3\ aa^2 \cos(\theta)^4$$
$$- 2\ r^2\ aa^2 \cos(\theta)^6\ M)\ (\mathrm{d}(\theta)\ \&^\wedge\ \mathrm{d}(\phi))) / (2\ r + M) + 4 \cos(\theta) \sin(\theta)\ r\ aa^2\ (-3\ r^3\ aa^2\ M - 4\ r^2 - 2\ r^2\ M^2\ aa^2$$
$$+ 6\ r^2\ M^2\ aa^2 \cos(\theta)^2 + 2\ r^2\ M^2\ aa^2 \cos(\theta)^6 - 6\ r^2\ M^2\ aa^2 \cos(\theta)^4 - 5\ r\ M - 2\ M^2 + 9\ r^3\ aa^2 \cos(\theta)^2\ M$$
$$+ 3 \cos(\theta)^6\ r^3\ aa^2\ M - 9\ r^3\ aa^2 \cos(\theta)^4\ M - \cos(\theta)^4\ M^2 + 3\ M^2 \cos(\theta)^2 + 4\ r^2 \cos(\theta)^2 + 6\ r \cos(\theta)^2\ M - \cos(\theta)^4\ r\ M$$
$$)\ (\mathrm{d}(r)\ \&^\wedge\ \mathrm{d}(\phi)) \Big/ (2\ r + M)^2 - aa \cos(\theta)\ (-M^2 + 16\ r^4\ aa^2 \cos(\theta)^2 + 20\ r^2\ M^2\ aa^2 \cos(\theta)^2 - 16\ r^2\ M^2\ aa^2 \cos(\theta)^4$$
$$- 8\ r^2\ M^2\ aa^2 - 16\ r^3\ aa^2\ M + 4\ r^2\ M^2\ aa^2 \cos(\theta)^6 + 36\ r^3\ aa^2 \cos(\theta)^2\ M - 8\ r^4\ aa^2 \cos(\theta)^4 - 24\ r^3\ aa^2 \cos(\theta)^4\ M$$
$$+ 4 \cos(\theta)^6\ r^3\ aa^2\ M + 12\ r^2 \sin(\theta)^2\ aa^2 \cos(\theta)^4\ M^2 - 20\ r^2 \sin(\theta)^2\ aa^2 \cos(\theta)^2\ M^2 + 8\ r^2 \sin(\theta)^2\ aa^2\ M^2$$
$$- 16\ r^4 \sin(\theta)^2\ aa^2 \cos(\theta)^2 + 8 \sin(\theta)^2\ r\ M - 8\ r^4\ aa^2 - 4\ r\ M - 4\ r^2 + 2 \sin(\theta)^2\ M^2 + 8 \sin(\theta)^2\ r^2$$
$$+ 16\ r^4 \sin(\theta)^2\ aa^2 - 48\ r^3 \sin(\theta)^2\ aa^2 \cos(\theta)^2\ M + 24\ r^3 \sin(\theta)^2\ aa^2 \cos(\theta)^4\ M + 24\ r^3 \sin(\theta)^2\ aa^2\ M)$$

$$\left(\mathrm{d}(\theta)\ \&^\wedge\ \mathrm{d}(r)\right)\Big/\left(2\,r+M\right)^2,\,0\right]$$

$$\left[4\,\frac{aa\,\sin(\theta)\,(\mathrm{d}(\theta)\ \&^\wedge\ \mathrm{d}(r))}{2\,r+M}-2\,\frac{r\,aa^2\,\cos(\theta)\,\sin(\theta)\,(-2\,r-3\,M+4\,M\cos(\theta)^2)\,(\mathrm{d}(\theta)\ \&^\wedge\ \mathrm{d}(\phi))}{2\,r+M}+((\right.$$

$$-4\,r^4\,aa^2\,\cos(\theta)^2+M^2+r^2\,M^2\,aa^2+4\,r^3\,aa^2\,M-4\,r^3\,aa^2\,\cos(\theta)^2\,M+2\,r^2\,M^2\,aa^2\,\cos(\theta)^4-3\,r^2\,M^2\,aa^2\,\cos(\theta)^2$$

$$+4\,r^4\,aa^2+4\,r\,M-4\,r^2)\,(\mathrm{d}(r)\ \&^\wedge\ \mathrm{d}(\phi)))\Big/\left(r^2\,(2\,r+M)^2\right),-((15\,\sin(\theta)^2\,aa^2\,r^2\,\cos(\theta)^4\,M$$

$$-12\,\sin(\theta)^2\,aa^2\,r^2\,\cos(\theta)^2\,M+3\,\sin(\theta)^2\,aa^2\,r^2\,M+2\,\sin(\theta)^2\,aa^2\,r^3-4\,\sin(\theta)^2\,aa^2\,r^3\,\cos(\theta)^2+\sin(\theta)^2\,M$$

$$+2\,\sin(\theta)^2\,r-2\,r\,\cos(\theta)^2+\cos(\theta)^4\,M-2\,r^3\,aa^2\,\cos(\theta)^2-M\,\cos(\theta)^2-3\,r^2\,M\,aa^2\,\cos(\theta)^2+8\,r^2\,M\,aa^2\,\cos(\theta)^4$$

$$-6\,\sin(\theta)^2\,r^2\,aa^2\,\cos(\theta)^6\,M+2\,\sin(\theta)^2\,aa^2\,r^3\,\cos(\theta)^4-2\,r^3\,aa^2\,\cos(\theta)^6+2\,\cos(\theta)^8\,r^2\,aa^2\,M+4\,r^3\,aa^2\,\cos(\theta)^4$$

$$+2\,\cos(\theta)^2\,\sin(\theta)^2\,r+\cos(\theta)^2\,\sin(\theta)^2\,M-7\,r^2\,aa^2\,\cos(\theta)^6\,M+2\,r\,\cos(\theta)^4)\,(\mathrm{d}(\theta)\ \&^\wedge\ \mathrm{d}(\phi)))\Big/(\right.$$

$$(-1+\cos(\theta)^2)\,(2\,r\,\cos(\theta)^2-2\,r-M+M\,\cos(\theta)^2))$$

$$-2\,\frac{(2\,r\,\cos(\theta)^2\,M+2\,M^2\,\cos(\theta)^2-6\,r\,M-4\,r^2-3\,M^2)\,\cos(\theta)\,r\,\sin(\theta)\,aa^2\,(\mathrm{d}(r)\ \&^\wedge\ \mathrm{d}(\phi))}{(2\,r+M)^2}-aa\,\cos(\theta)\,(4\,r^2$$

$$+4\,r\,M+3\,M^2-4\,r^2\,\cos(\theta)^2-4\,r\,\cos(\theta)^2\,M-5\,M^2\,\cos(\theta)^2+2\,\cos(\theta)^4\,M^2-8\,\sin(\theta)^2\,r^2-8\,\sin(\theta)^2\,r\,M$$

$$-2\,\sin(\theta)^2\,M^2)\,(\mathrm{d}(\theta)\ \&^\wedge\ \mathrm{d}(r))\Big/\left((-1+\cos(\theta)^2)^2\,(2\,r+M)^2\right),$$

$$-2\,\frac{r\,aa\,M\,\sin(\theta)\,(5\,r^2\,\cos(\theta)^4\,aa^2-6\,r^2\,\cos(\theta)^2\,aa^2+r^2\,aa^2+1-3\,\cos(\theta)^2)\,(\mathrm{d}(\theta)\ \&^\wedge\ \mathrm{d}(\phi))}{2\,r+M}+2\,M\,aa\,\cos(\theta)\,(\right.$$

$$4\,r^3\,aa^2\,\cos(\theta)^4+3\,r^2\,M\,aa^2\,\cos(\theta)^4-8\,r^3\,aa^2\,\cos(\theta)^2-6\,r^2\,M\,aa^2\,\cos(\theta)^2+4\,r^3\,aa^2+3\,r^2\,M\,aa^2+M-M\,\cos(\theta)^2$$

$$)\,(\mathrm{d}(r)\ \&^\wedge\ \mathrm{d}(\phi))\Big/\left(2\,r+M\right)^2-4\,\frac{\cos(\theta)\,r\,\sin(\theta)\,aa^2\,M\,(-r+3\,r\,\cos(\theta)^2+M\,\cos(\theta)^2)\,(\mathrm{d}(\theta)\ \&^\wedge\ \mathrm{d}(r))}{(2\,r+M)^2},\,0\right]$$

$$[\,0\,,\,0\,,\,0\,,\,0\,]$$

$$2\,r\,\cos(\theta)\,\sin(\theta)\,(-18\,r^4\,\cos(\theta)^4\,aa^4\,M+8\,r^4\,\cos(\theta)^6\,aa^4\,M-2\,r^4\,aa^4\,M+12\,r^4\,\cos(\theta)^2\,aa^4\,M$$

$$-6\,r^2\,M\,aa^2\,\cos(\theta)^4-2\,r-2\,r^2\,M\,aa^2+8\,r^3\,aa^2\,\cos(\theta)^2+M-8\,r^3\,aa^2+8\,r^2\,M\,aa^2\,\cos(\theta)^2)\,(\mathrm{d}(\theta)\ \&^\wedge\ \mathrm{d}(\phi))\,/\,($$

$$2\,r+M)-((24\,r^4\,aa^2\,\cos(\theta)^4-48\,r^4\,aa^2\,\cos(\theta)^2-M^2+16\,r^3\,aa^2\,M+M^2\,\cos(\theta)^2+32\,r^3\,aa^2\,\cos(\theta)^4\,M$$

$$-40\,r^3\,aa^2\,\cos(\theta)^2\,M-6\,r^2\,M^2\,aa^2\,\cos(\theta)^6-8\,\cos(\theta)^6\,r^3\,aa^2\,M+12\,r^2\,M^2\,aa^2\,\cos(\theta)^4-6\,r^2\,M^2\,aa^2\,\cos(\theta)^2$$

$$+24\,r^4\,aa^2+30\,r^4\,M^2\,aa^4\,\cos(\theta)^4+48\,r^5\,aa^4\,\cos(\theta)^4\,M-10\,r^4\,M^2\,aa^4\,\cos(\theta)^2-16\,r^5\,aa^4\,\cos(\theta)^2\,M$$

$$+10\,r^4\,M^2\,aa^4\,\cos(\theta)^8+16\,r^5\,aa^4\,\cos(\theta)^8\,M-30\,r^4\,M^2\,aa^4\,\cos(\theta)^6-48\,r^5\,aa^4\,\cos(\theta)^6\,M+4\,r\,M+4\,r^2$$

$$-4\,r^2\,\cos(\theta)^2-4\,r\,\cos(\theta)^2\,M)\,(\mathrm{d}(r)\ \&^\wedge\ \mathrm{d}(\phi)))\Big/\left(2\,r+M\right)^2+aa\,\sin(\theta)\,r\,(-4\,r^4\,aa^2\,\cos(\theta)^4+8\,r^4\,aa^2\,\cos(\theta)^2$$

$$-M^2-r^2\,M^2\,aa^2-4\,r^3\,aa^2\,M-M^2\,\cos(\theta)^2-20\,r^3\,aa^2\,\cos(\theta)^4\,M+8\,r^3\,aa^2\,\cos(\theta)^2\,M+6\,r^2\,M^2\,aa^2\,\cos(\theta)^6$$

$$+16\,\cos(\theta)^6\,r^3\,aa^2\,M-5\,r^2\,M^2\,aa^2\,\cos(\theta)^4-4\,r^4\,aa^2+4\,r\,M+4\,r^2+4\,r^2\,\cos(\theta)^2-4\,r\,\cos(\theta)^2\,M)$$

$$(\mathrm{d}(\theta)\ \&^\wedge\ \mathrm{d}(r))\Big/\left(2\,r+M\right)^2$$

$$OMEGAM :=$$

$$\left[\frac{r\,\sin(\theta)\,aa^2\,\cos(\theta)\,(-2\,r-2\,M\,\cos(\theta)^2+M+4\,r\,\cos(\theta)^2)\,(\mathrm{d}(r)\ \&^\wedge\ \mathrm{d}(\theta))}{2\,r+M}+((-6\,r^3\,aa^2\,\cos(\theta)^4\right.$$

$$+6\,r^3\,aa^2\,\cos(\theta)^2+7\,r^2\,M\,aa^2\,\cos(\theta)^2-17\,r^2\,M\,aa^2\,\cos(\theta)^4+10\,r^2\,aa^2\,\cos(\theta)^6\,M-4\,r\,\cos(\theta)^2+2\,M$$

$$-4\,M\,\cos(\theta)^2)\,aa\,\sin(\theta)\,r\,(\mathrm{d}(\theta)\ \&^\wedge\ \mathrm{d}(\phi)))\,/\,(2\,r+M)-2\,aa\,\cos(\theta)\,(4\,r^2-4\,r^2\,\cos(\theta)^2-M^2+M^2\,\cos(\theta)^2$$

$$-8\,r^3\,aa^2\,\cos(\theta)^4\,M-4\,r^4\,aa^2+2\,\cos(\theta)^6\,r^3\,aa^2\,M-4\,r^4\,aa^2\,\cos(\theta)^4+2\,r^2\,M^2\,aa^2\,\cos(\theta)^6+8\,r^4\,aa^2\,\cos(\theta)^2$$

$$-5\,r^2\,M^2\,aa^2\,\cos(\theta)^4-4\,r^3\,aa^2\,M+10\,r^3\,aa^2\,\cos(\theta)^2\,M+4\,r^2\,M^2\,aa^2\,\cos(\theta)^2-r^2\,M^2\,aa^2-2\,r\,M+2\,r\,\cos(\theta)^2\,M)$$

$$(\mathrm{d}(r)\ \&^\wedge\ \mathrm{d}(\phi))\Big/\left(4\,r^2+4\,r\,M+M^2\right),-((-6\,r^3\,aa^2\,\cos(\theta)^4+8\,r^3\,aa^2\,\cos(\theta)^2-2\,r^3\,aa^2+10\,r^2\,aa^2\,\cos(\theta)^6\,M$$

$$-3\,r^2\,M\,aa^2+18\,r^2\,M\,aa^2\,\cos(\theta)^2-25\,r^2\,M\,aa^2\,\cos(\theta)^4+2\,r-4\,r\,\cos(\theta)^2-8\,M\,\cos(\theta)^2+7\,M)\,aa\,\cos(\theta)\,r^2$$

$$(\mathrm{d}(\phi)\ \&^\wedge\ \mathrm{d}(\theta)))\,/\,(2\,r+M)-aa\,\sin(\theta)\,r\,(20\,r^3\,aa^2\,\cos(\theta)^2\,M+6\,r^2\,M^2\,aa^2\,\cos(\theta)^6-15\,r^2\,M^2\,aa^2\,\cos(\theta)^4$$

$$-28\,r^3\,aa^2\,\cos(\theta)^4\,M + 8\,\cos(\theta)^6\,r^3\,aa^2\,M - 4\,M^2\,\cos(\theta)^2 + 12\,r^4\,aa^2\,\cos(\theta)^2 - 12\,r^4\,aa^2\,\cos(\theta)^4 - 8\,r\,\cos(\theta)^2\,M$$
$$+ 9\,r^2\,M^2\,aa^2\,\cos(\theta)^2 - 8\,r^2\,\cos(\theta)^2 - 4\,r^2 + 3\,M^2)\,(d(\phi)\,\&^{\wedge}\,d(r))\Big/(4\,r^2 + 4\,r\,M + M^2)$$

$$+ 4\,\frac{r\,M\,(r\,M\,aa^2\,\cos(\theta)^4 + 1 - r\,M\,aa^2\,\cos(\theta)^2 + r^2\,\cos(\theta)^4\,aa^2 - r^2\,\cos(\theta)^2\,aa^2)\,(d(\theta)\,\&^{\wedge}\,d(r))}{4\,r^2 + 4\,r\,M + M^2},\ 2\,((\,$$

$$8\,r^4\,\cos(\theta)^2\,aa^4\,M - 13\,r^4\,\cos(\theta)^4\,aa^4\,M + 6\,r^4\,\cos(\theta)^6\,aa^4\,M - r^4\,aa^4\,M - 4\,r^3\,aa^2 + 4\,r^3\,aa^2\,\cos(\theta)^2 + r^2\,M\,aa^2$$
$$- 4\,r^2\,M\,aa^2\,\cos(\theta)^4 + 3\,r^2\,M\,aa^2\,\cos(\theta)^2 + 2\,M)\,r\,\cos(\theta)\,\sin(\theta)\,(d(\theta)\,\&^{\wedge}\,d(\phi)))/(2\,r+M) + r\,(\,$$
$$-24\,r^3\,M^2\,aa^4\,\cos(\theta)^6 + 20\,r^3\,aa^2 - aa^2\,r\,\cos(\theta)^2\,M^2 - 8\,r^3\,M^2\,aa^4\,\cos(\theta)^2 - 4\,aa^2\,r\,\cos(\theta)^6\,M^2$$
$$+ 24\,r^3\,M^2\,aa^4\,\cos(\theta)^4 + 6\,aa^2\,r\,\cos(\theta)^4\,M^2 - 36\,r^3\,aa^2\,\cos(\theta)^2 - aa^2\,r\,M^2 + 12\,r^2\,M\,aa^2 + 16\,r^3\,aa^2\,\cos(\theta)^4$$
$$+ 12\,r^4\,\cos(\theta)^8\,aa^4\,M + 8\,r^3\,M^2\,aa^4\,\cos(\theta)^8 - 12\,r^4\,\cos(\theta)^2\,aa^4\,M + 36\,r^4\,\cos(\theta)^4\,aa^4\,M - 36\,r^4\,\cos(\theta)^6\,aa^4\,M$$
$$- 4\,r^2\,aa^2\,\cos(\theta)^6\,M - 24\,r^2\,M\,aa^2\,\cos(\theta)^2 + 4\,M + 16\,r^2\,M\,aa^2\,\cos(\theta)^4 - 4\,M\,\cos(\theta)^2)\,(d(\phi)\,\&^{\wedge}\,d(r))\Big/(\,$$
$$4\,r^2 + 4\,r\,M + M^2) + aa\,\sin(\theta)\,r\,(-12\,r^3\,aa^2\,\cos(\theta)^4\,M + 4\,r^2\,M^2\,aa^2\,\cos(\theta)^6 - 2\,r^2\,M^2\,aa^2\,\cos(\theta)^4$$
$$+ 12\,\cos(\theta)^6\,r^3\,aa^2\,M - 4\,r\,\cos(\theta)^2\,M - 2\,r^2\,M^2\,aa^2\,\cos(\theta)^2 + 4\,r^2 - M^2 + 4\,r\,M)\,(d(\theta)\,\&^{\wedge}\,d(r))\Big/(\,$$
$$4\,r^2 + 4\,r\,M + M^2),\ 0\Bigg]$$

$$\Bigg[-((-6\,r^3\,aa^2 - 6\,r^3\,aa^2\,\cos(\theta)^4 + 12\,r^3\,aa^2\,\cos(\theta)^2 - 7\,r^2\,M\,aa^2 - 27\,r^2\,M\,aa^2\,\cos(\theta)^4 + 10\,r^2\,aa^2\,\cos(\theta)^6\,M$$

$$+ 24\,r^2\,M\,aa^2\,\cos(\theta)^2 - 4\,r\,\cos(\theta)^2 + 4\,r + 4\,M\,\cos(\theta)^2 - 4\,M)\,\cos(\theta)\,aa\,(d(\phi)\,\&^{\wedge}\,d(\theta)))/(2\,r+M) + \sin(\theta)$$
$$aa\,(4\,r^2\,M^2\,aa^2\,\cos(\theta)^6 - 16\,r^3\,aa^2\,\cos(\theta)^4\,M + 4\,\cos(\theta)^6\,r^3\,aa^2\,M - 8\,r^4\,aa^2\,\cos(\theta)^4 + 16\,r^4\,aa^2\,\cos(\theta)^2 - 8\,r^4\,aa^2$$
$$- 8\,r^3\,aa^2\,M - 2\,r^2\,M^2\,aa^2 + 20\,r^3\,aa^2\,\cos(\theta)^2\,M - 10\,r^2\,M^2\,aa^2\,\cos(\theta)^4 + 4\,r^2 + 8\,r^2\,M^2\,aa^2\,\cos(\theta)^2 + 4\,r\,\cos(\theta)^2\,M$$
$$- 4\,r\,M + 4\,M^2\,\cos(\theta)^2 - 8\,r^2\,\cos(\theta)^2 - 3\,M^2)\,(d(r)\,\&^{\wedge}\,d(\phi))\Big/(r\,(4\,r^2 + 4\,r\,M + M^2)) - ((\,$$
$$8\,r\,M + 2\,M^2 + 4\,r^4\,aa^2 - 2\,r^2\,M^2\,aa^2\,\cos(\theta)^4 - 12\,r^4\,aa^2\,\cos(\theta)^2 + 8\,r^4\,aa^2\,\cos(\theta)^4 + 3\,r^2\,M^2\,aa^2\,\cos(\theta)^2 - r^2\,M^2\,aa^2$$
$$)\,(d(\theta)\,\&^{\wedge}\,d(r)))\Big/((4\,r^2 + 4\,r\,M + M^2)\,r^2),$$

$$4\,\frac{aa^2\,\sin(\theta)\,r\,\cos(\theta)\,M\,(-r + r\,\cos(\theta)^2 + M\,\cos(\theta)^2 - M)\,(d(r)\,\&^{\wedge}\,d(\theta))}{4\,r^2 + 4\,r\,M + M^2} - ((8\,r^3\,aa^2\,\cos(\theta)^2 - 2\,r^3\,aa^2$$
$$- 6\,r^3\,aa^2\,\cos(\theta)^4 - 3\,r^2\,M\,aa^2 - 25\,r^2\,M\,aa^2\,\cos(\theta)^4 + 10\,r^2\,aa^2\,\cos(\theta)^6\,M + 18\,r^2\,M\,aa^2\,\cos(\theta)^2 - 4\,r\,\cos(\theta)^2$$
$$+ 2\,r - 3\,M)\,aa\,\sin(\theta)\,r\,(d(\theta)\,\&^{\wedge}\,d(\phi)))/(2\,r+M) + aa\,\cos(\theta)\,(4\,r^2 - 8\,r^2\,\cos(\theta)^2 - M^2 - 36\,r^3\,aa^2\,\cos(\theta)^4\,M$$
$$- 12\,r^4\,aa^2 + 8\,\cos(\theta)^6\,r^3\,aa^2\,M - 12\,r^4\,aa^2\,\cos(\theta)^4 + 6\,r^2\,M^2\,aa^2\,\cos(\theta)^6 + 24\,r^4\,aa^2\,\cos(\theta)^2 - 21\,r^2\,M^2\,aa^2\,\cos(\theta)^4$$
$$- 20\,r^3\,aa^2\,M + 48\,r^3\,aa^2\,\cos(\theta)^2\,M + 24\,r^2\,M^2\,aa^2\,\cos(\theta)^2 - 9\,r^2\,M^2\,aa^2 - 4\,r\,\cos(\theta)^2\,M)\,(d(r)\,\&^{\wedge}\,d(\phi))\Big/(\,$$
$$4\,r^2 + 4\,r\,M + M^2),\ ((2\,r^4\,aa^4\,M - 18\,r^4\,\cos(\theta)^2\,aa^4\,M + 42\,r^4\,\cos(\theta)^4\,aa^4\,M + 12\,r^4\,\cos(\theta)^8\,aa^4\,M$$
$$- 38\,r^4\,\cos(\theta)^6\,aa^4\,M + 8\,r^3\,aa^2\,\cos(\theta)^4 - 2\,r^3\,aa^2 - 6\,r^3\,aa^2\,\cos(\theta)^2 + 5\,r^2\,M\,aa^2 + 26\,r^2\,M\,aa^2\,\cos(\theta)^4$$
$$- 8\,r^2\,aa^2\,\cos(\theta)^6\,M - 23\,r^2\,M\,aa^2\,\cos(\theta)^2 - 2\,M\,\cos(\theta)^2 + 2\,M)\,(d(\theta)\,\&^{\wedge}\,d(\phi)))/(2\,r+M) + 2\,\sin(\theta)\,\cos(\theta)\,(\,$$
$$-12\,r^4\,M^2\,aa^4\,\cos(\theta)^4 - 5\,r^2\,M^2\,aa^2 + 7\,r^2\,M^2\,aa^2\,\cos(\theta)^2 + 12\,r^4\,M^2\,aa^4\,\cos(\theta)^2 + 14\,r^3\,aa^2\,\cos(\theta)^2\,M$$
$$+ 4\,r^4\,M^2\,aa^4\,\cos(\theta)^6 - 12\,r^3\,aa^2\,M - M^2 - 4\,r^4\,M^2\,aa^4 + 18\,r^5\,aa^4\,\cos(\theta)^2\,M + 6\,r^5\,aa^4\,\cos(\theta)^6\,M - 8\,r^4\,aa^2$$
$$+ 8\,r^4\,aa^2\,\cos(\theta)^2 - 18\,r^5\,aa^4\,\cos(\theta)^4\,M - 2\,r^3\,aa^2\,\cos(\theta)^4\,M - 6\,r^5\,M\,aa^4 - 2\,r^2\,M^2\,aa^2\,\cos(\theta)^4 - 2\,r\,M)$$
$$(d(r)\,\&^{\wedge}\,d(\phi))\Big/(r\,(4\,r^2 + 4\,r\,M + M^2)) + \cos(\theta)\,aa\,(-4\,r\,M + 4\,r^2\,M^2\,aa^2\,\cos(\theta)^6 + 12\,\cos(\theta)^6\,r^3\,aa^2\,M$$
$$- 24\,r^3\,aa^2\,\cos(\theta)^4\,M + 4\,M^2\,\cos(\theta)^2 + 12\,r^3\,aa^2\,\cos(\theta)^2\,M - 3\,M^2 + 4\,r^2 - 6\,r^2\,M^2\,aa^2\,\cos(\theta)^4 + 2\,r^2\,M^2\,aa^2$$
$$+ 8\,r\,\cos(\theta)^2\,M)\,(d(\theta)\,\&^{\wedge}\,d(r))\Big/(4\,r^2 + 4\,r\,M + M^2),\ 0\Bigg]$$

$$\Bigg[\frac{(-2\,r - 2\,M\,\cos(\theta)^2 + M + 4\,r\,\cos(\theta)^2)\,aa\,(d(r)\,\&^{\wedge}\,d(\theta))}{(2\,r+M)\,\sin(\theta)\,r}$$

$$+ \frac{(-6\,r\cos(\theta)^2 + 6\,r + 7\,M + 10\cos(\theta)^4 M - 17\,M\cos(\theta)^2)\cos(\theta)\,r\,aa^2\,(\mathrm{d}(\theta)\,\&^\wedge\,\mathrm{d}(\phi))}{\sin(\theta)\,(2\,r+M)} + 2\,((4\,r^4\,aa^2$$

$$- 6\,r^3\,aa^2\cos(\theta)^2 M - 3\,r^2\,M^2\,aa^2\cos(\theta)^2 + 2\,r^2\,M^2\,aa^2\cos(\theta)^4 + 4\,r\,M - 4\,r^4\,aa^2\cos(\theta)^2 + 2\,r^3\,aa^2\cos(\theta)^4 M$$

$$+ 4\,r^3\,aa^2\,M + r^2\,M^2\,aa^2 + M^2)\,(\mathrm{d}(r)\,\&^\wedge\,\mathrm{d}(\phi)))\Big/ ((4\,r^2 + 4\,r\,M + M^2)\,r^2)\,,$$

$$- \frac{(10\,r^2\,M\,aa^2\cos(\theta)^4 - 6\,r^3\,aa^2\cos(\theta)^2 - 15\,r^2\,M\,aa^2\cos(\theta)^2 + 2\,r^3\,aa^2 + 3\,r^2\,M\,aa^2 + 2\,M)\,(\mathrm{d}(\theta)\,\&^\wedge\,\mathrm{d}(\phi))}{2\,r+M} -$$

$$\cos(\theta)\sin(\theta)\,(6\,r^2\,M^2\,aa^2\cos(\theta)^4 + 2\,M^2 + 8\,r^3\,aa^2\cos(\theta)^4 M + 4\,r\,M - 12\,r^4\,aa^2\cos(\theta)^2 - 28\,r^3\,aa^2\cos(\theta)^2 M$$

$$+ 20\,r^3\,aa^2\,M + 12\,r^4\,aa^2 + 9\,r^2\,M^2\,aa^2 - 15\,r^2\,M^2\,aa^2\cos(\theta)^2)\,(\mathrm{d}(r)\,\&^\wedge\,\mathrm{d}(\phi))\Big/ ($$

$$(4\,r^2\cos(\theta)^2 + 4\,r\cos(\theta)^2 M + M^2\cos(\theta)^2 - 4\,r^2 - 4\,r\,M - M^2)\,r) - 4\,\frac{(r+M)\,aa\cos(\theta)\,M\,(\mathrm{d}(\theta)\,\&^\wedge\,\mathrm{d}(r))}{4\,r^2 + 4\,r\,M + M^2}\,,\ -2$$

$$M\cos(\theta)\,(-2\,r - M + 6\,r^3\,aa^2\cos(\theta)^4 - r^2\,M\,aa^2\cos(\theta)^2 + 2\,r^2\,M\,aa^2\cos(\theta)^4 - r^2\,M\,aa^2 - 6\,r^3\,aa^2\cos(\theta)^2)$$

$$(\mathrm{d}(r)\,\&^\wedge\,\mathrm{d}(\theta))\Big/ (r\sin(\theta)\,(4\,r^2 + 4\,r\,M + M^2)) + ((-26\,r^2\,M\,aa^2\cos(\theta)^4 - 2\,r^2\,M\,aa^2 + 16\,r^2\,M\,aa^2\cos(\theta)^2$$

$$+ 12\,r^2\,aa^2\cos(\theta)^6 M + 2\,r - 2\,r\cos(\theta)^2 - 10\cos(\theta)^4 M + 13\,M\cos(\theta)^2 - 3\,M)\,r\,aa\,(\mathrm{d}(\theta)\,\&^\wedge\,\mathrm{d}(\phi)))/($$

$$(2\,r + M)\sin(\theta)) + aa\cos(\theta)\,(-4\,r^2 + 5\,M^2 - 6\,M^2\cos(\theta)^2 + 12\,r^3\,aa^2\cos(\theta)^4 M + 8\,r^2\,M^2\,aa^2\cos(\theta)^4$$

$$+ 12\,r^3\,aa^2\,M - 24\,r^3\,aa^2\cos(\theta)^2 M - 16\,r^2\,M^2\,aa^2\cos(\theta)^2 + 8\,r^2\,M^2\,aa^2 + 4\,r\,M - 8\,r\cos(\theta)^2 M)\,(\mathrm{d}(r)\,\&^\wedge\,\mathrm{d}(\phi))$$

$$\Big/ (4\,r^2 + 4\,r\,M + M^2)\,,\ 0\Big]$$

$$[\,0\,,\,0\,,\,0\,,\,0\,]$$

WHAT AN ALGEBRAIC MESS. BUT NOW COMPUTE the trace:

```
> TRACEOMEGA_noTorsion_smallmass:=subs(aa=0,M^2=0,2*r+M=2*r,simpform(wcollect(OMEG
  AM[1,1]+OMEGAM[2,2]+OMEGAM[3,3]+OMEGAM[4,4])));JJ:=subs(d(TRACEOMEGA_noTorsion_s
  mallmass));
```

$$TRACEOMEGA\_noTorsion\_smallmass := \frac{M\cos(\theta)\,(\mathrm{d}(r)\,\&^\wedge\,\mathrm{d}(\theta))}{r^2\sin(\theta)}$$

$$JJ := 0$$

```
> TRACEOMEGA_smallTorsion_nomass:=subs(aa^2=0,M=0,simpform(wcollect(OMEGAM[1,1]+OM
  EGAM[2,2]+OMEGAM[3,3]+OMEGAM[4,4])));JJ:=simplify(d(TRACEOMEGA_smallTorsion_noma
  ss));
```

$$TRACEOMEGA\_smallTorsion\_nomass :=$$

$$\frac{1}{2}\,\frac{aa\,(2\,r - 2\,r\cos(\theta)^2 - 2\sin(\theta)^2 r)\,(\mathrm{d}(\theta)\,\&^\wedge\,\mathrm{d}(\phi))}{\sin(\theta)} - 2\,aa\cos(\theta)\,(\mathrm{d}(r)\,\&^\wedge\,\mathrm{d}(\phi))$$

$$JJ := -2\,aa\,\&^\wedge(\mathrm{d}(r),\mathrm{d}(\theta),\mathrm{d}(\phi))\sin(\theta)$$

```
> TRACEOMEGA:=((subs(aa^2=0,M^2=0,2*r+M=2*r,simpform(wcollect(OMEGAM[1,1]+OMEGAM[2
  ,2]+OMEGAM[3,3]+OMEGAM[4,4]))))));JJ:=simplify(subs(M^2=0,d(TRACEOMEGA)));
```

$$TRACEOMEGA := \frac{M\cos(\theta)\,(\mathrm{d}(r)\,\&^\wedge\,\mathrm{d}(\theta))}{r^2\sin(\theta)} + \frac{1}{2}\,aa$$

$$(-10\cos(\theta)^4 M + 13\,M\cos(\theta)^2 + 2\,r - 3\,M - 2\,r\cos(\theta)^2 - 2\sin(\theta)^2 r + 5\sin(\theta)^2 M - 4\cos(\theta)^2\sin(\theta)^2 M)$$

$$(\mathrm{d}(\theta)\,\&^\wedge\,\mathrm{d}(\phi))\,/\sin(\theta) + \frac{1}{2}\,\frac{(-4\,r - 8\,M\cos(\theta)^2 + 6\,M)\,aa\cos(\theta)\,(\mathrm{d}(r)\,\&^\wedge\,\mathrm{d}(\phi))}{r}$$

$$JJ := -\frac{\sin(\theta)\,(12\,M\cos(\theta)^2 + 2\,r - 3\,M)\,aa\,\&^\wedge(\mathrm{d}(r),\mathrm{d}(\theta),\mathrm{d}(\phi))}{r}$$

The matrix of curvature 2-forms computed from the formula d{gamma] +

[C]^[Gamma] is not zero, and does NOT HAVE A ZERO TRACE. HENCE THE NON- ZERO TRACE IS DUE TO THE INTERACTION TERMS [T]^[Gamma].

MOREOVER, there is a 4 CURRENT JJ if the TORSION pertubation aa is not ZERO !!!! In this example there appears to exist a density distribution which depends upon the torsion coefficients. There is no "3- current" in the example.

```
> dtheta_dphi:=simplify(factor(subs(aa^2=0,(getcoeff(TRACEOMEGA&^d(r)&^d(tau))))))
  ;dr_dphi:=simplify(factor(subs(aa^2=0,(getcoeff(TRACEOMEGA&^d(theta)&^d(tau)))))
  );dr_dtheta:=simplify(factor(subs(aa^2=0,(getcoeff(TRACEOMEGA&^d(phi)&^d(tau))))
  )));simplify(factor(subs(aa^2=0,(getcoeff(TRACEOMEGA&^d(phi)&^d(theta))))));simpl
  ify(factor(subs(aa^2=0,(getcoeff(TRACEOMEGA&^d(phi)&^d(r))))));simplify(factor(s
  ubs(aa^2=0,(getcoeff(TRACEOMEGA&^d(theta)&^d(r)))))));
```

Small torsion and finite mass approximation:

>

$$dtheta\_dphi := \sin(\theta)\,(3\cos(\theta)^2 + 1)\,aa\,M$$

$$dr\_dphi := -\frac{(2\,r + 4\,M\cos(\theta)^2 - 3\,M)\,aa\,\cos(\theta)}{r}$$

$$dr\_dtheta := \frac{M\cos(\theta)}{r^2\sin(\theta)}$$

$$0$$
$$0$$
$$0$$

As aa goes to zero, there is a non-zero contribution to the trace of the of the curvature 2-forms which appears like an H field, as there is a possibility of a current 3-form.!!!!

>

The Right Cartan matrix is often defined as the sum of Christoffel Symbols and Rotation coefficients. Here the sign is changed for convenience

CartanRight(ijk) = ChristoffelGamma(ijk) - T(ijk)

Compute the T(i,j,k): These coefficients - if not zero - indicate the effects of the pertubations on the metric and Basis.   If there is no difference

between the Christoffel symbols and the Cartan connection symbols, then the T(i,j,k) are zero.

```
> for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do ss:=0; ss
  := (-CC[i,j,k]+C2S[i,j,k]); SHIPTR[i,j,k]:=simplify(ss)  od od od    ;
>
>
> for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do if
  C2S[i,j,k]=0 and CC[i,j,k]=0 then else print(`T`(i,j,k)=simplify(SHIPTR[i,j,k]))
  fi od od od ;
```

T(ijk) index (1,-1,-1) with mass and torsion perturbations.

$$T(1,1,1) = 2\,\frac{-2\,r^3\,aa^2\cos(\theta)^2 + 2\,r^3\,aa^2\cos(\theta)^4 - M}{r\,(2\,r+M)}$$

$$T(1,1,2) = \cos(\theta)\,aa^2\,r^2\sin(\theta) - 2\cos(\theta)^3\,aa^2\,r^2\sin(\theta)$$

$$T(1,1,3) = aa\cos(\theta)\,r\,(-2\,M + 2\,M\cos(\theta)^2 + 2\,r^3\,aa^2 - 4\,r^3\,aa^2\cos(\theta)^2 + 2\,r^3\,aa^2\cos(\theta)^4 + 5\,r^2\,M\,aa^2\cos(\theta)^4$$
$$-\,2\,r^2\,aa^2\cos(\theta)^6\,M + r^2\,M\,aa^2 - 4\,r^2\,M\,aa^2\cos(\theta)^2)\,/\,(2\,r+M)$$

$$T(1,2,1) = \cos(\theta)\,aa^2\,r^2\sin(\theta) - 2\cos(\theta)^3\,aa^2\,r^2\sin(\theta)$$

$$T(1,2,2) = -2\,\frac{r\,(-M - 2\,r^3\,aa^2\cos(\theta)^2 + 2\,r^3\,aa^2\cos(\theta)^4 + 2\,r^2\,M\,aa^2\cos(\theta)^4 - 2\,r^2\,M\,aa^2\cos(\theta)^2)}{2\,r+M}$$

$$T(1,2,3) = -\sin(\theta)\,r^2\,aa\,(2\,M\cos(\theta)^2 - 2\,r^3\,aa^2\cos(\theta)^2 + 2\,r^3\,aa^2\cos(\theta)^4 - 3\,r^2\,M\,aa^2\cos(\theta)^2$$
$$+\,5\,r^2\,M\,aa^2\cos(\theta)^4 - 2\,r^2\,aa^2\cos(\theta)^6\,M + 2\,r - M)\,/\,(2\,r+M)$$

$$T(1,3,1) = r^2\cos(\theta)\,aa\,(4 - 4\cos(\theta)^2 + 2\,r^2\,aa^2 - 4\,r^2\cos(\theta)^2\,aa^2 + 2\,r^2\cos(\theta)^4\,aa^2 + 5\,r\,M\,aa^2\cos(\theta)^4$$
$$-\,2\,r\,aa^2\cos(\theta)^6\,M + r\,M\,aa^2 - 4\,r\,M\,aa^2\cos(\theta)^2)\,/\,(2\,r+M)$$

$$T(1,3,2) = -\sin(\theta)\,r^2\,aa\,(-4\,r\cos(\theta)^2 - 2\,r^3\,aa^2\cos(\theta)^2 + 2\,r^3\,aa^2\cos(\theta)^4 - 3\,r^2\,M\,aa^2\cos(\theta)^2$$
$$+\,5\,r^2\,M\,aa^2\cos(\theta)^4 - 2\,r^2\,aa^2\cos(\theta)^6\,M + 2\,r - M)\,/\,(2\,r+M)$$

$$T(1,3,3) = -2\,r\,(-M + M\cos(\theta)^2 - 3\,r^4\cos(\theta)^6\,aa^4\,M + r^4\cos(\theta)^8\,aa^4\,M - r^4\cos(\theta)^2\,aa^4\,M + 3\,r^4\cos(\theta)^4\,aa^4\,M$$
$$-\,r^2\,aa^2\cos(\theta)^6\,M + 2\,r^3\,aa^2 - r^2\,M\,aa^2\cos(\theta)^2 + 2\,r^3\,aa^2\cos(\theta)^4 - 4\,r^3\,aa^2\cos(\theta)^2 + 2\,r^2\,M\,aa^2\cos(\theta)^4)\,/\,($$
$$2\,r+M)$$

$$T(2,1,1) = -4\,\frac{(-1 + \cos(\theta)^2)\,r\,aa^2\cos(\theta)\sin(\theta)}{2\,r+M}$$

$$T(2,1,2) =$$
$$-\,\frac{2\,M - 6\,r^3\,aa^2\cos(\theta)^2 + 2\,r^3\,aa^2 + 4\,r^3\,aa^2\cos(\theta)^4 - 3\,r^2\,M\,aa^2\cos(\theta)^2 + 2\,r^2\,M\,aa^2\cos(\theta)^4 + r^2\,M\,aa^2}{r\,(2\,r+M)}$$

$$T(2,1,3) = -\sin(\theta)\,aa\,(-2\,r - M - 4\,r^3\,aa^2\cos(\theta)^2 + 2\,r^3\,aa^2 + 2\,r^3\,aa^2\cos(\theta)^4 - 2\,r^2\,aa^2\cos(\theta)^6\,M + r^2\,M\,aa^2$$
$$+\,5\,r^2\,M\,aa^2\cos(\theta)^4 - 4\,r^2\,M\,aa^2\cos(\theta)^2)\,/\,(2\,r+M)$$

$$T(2,2,1) =$$
$$-\,\frac{2\,M - 6\,r^3\,aa^2\cos(\theta)^2 + 2\,r^3\,aa^2 + 4\,r^3\,aa^2\cos(\theta)^4 - 3\,r^2\,M\,aa^2\cos(\theta)^2 + 2\,r^2\,M\,aa^2\cos(\theta)^4 + r^2\,M\,aa^2}{r\,(2\,r+M)}$$

$$T(2, 2, 2) = 4\,\frac{\cos(\theta)\,aa^2\,r^2\,\sin(\theta)\,(-r + r\cos(\theta)^2 + M\cos(\theta)^2 - M)}{2\,r + M}$$

$$T(2, 2, 3) = -\frac{r^3\cos(\theta)\,aa^3\,(2\,r - 4\,r\cos(\theta)^2 + 2\,r\cos(\theta)^4 + 3\,M - 8\,M\cos(\theta)^2 + 7\cos(\theta)^4\,M - 2\cos(\theta)^6\,M)}{2\,r + M}$$

$$T(2, 3, 1) = -\sin(\theta)\,aa\,(2\,r + M - 4\,r\cos(\theta)^2 - 2\,M\cos(\theta)^2 - 4\,r^3\,aa^2\cos(\theta)^2 + 2\,r^3\,aa^2 + 2\,r^3\,aa^2\cos(\theta)^4$$
$$- 2\,r^2\,aa^2\cos(\theta)^6\,M + r^2\,M\,aa^2 + 5\,r^2\,M\,aa^2\cos(\theta)^4 - 4\,r^2\,M\,aa^2\cos(\theta)^2)\,/\,(2\,r + M)$$

$$T(2, 3, 2) = -r\cos(\theta)\,aa\,(4\,r + 2\,M - 4\,r\cos(\theta)^2 - 2\,M\cos(\theta)^2 + 2\,r^3\,aa^2 - 4\,r^3\,aa^2\cos(\theta)^2 + 2\,r^3\,aa^2\cos(\theta)^4$$
$$+ 3\,r^2\,M\,aa^2 - 8\,r^2\,M\,aa^2\cos(\theta)^2 + 7\,r^2\,M\,aa^2\cos(\theta)^4 - 2\,r^2\,aa^2\cos(\theta)^6\,M)\,/\,(2\,r + M)$$

$$T(2, 3, 3) = 2\,\sin(\theta)\cos(\theta)\,r^2\,aa^2\,(r^2\,aa^2\cos(\theta)^6\,M - r^2\,M\,aa^2 - 3\,r^2\,M\,aa^2\cos(\theta)^4 + 3\,r^2\,M\,aa^2\cos(\theta)^2$$
$$+ 2\,r\cos(\theta)^2 - 2\,r - 2\,M - \cos(\theta)^4\,M + 3\,M\cos(\theta)^2)\,/\,(2\,r + M)$$

$$T(3, 1, 1) = -4\,\frac{\cos(\theta)\,aa}{2\,r + M}$$

$$T(3, 1, 2) = \frac{\sin(\theta)\,aa\,(-1 + 2\cos(\theta)^2)}{-1 + \cos(\theta)^2}$$

$$T(3, 1, 3) = -\frac{2\,M + 2\,r^3\,aa^2\cos(\theta)^2 + 3\,r^2\,M\,aa^2\cos(\theta)^2 - 2\,r^2\,M\,aa^2\cos(\theta)^4 - 2\,r^3\,aa^2 - r^2\,M\,aa^2}{r\,(2\,r + M)}$$

$$T(3, 2, 1) = \frac{\sin(\theta)\,aa\,(-1 + 2\cos(\theta)^2)}{-1 + \cos(\theta)^2}$$

$$T(3, 2, 2) = 4\,\frac{(r + M)\cos(\theta)\,r\,aa}{2\,r + M}$$

$$T(3, 2, 3) = \frac{(2\,r - 2\,M\cos(\theta)^2 + 3\,M)\,r^2\,aa^2\cos(\theta)\sin(\theta)}{2\,r + M}$$

$$T(3, 3, 1) = -\frac{2\,M + 2\,r^3\,aa^2\cos(\theta)^2 + 3\,r^2\,M\,aa^2\cos(\theta)^2 - 2\,r^2\,M\,aa^2\cos(\theta)^4 - 2\,r^3\,aa^2 - r^2\,M\,aa^2}{r\,(2\,r + M)}$$

$$T(3, 3, 2) = \frac{(2\,r - 2\,M\cos(\theta)^2 + 3\,M)\,r^2\,aa^2\cos(\theta)\sin(\theta)}{2\,r + M}$$

$$T(3, 3, 3) = 2\,\frac{\cos(\theta)\,r\,aa\,M\,(r^2\cos(\theta)^4\,aa^2 - 2\,r^2\cos(\theta)^2\,aa^2 + r^2\,aa^2 + 1 - \cos(\theta)^2)}{2\,r + M}$$

```
>
>
> for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do if
  C2S[i,j,k]=90 and CC[i,j,k]=0 then else
  print(`T`(i,j,k)=simplify(subs(aa=0,SHIPTR[i,j,k]))) fi od od od ;
```

T(ijk) index (1,-1,-1) with mass perturbations only.

$$T(1, 1, 1) = -2\,\frac{M}{r\,(2\,r + M)}$$

$$T(1, 1, 2) = 0$$

$$T(1, 1, 3) = 0$$

$$T(1, 1, 4) = 0$$

$$T(1, 2, 1) = 0$$

$$T(1, 2, 2) = 2\,\frac{r\,M}{2\,r + M}$$

$$T(1, 2, 3) = 0$$

$$T(1, 2, 4) = 0$$

$$T(1, 3, 1) = 0$$

$$T(1, 3, 2) = 0$$

$$T(1, 3, 3) = -2\frac{rM(-1+\cos(\theta)^2)}{2r+M}$$

$$T(1, 3, 4) = 0$$

$$T(1, 4, 1) = 0$$

$$T(1, 4, 2) = 0$$

$$T(1, 4, 3) = 0$$

$$T(1, 4, 4) = 0$$

$$T(2, 1, 1) = 0$$

$$T(2, 1, 2) = -2\frac{M}{r(2r+M)}$$

$$T(2, 1, 3) = 0$$

$$T(2, 1, 4) = 0$$

$$T(2, 2, 1) = -2\frac{M}{r(2r+M)}$$

$$T(2, 2, 2) = 0$$

$$T(2, 2, 3) = 0$$

$$T(2, 2, 4) = 0$$

$$T(2, 3, 1) = 0$$

$$T(2, 3, 2) = 0$$

$$T(2, 3, 3) = 0$$

$$T(2, 3, 4) = 0$$

$$T(2, 4, 1) = 0$$

$$T(2, 4, 2) = 0$$

$$T(2, 4, 3) = 0$$

$$T(2, 4, 4) = 0$$

$$T(3, 1, 1) = 0$$

$$T(3, 1, 2) = 0$$

$$T(3, 1, 3) = -2\frac{M}{r(2r+M)}$$

$$T(3, 1, 4) = 0$$

$$T(3, 2, 1) = 0$$

$$T(3, 2, 2) = 0$$

$$T(3, 2, 3) = 0$$

$$T(3, 2, 4) = 0$$

$$T(3, 3, 1) = -2\frac{M}{r(2r+M)}$$

$$T(3, 3, 2) = 0$$

$$T(3, 3, 3) = 0$$

$$T(3, 3, 4) = 0$$

$$T(3, 4, 1) = 0$$
$$T(3, 4, 2) = 0$$
$$T(3, 4, 3) = 0$$
$$T(3, 4, 4) = 0$$
$$T(4, 1, 1) = 0$$
$$T(4, 1, 2) = 0$$
$$T(4, 1, 3) = 0$$
$$T(4, 1, 4) = 0$$
$$T(4, 2, 1) = 0$$
$$T(4, 2, 2) = 0$$
$$T(4, 2, 3) = 0$$
$$T(4, 2, 4) = 0$$
$$T(4, 3, 1) = 0$$
$$T(4, 3, 2) = 0$$
$$T(4, 3, 3) = 0$$
$$T(4, 3, 4) = 0$$
$$T(4, 4, 1) = 0$$
$$T(4, 4, 2) = 0$$
$$T(4, 4, 3) = 0$$
$$T(4, 4, 4) = 0$$

```
>
> for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do if
  C2S[i,j,k]=90 and CC[i,j,k]=0 then else
  print(`T`(i,j,k)=simplify(subs(M=0,SHIPTR[i,j,k]))) fi od od od ;
```

T(ijk) index (1,-1,-1) with torsion perturbations only.

$$T(1, 1, 1) = -2\, r \cos(\theta)^2\, aa^2 + 2\, r \cos(\theta)^4\, aa^2$$
$$T(1, 1, 2) = \cos(\theta)\, aa^2\, r^2 \sin(\theta) - 2 \cos(\theta)^3\, aa^2\, r^2 \sin(\theta)$$
$$T(1, 1, 3) = aa^3 \cos(\theta)\, r^3 - 2\, aa^3 \cos(\theta)^3\, r^3 + aa^3 \cos(\theta)^5\, r^3$$
$$T(1, 1, 4) = 0$$
$$T(1, 2, 1) = \cos(\theta)\, aa^2\, r^2 \sin(\theta) - 2 \cos(\theta)^3\, aa^2\, r^2 \sin(\theta)$$
$$T(1, 2, 2) = 2\, r^3\, aa^2 \cos(\theta)^2 - 2\, r^3\, aa^2 \cos(\theta)^4$$
$$T(1, 2, 3) = \sin(\theta)\, r^4\, aa^3 \cos(\theta)^2 - \sin(\theta)\, r^4\, aa^3 \cos(\theta)^4 - \sin(\theta)\, r^2\, aa$$
$$T(1, 2, 4) = 0$$
$$T(1, 3, 1) = 2\, r \cos(\theta)\, aa - 2\, r \cos(\theta)^3\, aa + aa^3 \cos(\theta)\, r^3 - 2\, aa^3 \cos(\theta)^3\, r^3 + aa^3 \cos(\theta)^5\, r^3$$
$$T(1, 3, 2) = 2 \sin(\theta)\, r^2 \cos(\theta)^2\, aa + \sin(\theta)\, r^4\, aa^3 \cos(\theta)^2 - \sin(\theta)\, r^4\, aa^3 \cos(\theta)^4 - \sin(\theta)\, r^2\, aa$$
$$T(1, 3, 3) = -2\, r^3\, aa^2 + 4\, r^3\, aa^2 \cos(\theta)^2 - 2\, r^3\, aa^2 \cos(\theta)^4$$
$$T(1, 3, 4) = 0$$
$$T(1, 4, 1) = 0$$
$$T(1, 4, 2) = 0$$
$$T(1, 4, 3) = 0$$
$$T(1, 4, 4) = 0$$

$$T(2, 1, 1) = 2\ aa^2 \cos(\theta) \sin(\theta) - 2\ aa^2 \cos(\theta)^3 \sin(\theta)$$

$$T(2, 1, 2) = 3\ r \cos(\theta)^2 aa^2 - aa^2 r - 2\ r \cos(\theta)^4 aa^2$$

$$T(2, 1, 3) = \sin(\theta)\ aa + 2 \sin(\theta)\ aa^3 r^2 \cos(\theta)^2 - \sin(\theta)\ aa^3 r^2 - \sin(\theta)\ aa^3 r^2 \cos(\theta)^4$$

$$T(2, 1, 4) = 0$$

$$T(2, 2, 1) = 3\ r \cos(\theta)^2 aa^2 - aa^2 r - 2\ r \cos(\theta)^4 aa^2$$

$$T(2, 2, 2) = -2 \cos(\theta)\ aa^2 r^2 \sin(\theta) + 2 \cos(\theta)^3 aa^2 r^2 \sin(\theta)$$

$$T(2, 2, 3) = -aa^3 \cos(\theta)\ r^3 + 2\ aa^3 \cos(\theta)^3 r^3 - aa^3 \cos(\theta)^5 r^3$$

$$T(2, 2, 4) = 0$$

$$T(2, 3, 1) = -\sin(\theta)\ aa + 2 \sin(\theta)\ aa \cos(\theta)^2 + 2 \sin(\theta)\ aa^3 r^2 \cos(\theta)^2 - \sin(\theta)\ aa^3 r^2 - \sin(\theta)\ aa^3 r^2 \cos(\theta)^4$$

$$T(2, 3, 2) = -2\ r \cos(\theta)\ aa + 2\ r \cos(\theta)^3 aa - aa^3 \cos(\theta)\ r^3 + 2\ aa^3 \cos(\theta)^3 r^3 - aa^3 \cos(\theta)^5 r^3$$

$$T(2, 3, 3) = -2 \cos(\theta)\ aa^2 r^2 \sin(\theta) + 2 \cos(\theta)^3 aa^2 r^2 \sin(\theta)$$

$$T(2, 3, 4) = 0$$

$$T(2, 4, 1) = 0$$

$$T(2, 4, 2) = 0$$

$$T(2, 4, 3) = 0$$

$$T(2, 4, 4) = 0$$

$$T(3, 1, 1) = -2\ \frac{\cos(\theta)\ aa}{r}$$

$$T(3, 1, 2) = \frac{\sin(\theta)\ aa\ (-1 + 2 \cos(\theta)^2)}{-1 + \cos(\theta)^2}$$

$$T(3, 1, 3) = aa^2 r - r \cos(\theta)^2 aa^2$$

$$T(3, 1, 4) = 0$$

$$T(3, 2, 1) = \frac{\sin(\theta)\ aa\ (-1 + 2 \cos(\theta)^2)}{-1 + \cos(\theta)^2}$$

$$T(3, 2, 2) = 2\ r \cos(\theta)\ aa$$

$$T(3, 2, 3) = \cos(\theta)\ aa^2 r^2 \sin(\theta)$$

$$T(3, 2, 4) = 0$$

$$T(3, 3, 1) = aa^2 r - r \cos(\theta)^2 aa^2$$

$$T(3, 3, 2) = \cos(\theta)\ aa^2 r^2 \sin(\theta)$$

$$T(3, 3, 3) = 0$$

$$T(3, 3, 4) = 0$$

$$T(3, 4, 1) = 0$$

$$T(3, 4, 2) = 0$$

$$T(3, 4, 3) = 0$$

$$T(3, 4, 4) = 0$$

$$T(4, 1, 1) = 0$$

$$T(4, 1, 2) = 0$$

$$T(4, 1, 3) = 0$$

$$T(4, 1, 4) = 0$$

$$T(4, 2, 1) = 0$$

$$T(4, 2, 2) = 0$$

$$T(4, 2, 3) = 0$$

$$T(4, 2, 4) = 0$$
$$T(4, 3, 1) = 0$$
$$T(4, 3, 2) = 0$$
$$T(4, 3, 3) = 0$$
$$T(4, 3, 4) = 0$$
$$T(4, 4, 1) = 0$$
$$T(4, 4, 2) = 0$$
$$T(4, 4, 3) = 0$$
$$T(4, 4, 4) = 0$$

NOte that there are first and higher order terms in the connection perturbation coefficient aa.

\>

```
> for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do if
  C2S[i,j,k]=90 and CC[i,j,k]=0 then else
  print(`T`(i,j,k)=simplify(subs(M=0,aa=0,SHIPTR[i,j,k]))) fi od od od ;
```

T(ijk) index (1,-1,-1) with no torsion and no mass perturbations.

$$T(1, 1, 1) = 0$$
$$T(1, 1, 2) = 0$$
$$T(1, 1, 3) = 0$$
$$T(1, 1, 4) = 0$$
$$T(1, 2, 1) = 0$$
$$T(1, 2, 2) = 0$$
$$T(1, 2, 3) = 0$$
$$T(1, 2, 4) = 0$$
$$T(1, 3, 1) = 0$$
$$T(1, 3, 2) = 0$$
$$T(1, 3, 3) = 0$$
$$T(1, 3, 4) = 0$$
$$T(1, 4, 1) = 0$$
$$T(1, 4, 2) = 0$$
$$T(1, 4, 3) = 0$$
$$T(1, 4, 4) = 0$$
$$T(2, 1, 1) = 0$$
$$T(2, 1, 2) = 0$$
$$T(2, 1, 3) = 0$$
$$T(2, 1, 4) = 0$$
$$T(2, 2, 1) = 0$$
$$T(2, 2, 2) = 0$$
$$T(2, 2, 3) = 0$$
$$T(2, 2, 4) = 0$$
$$T(2, 3, 1) = 0$$
$$T(2, 3, 2) = 0$$
$$T(2, 3, 3) = 0$$
$$T(2, 3, 4) = 0$$

$$T(2, 4, 1) = 0$$
$$T(2, 4, 2) = 0$$
$$T(2, 4, 3) = 0$$
$$T(2, 4, 4) = 0$$
$$T(3, 1, 1) = 0$$
$$T(3, 1, 2) = 0$$
$$T(3, 1, 3) = 0$$
$$T(3, 1, 4) = 0$$
$$T(3, 2, 1) = 0$$
$$T(3, 2, 2) = 0$$
$$T(3, 2, 3) = 0$$
$$T(3, 2, 4) = 0$$
$$T(3, 3, 1) = 0$$
$$T(3, 3, 2) = 0$$
$$T(3, 3, 3) = 0$$
$$T(3, 3, 4) = 0$$
$$T(3, 4, 1) = 0$$
$$T(3, 4, 2) = 0$$
$$T(3, 4, 3) = 0$$
$$T(3, 4, 4) = 0$$
$$T(4, 1, 1) = 0$$
$$T(4, 1, 2) = 0$$
$$T(4, 1, 3) = 0$$
$$T(4, 1, 4) = 0$$
$$T(4, 2, 1) = 0$$
$$T(4, 2, 2) = 0$$
$$T(4, 2, 3) = 0$$
$$T(4, 2, 4) = 0$$
$$T(4, 3, 1) = 0$$
$$T(4, 3, 2) = 0$$
$$T(4, 3, 3) = 0$$
$$T(4, 3, 4) = 0$$
$$T(4, 4, 1) = 0$$
$$T(4, 4, 2) = 0$$
$$T(4, 4, 3) = 0$$
$$T(4, 4, 4) = 0$$

>

Right Cartan(ijk) = ChristoffelGamma(ijk) - T(ijk)

NOW COMPUTE THE MATRIX ELEMENTS OF THE T(ijk) CONNECTION

```
> TGamma11:=SHIPTR[1,1,1]*d(r)+SHIPTR[1,1,2]*d(theta)+SHIPTR[1,1,3]*d(phi)+SHIPTR[
  1,1,4]*d(tau);
```

> `TGamma12:=SHIPTR[1,2,1]*d(r)+SHIPTR[1,2,2]*d(theta)+SHIPTR[1,2,3]*d(phi)+SHIPTR[1,2,4]*d(tau);`

> `TGamma13:=SHIPTR[1,3,1]*d(r)+SHIPTR[1,3,2]*d(theta)+SHIPTR[1,3,3]*d(phi)+SHIPTR[1,3,4]*d(tau);`

> `TGamma14:=SHIPTR[1,4,1]*d(r)+SHIPTR[1,4,2]*d(theta)+SHIPTR[1,4,3]*d(phi)+SHIPTR[1,4,4]*d(tau);`

$$TGamma11 := 2\,\frac{(-2\,r^3\,aa^2\cos(\theta)^2 + 2\,r^3\,aa^2\cos(\theta)^4 - M)\,\mathrm{d}(r)}{r\,(2\,r + M)}$$

$$+ (\cos(\theta)\,aa^2\,r^2\sin(\theta) - 2\cos(\theta)^3\,aa^2\,r^2\sin(\theta))\,\mathrm{d}(\theta) - aa\cos(\theta)\,r\,(2\,M - 2\,M\cos(\theta)^2 - 2\,r^3\,aa^2$$

$$+ 4\,r^3\,aa^2\cos(\theta)^2 - 2\,r^3\,aa^2\cos(\theta)^4 - 5\,r^2\,M\,aa^2\cos(\theta)^4 + 2\,r^2\,aa^2\cos(\theta)^6\,M - r^2\,M\,aa^2 + 4\,r^2\,M\,aa^2\cos(\theta)^2)$$

$$\mathrm{d}(\phi)\,/\,(2\,r + M)$$

$$TGamma12 := (\cos(\theta)\,aa^2\,r^2\sin(\theta) - 2\cos(\theta)^3\,aa^2\,r^2\sin(\theta))\,\mathrm{d}(r)$$

$$- 2\,\frac{r\,(-M - 2\,r^3\,aa^2\cos(\theta)^2 + 2\,r^3\,aa^2\cos(\theta)^4 + 2\,r^2\,M\,aa^2\cos(\theta)^4 - 2\,r^2\,M\,aa^2\cos(\theta)^2)\,\mathrm{d}(\theta)}{2\,r + M} + \sin(\theta)\,r^2\,aa\,($$

$$-2\,M\cos(\theta)^2 + 2\,r^3\,aa^2\cos(\theta)^2 - 2\,r^3\,aa^2\cos(\theta)^4 + 3\,r^2\,M\,aa^2\cos(\theta)^2 - 5\,r^2\,M\,aa^2\cos(\theta)^4 + 2\,r^2\,aa^2\cos(\theta)^6\,M$$

$$- 2\,r + M)\,\mathrm{d}(\phi)\,/\,(2\,r + M)$$

$$TGamma13 := r^2\cos(\theta)\,aa\,(4 - 4\cos(\theta)^2 + 2\,r^2\,aa^2 - 4\,r^2\cos(\theta)^2\,aa^2 + 2\,r^2\cos(\theta)^4\,aa^2 + 5\,r\,M\,aa^2\cos(\theta)^4$$

$$- 2\,r\,aa^2\cos(\theta)^6\,M + r\,M\,aa^2 - 4\,r\,M\,aa^2\cos(\theta)^2)\,\mathrm{d}(r)\,/\,(2\,r + M) - \sin(\theta)\,r^2\,aa\,(-4\,r\cos(\theta)^2 - 2\,r^3\,aa^2\cos(\theta)^2$$

$$+ 2\,r^3\,aa^2\cos(\theta)^4 - 3\,r^2\,M\,aa^2\cos(\theta)^2 + 5\,r^2\,M\,aa^2\cos(\theta)^4 - 2\,r^2\,aa^2\cos(\theta)^6\,M + 2\,r - M)\,\mathrm{d}(\theta)\,/\,(2\,r + M) - 2\,r$$

$$(-M + M\cos(\theta)^2 - 3\,r^4\cos(\theta)^6\,aa^4\,M + r^4\cos(\theta)^8\,aa^4\,M - r^4\cos(\theta)^2\,aa^4\,M + 3\,r^4\cos(\theta)^4\,aa^4\,M - r^2\,aa^2\cos(\theta)^6\,M$$

$$+ 2\,r^3\,aa^2 - r^2\,M\,aa^2\cos(\theta)^2 + 2\,r^3\,aa^2\cos(\theta)^4 - 4\,r^3\,aa^2\cos(\theta)^2 + 2\,r^2\,M\,aa^2\cos(\theta)^4)\,\mathrm{d}(\phi)\,/\,(2\,r + M)$$

$$TGamma14 := 0$$

> `TGamma21:=SHIPTR[2,1,1]*d(r)+SHIPTR[2,1,2]*d(theta)+SHIPTR[2,1,3]*d(phi)+SHIPTR[2,1,4]*d(tau);`

> `TGamma22:=SHIPTR[2,2,1]*d(r)+SHIPTR[2,2,2]*d(theta)+SHIPTR[2,2,3]*d(phi)+SHIPTR[2,2,4]*d(tau);`

> `TGamma23:=SHIPTR[2,3,1]*d(r)+SHIPTR[2,3,2]*d(theta)+SHIPTR[2,3,3]*d(phi)+SHIPTR[2,3,4]*d(tau);`

> `TGamma24:=SHIPTR[2,4,1]*d(r)+SHIPTR[2,4,2]*d(theta)+SHIPTR[2,4,3]*d(phi)+SHIPTR[2,4,4]*d(tau);`

$$TGamma21 := -4\,\frac{(-1 + \cos(\theta)^2)\,r\,aa^2\cos(\theta)\sin(\theta)\,\mathrm{d}(r)}{2\,r + M}$$

$$- \frac{(2\,M - 6\,r^3\,aa^2\cos(\theta)^2 + 2\,r^3\,aa^2 + 4\,r^3\,aa^2\cos(\theta)^4 - 3\,r^2\,M\,aa^2\cos(\theta)^2 + 2\,r^2\,M\,aa^2\cos(\theta)^4 + r^2\,M\,aa^2)\,\mathrm{d}(\theta)}{r\,(2\,r + M)}$$

$$- \sin(\theta)\,aa\,(-2\,r - M - 4\,r^3\,aa^2\cos(\theta)^2 + 2\,r^3\,aa^2 + 2\,r^3\,aa^2\cos(\theta)^4 - 2\,r^2\,aa^2\cos(\theta)^6\,M + r^2\,M\,aa^2$$

$$+ 5\,r^2\,M\,aa^2\cos(\theta)^4 - 4\,r^2\,M\,aa^2\cos(\theta)^2)\,\mathrm{d}(\phi)\,/\,(2\,r + M)$$

$$TGamma22 :=$$

$$- \frac{(2\,M - 6\,r^3\,aa^2\cos(\theta)^2 + 2\,r^3\,aa^2 + 4\,r^3\,aa^2\cos(\theta)^4 - 3\,r^2\,M\,aa^2\cos(\theta)^2 + 2\,r^2\,M\,aa^2\cos(\theta)^4 + r^2\,M\,aa^2)\,\mathrm{d}(r)}{r\,(2\,r + M)}$$

$$+ 4\,\frac{\cos(\theta)\,aa^2\,r^2\sin(\theta)\,(-r + r\cos(\theta)^2 + M\cos(\theta)^2 - M)\,\mathrm{d}(\theta)}{2\,r + M}$$

$$- \frac{r^3\cos(\theta)\,aa^3\,(2\,r - 4\,r\cos(\theta)^2 + 2\,r\cos(\theta)^4 + 3\,M - 8\,M\cos(\theta)^2 + 7\cos(\theta)^4\,M - 2\cos(\theta)^6\,M)\,\mathrm{d}(\phi)}{2\,r + M}$$

$$TGamma23 := -\sin(\theta)\,aa\,(2\,r + M - 4\,r\cos(\theta)^2 - 2\,M\cos(\theta)^2 - 4\,r^3\,aa^2\cos(\theta)^2 + 2\,r^3\,aa^2 + 2\,r^3\,aa^2\cos(\theta)^4$$

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$$-2\,r^2\,aa^2\cos(\theta)^6\,M + r^2\,M\,aa^2 + 5\,r^2\,M\,aa^2\cos(\theta)^4 - 4\,r^2\,M\,aa^2\cos(\theta)^2)\,\mathrm{d}(r)\,/\,(2\,r+M) - r\cos(\theta)\,aa\,(4\,r$$
$$+2\,M - 4\,r\cos(\theta)^2 - 2\,M\cos(\theta)^2 + 2\,r^3\,aa^2 - 4\,r^3\,aa^2\cos(\theta)^2 + 2\,r^3\,aa^2\cos(\theta)^4 + 3\,r^2\,M\,aa^2 - 8\,r^2\,M\,aa^2\cos(\theta)^2$$
$$+7\,r^2\,M\,aa^2\cos(\theta)^4 - 2\,r^2\,aa^2\cos(\theta)^6\,M)\,\mathrm{d}(\theta)\,/\,(2\,r+M) + 2\sin(\theta)\cos(\theta)\,r^2\,aa^2\,(r^2\,aa^2\cos(\theta)^6\,M - r^2\,M\,aa^2$$
$$-3\,r^2\,M\,aa^2\cos(\theta)^4 + 3\,r^2\,M\,aa^2\cos(\theta)^2 + 2\,r\cos(\theta)^2 - 2\,r - 2\,M - \cos(\theta)^4\,M + 3\,M\cos(\theta)^2)\,\mathrm{d}(\phi)\,/\,(2\,r+M)$$

$$TGamma24 := 0$$

```
> TGamma31:=SHIPTR[3,1,1]*d(r)+SHIPTR[3,1,2]*d(theta)+SHIPTR[3,1,3]*d(phi)+SHIPTR[
  3,1,4]*d(tau);
> TGamma32:=SHIPTR[3,2,1]*d(r)+SHIPTR[3,2,2]*d(theta)+SHIPTR[3,2,3]*d(phi)+SHIPTR[
  3,2,4]*d(tau);
> TGamma33:=SHIPTR[3,3,1]*d(r)+SHIPTR[3,3,2]*d(theta)+SHIPTR[3,3,3]*d(phi)+SHIPTR[
  3,3,4]*d(tau);
> TGamma34:=SHIPTR[3,4,1]*d(r)+SHIPTR[3,4,2]*d(theta)+SHIPTR[3,4,3]*d(phi)+SHIPTR[
  3,4,4]*d(tau);
```

$$TGamma31 := -4\,\frac{\cos(\theta)\,aa\,\mathrm{d}(r)}{2\,r+M} + \frac{\sin(\theta)\,aa\,(-1+2\cos(\theta)^2)\,\mathrm{d}(\theta)}{-1+\cos(\theta)^2}$$
$$-\frac{(2\,M + 2\,r^3\,aa^2\cos(\theta)^2 + 3\,r^2\,M\,aa^2\cos(\theta)^2 - 2\,r^2\,M\,aa^2\cos(\theta)^4 - 2\,r^3\,aa^2 - r^2\,M\,aa^2)\,\mathrm{d}(\phi)}{r\,(2\,r+M)}$$

$$TGamma32 := \frac{\sin(\theta)\,aa\,(-1+2\cos(\theta)^2)\,\mathrm{d}(r)}{-1+\cos(\theta)^2} + 4\,\frac{(r+M)\cos(\theta)\,r\,aa\,\mathrm{d}(\theta)}{2\,r+M}$$
$$+\frac{(2\,r - 2\,M\cos(\theta)^2 + 3\,M)\,r^2\,aa^2\cos(\theta)\sin(\theta)\,\mathrm{d}(\phi)}{2\,r+M}$$

$$TGamma33 := -\frac{(2\,M + 2\,r^3\,aa^2\cos(\theta)^2 + 3\,r^2\,M\,aa^2\cos(\theta)^2 - 2\,r^2\,M\,aa^2\cos(\theta)^4 - 2\,r^3\,aa^2 - r^2\,M\,aa^2)\,\mathrm{d}(r)}{r\,(2\,r+M)}$$
$$+\frac{(2\,r - 2\,M\cos(\theta)^2 + 3\,M)\,r^2\,aa^2\cos(\theta)\sin(\theta)\,\mathrm{d}(\theta)}{2\,r+M}$$
$$+2\,\frac{\cos(\theta)\,r\,aa\,M\,(r^2\cos(\theta)^4\,aa^2 - 2\,r^2\cos(\theta)^2\,aa^2 + r^2\,aa^2 + 1 - \cos(\theta)^2)\,\mathrm{d}(\phi)}{2\,r+M}$$

$$TGamma34 := 0$$

```
> TGamma41:=SHIPTR[4,1,1]*d(r)+SHIPTR[4,1,2]*d(theta)+SHIPTR[4,1,3]*d(phi)+SHIPTR[
  4,1,4]*d(tau);
> TGamma42:=SHIPTR[4,2,1]*d(r)+SHIPTR[4,2,2]*d(theta)+SHIPTR[4,2,3]*d(phi)+SHIPTR[
  4,2,4]*d(tau);
> TGamma43:=SHIPTR[4,3,1]*d(r)+SHIPTR[4,3,2]*d(theta)+SHIPTR[4,3,3]*d(phi)+SHIPTR[
  4,3,4]*d(tau);
> TGamma44:=SHIPTR[4,4,1]*d(r)+SHIPTR[4,4,2]*d(theta)+SHIPTR[4,4,3]*d(phi)+SHIPTR[
  4,4,4]*d(tau);
```

$$TGamma41 := 0$$
$$TGamma42 := 0$$
$$TGamma43 := 0$$
$$TGamma44 := 0$$

```
> 
> 
```

Compute the inertial curvature d[T] + [C]^[T]
```
> OmegaCC11:=simpform(simplify(CGamma11&^TGamma11+CGamma12&^TGamma21+CGamma13&^TGa
```

```
mma31+CGamma14&^TGamma41));OmegaCC12:=simpform(simplify(CGamma11&^TGamma12+CGamm
a12&^TGamma22+CGamma13&^TGamma32+CGamma14&^TGamma42));OmegaCC13:=simpform(simpli
fy(CGamma11&^TGamma13+CGamma12&^TGamma23+CGamma13&^TGamma33+CGamma14&^TGamma43))
;OmegaCC14:=simpform(simplify(CGamma11&^TGamma14+CGamma12&^TGamma24+CGamma13&^TG
amma34+CGamma14&^TGamma44));
```

$OmegaCC11 := -\dfrac{r\sin(\theta)\,aa^2\cos(\theta)\,(-2\,r-2\,M\cos(\theta)^2+M+4\,r\cos(\theta)^2)\,(\mathrm{d}(r)\,\&^\wedge\,\mathrm{d}(\theta))}{2\,r+M}+r\sin(\theta)\,aa\,($

$\quad -4\,r\cos(\theta)^2-5\,r^2\,M\,aa^2\cos(\theta)^2-4\,r^2\,aa^2\cos(\theta)^6\,M-6\,r^3\,aa^2\cos(\theta)^2+8\,r^2\,M\,aa^2\cos(\theta)^4+4\,r^3\,aa^2\cos(\theta)^4$

$\quad +2\,M\cos(\theta)^2+r^2\,M\,aa^2+2\,r^3\,aa^2)\,(\mathrm{d}(\theta)\,\&^\wedge\,\mathrm{d}(\phi))\,/\,(2\,r+M)-\cos(\theta)\,aa\,(5\,r^2\,M\,aa^2\cos(\theta)^4$

$\quad -4\,r^2\,M\,aa^2\cos(\theta)^2+2\,r^3\,aa^2-2\,r^2\,aa^2\cos(\theta)^6\,M+2\,r^3\,aa^2\cos(\theta)^4-4\,r^3\,aa^2\cos(\theta)^2-4\,r\cos(\theta)^2+4\,r$

$\quad +r^2\,M\,aa^2+4\,M\cos(\theta)^2-4\,M)\,(\mathrm{d}(r)\,\&^\wedge\,\mathrm{d}(\phi))\,/\,(2\,r+M)$

$OmegaCC12 := -r^2\cos(\theta)\,aa\,(4\,r^3\,aa^2\cos(\theta)^4-11\,r^2\,M\,aa^2\cos(\theta)^2+12\,r^2\,M\,aa^2\cos(\theta)^4-6\,r^3\,aa^2\cos(\theta)^2$

$\quad -2\,M\cos(\theta)^2+2\,M+3\,r^2\,M\,aa^2+2\,r^3\,aa^2-4\,r\cos(\theta)^2+4\,r-4\,r^2\,aa^2\cos(\theta)^6\,M)\,(\mathrm{d}(\phi)\,\&^\wedge\,\mathrm{d}(\theta))\,/\,(2\,r+M)$

$\quad -r\sin(\theta)\,aa\,(2\,r^3\,aa^2\cos(\theta)^4+5\,r^2\,M\,aa^2\cos(\theta)^4-2\,r^2\,aa^2\cos(\theta)^6\,M-3\,r^2\,M\,aa^2\cos(\theta)^2-4\,r\cos(\theta)^2+M$

$\quad -2\,r^3\,aa^2\cos(\theta)^2+2\,r)\,(\mathrm{d}(\phi)\,\&^\wedge\,\mathrm{d}(r))\,/\,(2\,r+M)$

$\quad +\dfrac{(2\,r^3\,aa^2+2\,M-4\,r^3\,aa^2\cos(\theta)^2+r^2\,M\,aa^2+4\,r^3\,aa^2\cos(\theta)^4)\,(\mathrm{d}(\theta)\,\&^\wedge\,\mathrm{d}(r))}{2\,r+M}$

$OmegaCC13 := 2\,r^3\cos(\theta)\,aa^2\sin(\theta)\,(4\,r^2\,M\,aa^2\cos(\theta)^2-5\,r^2\,M\,aa^2\cos(\theta)^4+2\,r^2\,aa^2\cos(\theta)^6\,M-r^2\,M\,aa^2$

$\quad -4\,r+4\,r\cos(\theta)^2+5\,M\cos(\theta)^2-2\cos(\theta)^4\,M-3\,M)\,(\mathrm{d}(\phi)\,\&^\wedge\,\mathrm{d}(\theta))\,/\,(2\,r+M)-((6\,r^2\,M\,aa^2\cos(\theta)^4$

$\quad -2\,r^4\cos(\theta)^2\,aa^4\,M-2\,M+4\,r^3\,aa^2\cos(\theta)^4-6\,r^3\,aa^2\cos(\theta)^2+2\,r^3\,aa^2+2\,M\cos(\theta)^2+r^2\,M\,aa^2$

$\quad +6\,r^4\cos(\theta)^4\,aa^4\,M+2\,r^4\cos(\theta)^8\,aa^4\,M-6\,r^4\cos(\theta)^6\,aa^4\,M-5\,r^2\,M\,aa^2\cos(\theta)^2-2\,r^2\,aa^2\cos(\theta)^6\,M)$

$\quad (\mathrm{d}(\phi)\,\&^\wedge\,\mathrm{d}(r)))\,/\,(2\,r+M)+aa\sin(\theta)\,r\,(2\,r^3\,aa^2\cos(\theta)^4+r^2\,M\,aa^2-4\,r^3\,aa^2\cos(\theta)^2-2\,r^2\,M\,aa^2\cos(\theta)^2$

$\quad +3\,r^2\,M\,aa^2\cos(\theta)^4-4\,r\cos(\theta)^2-2\,r^2\,aa^2\cos(\theta)^6\,M+M+2\,r+2\,r^3\,aa^2)\,(\mathrm{d}(\theta)\,\&^\wedge\,\mathrm{d}(r))\,/\,(2\,r+M)$

$$OmegaCC14 := 0$$

```
> OmegaCC21:=simpform(simplify(CGamma21&^TGamma11+CGamma22&^TGamma21+CGamma23&^TGa
mma31+CGamma24&^TGamma41));OmegaCC22:=simpform(simplify(CGamma21&^TGamma12+CGamm
a22&^TGamma22+CGamma23&^TGamma32+CGamma24&^TGamma42));OmegaCC23:=simpform(simpli
fy(CGamma21&^TGamma13+CGamma22&^TGamma23+CGamma23&^TGamma33+CGamma24&^TGamma43))
;OmegaCC24:=simpform(simplify(CGamma21&^TGamma14+CGamma22&^TGamma24+CGamma23&^TG
amma34+CGamma24&^TGamma44));
```

$OmegaCC21 := -\cos(\theta)\,aa\,(-5\,M-4\,r^2\,aa^2\cos(\theta)^6\,M+4\,r^3\,aa^2\cos(\theta)^4-8\,r^2\,M\,aa^2\cos(\theta)^2+4\,M\cos(\theta)^2$

$\quad +10\,r^2\,M\,aa^2\cos(\theta)^4+2\,r^2\,M\,aa^2+4\,r^3\,aa^2-8\,r^3\,aa^2\cos(\theta)^2-4\,r\cos(\theta)^2+2\,r)\,(\mathrm{d}(\phi)\,\&^\wedge\,\mathrm{d}(\theta))\,/\,(2\,r+M)$

$\quad +\dfrac{\sin(\theta)\,aa\,(-4\,M\cos(\theta)^2+3\,M-2\,r+4\,r\cos(\theta)^2)\,(\mathrm{d}(\phi)\,\&^\wedge\,\mathrm{d}(r))}{r\,(2\,r+M)}$

$\quad +8\,\dfrac{(-1+\cos(\theta)^2)\,r\,aa^2\cos(\theta)^2\,(\mathrm{d}(\theta)\,\&^\wedge\,\mathrm{d}(r))}{2\,r+M}$

$OmegaCC22 := 2\cos(\theta)\,aa^2\,r\sin(\theta)\,(-1+2\cos(\theta)^2)\,(\mathrm{d}(r)\,\&^\wedge\,\mathrm{d}(\theta))-r\sin(\theta)\,aa\,(-4\,r\cos(\theta)^2$

$\quad -6\,r^2\,M\,aa^2\cos(\theta)^2-4\,r^2\,aa^2\cos(\theta)^6\,M-4\,r^3\,aa^2\cos(\theta)^2+10\,r^2\,M\,aa^2\cos(\theta)^4+4\,r^3\,aa^2\cos(\theta)^4+2\,r-3\,M)$

$\quad (\mathrm{d}(\theta)\,\&^\wedge\,\mathrm{d}(\phi))\,/\,(2\,r+M)-\dfrac{\cos(\theta)\,aa\,(4\,r\cos(\theta)^2-2\,r+M)\,(\mathrm{d}(r)\,\&^\wedge\,\mathrm{d}(\phi))}{2\,r+M}$

$OmegaCC23 := -2\,r\cos(\theta)\,aa\,(4-4\cos(\theta)^2+2\,r^2\,aa^2-4\,r^2\cos(\theta)^2\,aa^2+2\,r^2\cos(\theta)^4\,aa^2+5\,r\,M\,aa^2\cos(\theta)^4$

$\quad -2\,r\,aa^2\cos(\theta)^6\,M+r\,M\,aa^2-4\,r\,M\,aa^2\cos(\theta)^2)\,(\mathrm{d}(r)\,\&^\wedge\,\mathrm{d}(\theta))\,/\,(2\,r+M)+((-4\,r^4\cos(\theta)^2\,aa^4\,M$

$\quad -14\,r^3\,aa^2\cos(\theta)^2+12\,r^4\cos(\theta)^4\,aa^4\,M+6\,r^3\,aa^2+8\,r^3\,aa^2\cos(\theta)^4+4\,r^4\cos(\theta)^8\,aa^4\,M+8\,r^2\,M\,aa^2\cos(\theta)^4$

$$- 12\,r^4 \cos(\theta)^6\,aa^4\,M + 2\,M \cos(\theta)^2 - r^2\,M\,aa^2 - 3\,r^2\,M\,aa^2 \cos(\theta)^2 - 4\,r^2\,aa^2 \cos(\theta)^6\,M - 2\,M)\,(\mathrm{d}(\phi)\ \&^\wedge\ \mathrm{d}(\theta)))$$
$$/\,(2\,r + M) + 2\,\frac{\cos(\theta) \sin(\theta)\,M\,(r^2 \cos(\theta)^2\,aa^2 - r^2\,aa^2 - 1)\,(\mathrm{d}(r)\ \&^\wedge\ \mathrm{d}(\phi))}{r\,(2\,r + M)}$$
$$OmegaCC24 := 0$$

```
> OmegaCC31:=simpform(simplify(CGamma31&^TGamma11+CGamma32&^TGamma21+CGamma33&^TGa
  mma31+CGamma34&^TGamma41));OmegaCC32:=simpform(simplify(CGamma31&^TGamma12+CGamm
  a32&^TGamma22+CGamma33&^TGamma32+CGamma34&^TGamma42));OmegaCC33:=simpform(simpli
  fy(CGamma31&^TGamma13+CGamma32&^TGamma23+CGamma33&^TGamma33+CGamma34&^TGamma43))
  ;OmegaCC34:=simpform(simplify(CGamma31&^TGamma14+CGamma32&^TGamma24+CGamma33&^TG
  amma34+CGamma34&^TGamma44));
```

$$OmegaCC31 := \frac{aa\,(2\,r + M - 2\,M \cos(\theta)^2)\,(\mathrm{d}(r)\ \&^\wedge\ \mathrm{d}(\theta))}{r \sin(\theta)\,(2\,r + M)}$$
$$- \frac{\cos(\theta)\,r\,aa^2\,(-2 \cos(\theta)^4\,M + 2\,r \cos(\theta)^2 + 3\,M \cos(\theta)^2 - M - 2\,r)\,(\mathrm{d}(\theta)\ \&^\wedge\ \mathrm{d}(\phi))}{\sin(\theta)\,(2\,r + M)}$$
$$- \frac{aa^2\,(-2 \cos(\theta)^4\,M + 2\,r \cos(\theta)^2 + 3\,M \cos(\theta)^2 - M - 2\,r)\,(\mathrm{d}(r)\ \&^\wedge\ \mathrm{d}(\phi))}{2\,r + M}$$

$$OmegaCC32 := - \frac{\cos(\theta)\,aa\,(2\,r - 2\,M \cos(\theta)^2 + 3\,M)\,(\mathrm{d}(r)\ \&^\wedge\ \mathrm{d}(\theta))}{2\,r \cos(\theta)^2 - 2\,r - M + M \cos(\theta)^2}$$
$$+ \frac{(-2\,r^2\,M\,aa^2 \cos(\theta)^4 + 3\,r^2\,M\,aa^2 \cos(\theta)^2 + 2\,r^3\,aa^2 \cos(\theta)^2 - 2\,M)\,(\mathrm{d}(\theta)\ \&^\wedge\ \mathrm{d}(\phi))}{2\,r + M} + \sin(\theta) \cos(\theta)$$
$$(2\,r^3\,aa^2 \cos(\theta)^2 - 2\,r^3\,aa^2 - 2\,r^2\,M\,aa^2 \cos(\theta)^4 + 5\,r^2\,M\,aa^2 \cos(\theta)^2 - 3\,r^2\,M\,aa^2 - 2\,M)\,(\mathrm{d}(r)\ \&^\wedge\ \mathrm{d}(\phi)) \Big/ ($$
$$r\,(2\,r \cos(\theta)^2 - 2\,r - M + M \cos(\theta)^2))$$

$$OmegaCC33 := -2\,\frac{\cos(\theta)\,M\,(r^2 \cos(\theta)^2\,aa^2 - r^2\,aa^2 - 1)\,(\mathrm{d}(r)\ \&^\wedge\ \mathrm{d}(\theta))}{\sin(\theta)\,r\,(2\,r + M)} + r\,aa$$
$$(2\,r^2\,aa^2 \cos(\theta)^6\,M - 4 \cos(\theta)^4\,M - 4\,r^2\,M\,aa^2 \cos(\theta)^4 - 2\,r \cos(\theta)^2 + 2\,r^2\,M\,aa^2 \cos(\theta)^2 + 5\,M \cos(\theta)^2 - M + 2\,r)$$
$$(\mathrm{d}(\theta)\ \&^\wedge\ \mathrm{d}(\phi)) / (\sin(\theta)\,(2\,r + M))$$
$$+ \frac{\cos(\theta)\,aa\,(2\,r^2\,M\,aa^2 \cos(\theta)^4 - 4\,M \cos(\theta)^2 - 4\,r^2\,M\,aa^2 \cos(\theta)^2 + 3\,M - 2\,r + 2\,r^2\,M\,aa^2)\,(\mathrm{d}(r)\ \&^\wedge\ \mathrm{d}(\phi))}{2\,r + M}$$

$$OmegaCC34 := 0$$

```
> OmegaCC41:=simpform(simplify(CGamma41&^TGamma11+CGamma42&^TGamma21+CGamma43&^TGa
  mma31+CGamma44&^TGamma41));OmegaCC42:=simpform(simplify(CGamma41&^TGamma12+CGamm
  a42&^TGamma22+CGamma43&^TGamma32+CGamma44&^TGamma42));OmegaCC43:=simpform(simpli
  fy(CGamma41&^TGamma13+CGamma42&^TGamma23+CGamma43&^TGamma33+CGamma44&^TGamma43))
  ;OmegaCC44:=simpform(simplify(CGamma41&^TGamma14+CGamma42&^TGamma24+CGamma43&^TG
  amma34+CGamma44&^TGamma44));
```

$$OmegaCC41 := 0$$
$$OmegaCC42 := 0$$
$$OmegaCC43 := 0$$
$$OmegaCC44 := 0$$

```
> CartanT:=array([[TGamma11,TGamma12,TGamma13,TGamma14],[TGamma21,TGamma22,TGamma2
  3,TGamma24],[TGamma31,TGamma32,TGamma33,TGamma34],[TGamma41,TGamma42,TGamma43,TG
  amma44]]);
```

$$CartanT :=$$

$$\left[\, 2\,\frac{(-2\,r^3\,aa^2\cos(\theta)^2 + 2\,r^3\,aa^2\cos(\theta)^4 - M)\,d(r)}{r\,(2\,r+M)} + (\cos(\theta)\,aa^2\,r^2\sin(\theta) - 2\cos(\theta)^3\,aa^2\,r^2\sin(\theta))\,d(\theta) - aa\right.$$

$$\cos(\theta)\,r\,(2\,M - 2\,M\cos(\theta)^2 - 2\,r^3\,aa^2 + 4\,r^3\,aa^2\cos(\theta)^2 - 2\,r^3\,aa^2\cos(\theta)^4 - 5\,r^2\,M\,aa^2\cos(\theta)^4$$

$$+\,2\,r^2\,aa^2\cos(\theta)^6\,M - r^2\,M\,aa^2 + 4\,r^2\,M\,aa^2\cos(\theta)^2)\,d(\phi)\,/\,(2\,r+M)\,,$$

$$(\cos(\theta)\,aa^2\,r^2\sin(\theta) - 2\cos(\theta)^3\,aa^2\,r^2\sin(\theta))\,d(r)$$

$$-\,2\,\frac{r\,(-M - 2\,r^3\,aa^2\cos(\theta)^4 + 2\,r^3\,aa^2\cos(\theta)^4 + 2\,r^2\,M\,aa^2\cos(\theta)^4 - 2\,r^2\,M\,aa^2\cos(\theta)^2)\,d(\theta)}{2\,r+M} + \sin(\theta)\,r^2\,aa\,($$

$$-2\,M\cos(\theta)^2 + 2\,r^3\,aa^2\cos(\theta)^2 - 2\,r^3\,aa^2\cos(\theta)^4 + 3\,r^2\,M\,aa^2\cos(\theta)^2 - 5\,r^2\,M\,aa^2\cos(\theta)^4 + 2\,r^2\,aa^2\cos(\theta)^6\,M$$

$$-\,2\,r+M)\,d(\phi)\,/\,(2\,r+M)\,,\ r^2\cos(\theta)\,aa\,(4 - 4\cos(\theta)^2 + 2\,r^2\,aa^2 - 4\,r^2\cos(\theta)^2\,aa^2 + 2\,r^2\cos(\theta)^4\,aa^2$$

$$+\,5\,r\,M\,aa^2\cos(\theta)^4 - 2\,r\,aa^2\cos(\theta)^6\,M + r\,M\,aa^2 - 4\,r\,M\,aa^2\cos(\theta)^2)\,d(r)\,/\,(2\,r+M) - \sin(\theta)\,r^2\,aa\,($$

$$-4\,r\cos(\theta)^2 - 2\,r^3\,aa^2\cos(\theta)^2 + 2\,r^3\,aa^2\cos(\theta)^4 - 3\,r^2\,M\,aa^2\cos(\theta)^2 + 5\,r^2\,M\,aa^2\cos(\theta)^4 - 2\,r^2\,aa^2\cos(\theta)^6\,M$$

$$+\,2\,r-M)\,d(\theta)\,/\,(2\,r+M) - 2\,r\,(-M + M\cos(\theta)^2 - 3\,r^4\cos(\theta)^6\,aa^4\,M + r^4\cos(\theta)^8\,aa^4\,M - r^4\cos(\theta)^2\,aa^4\,M$$

$$+\,3\,r^4\cos(\theta)^4\,aa^4\,M - r^2\,aa^2\cos(\theta)^6\,M + 2\,r^3\,aa^2 - r^2\,M\,aa^2\cos(\theta)^2 + 2\,r^3\,aa^2\cos(\theta)^4 - 4\,r^3\,aa^2\cos(\theta)^2$$

$$\left. +\,2\,r^2\,M\,aa^2\cos(\theta)^4)\,d(\phi)\,/\,(2\,r+M)\,,\ 0\,\right]$$

$$\left[\, -4\,\frac{(-1+\cos(\theta)^2)\,r\,aa^2\cos(\theta)\sin(\theta)\,d(r)}{2\,r+M}\right.$$

$$-\,\frac{(2\,M - 6\,r^3\,aa^2\cos(\theta)^2 + 2\,r^3\,aa^2 + 4\,r^3\,aa^2\cos(\theta)^4 - 3\,r^2\,M\,aa^2\cos(\theta)^2 + 2\,r^2\,M\,aa^2\cos(\theta)^4 + r^2\,M\,aa^2)\,d(\theta)}{r\,(2\,r+M)}$$

$$-\,\sin(\theta)\,aa\,(-2\,r - M - 4\,r^3\,aa^2\cos(\theta)^2 + 2\,r^3\,aa^2 + 2\,r^3\,aa^2\cos(\theta)^4 - 2\,r^2\,aa^2\cos(\theta)^6\,M + r^2\,M\,aa^2$$

$$+\,5\,r^2\,M\,aa^2\cos(\theta)^4 - 4\,r^2\,M\,aa^2\cos(\theta)^2)\,d(\phi)\,/\,(2\,r+M)\,,$$

$$-\,\frac{(2\,M - 6\,r^3\,aa^2\cos(\theta)^2 + 2\,r^3\,aa^2 + 4\,r^3\,aa^2\cos(\theta)^4 - 3\,r^2\,M\,aa^2\cos(\theta)^2 + 2\,r^2\,M\,aa^2\cos(\theta)^4 + r^2\,M\,aa^2)\,d(r)}{r\,(2\,r+M)}$$

$$+\,4\,\frac{\cos(\theta)\,aa^2\,r^2\sin(\theta)\,(-r + r\cos(\theta)^2 + M\cos(\theta)^2 - M)\,d(\theta)}{2\,r+M}$$

$$-\,\frac{r^3\cos(\theta)\,aa^3\,(2\,r - 4\,r\cos(\theta)^2 + 2\,r\cos(\theta)^4 + 3\,M - 8\,M\cos(\theta)^2 + 7\cos(\theta)^4\,M - 2\cos(\theta)^6\,M)\,d(\phi)}{2\,r+M}\,,\ -\sin(\theta)$$

$$aa\,(2\,r + M - 4\,r\cos(\theta)^2 - 2\,M\cos(\theta)^2 - 4\,r^3\,aa^2\cos(\theta)^2 + 2\,r^3\,aa^2 + 2\,r^3\,aa^2\cos(\theta)^4 - 2\,r^2\,aa^2\cos(\theta)^6\,M$$

$$+\,r^2\,M\,aa^2 + 5\,r^2\,M\,aa^2\cos(\theta)^4 - 4\,r^2\,M\,aa^2\cos(\theta)^2)\,d(r)\,/\,(2\,r+M) - r\cos(\theta)\,aa\,(4\,r + 2\,M - 4\,r\cos(\theta)^2$$

$$-\,2\,M\cos(\theta)^2 + 2\,r^3\,aa^2 - 4\,r^3\,aa^2\cos(\theta)^2 + 2\,r^3\,aa^2\cos(\theta)^4 + 3\,r^2\,M\,aa^2 - 8\,r^2\,M\,aa^2\cos(\theta)^2 + 7\,r^2\,M\,aa^2\cos(\theta)^4$$

$$-\,2\,r^2\,aa^2\cos(\theta)^6\,M)\,d(\theta)\,/\,(2\,r+M) + 2\sin(\theta)\cos(\theta)\,r^2\,aa^2\,(r^2\,aa^2\cos(\theta)^6\,M - r^2\,M\,aa^2 - 3\,r^2\,M\,aa^2\cos(\theta)^4$$

$$\left. +\,3\,r^2\,M\,aa^2\cos(\theta)^2 + 2\,r\cos(\theta)^2 - 2\,r - 2\,M - \cos(\theta)^4\,M + 3\,M\cos(\theta)^2)\,d(\phi)\,/\,(2\,r+M)\,,\ 0\,\right]$$

$$\left[\, -4\,\frac{\cos(\theta)\,aa\,d(r)}{2\,r+M} + \frac{\sin(\theta)\,aa\,(-1 + 2\cos(\theta)^2)\,d(\theta)}{-1+\cos(\theta)^2}\right.$$

$$-\,\frac{(2\,M + 2\,r^3\,aa^2\cos(\theta)^2 + 3\,r^2\,M\,aa^2\cos(\theta)^2 - 2\,r^2\,M\,aa^2\cos(\theta)^4 - 2\,r^3\,aa^2 - r^2\,M\,aa^2)\,d(\phi)}{r\,(2\,r+M)}\,,$$

$$\frac{\sin(\theta)\,aa\,(-1 + 2\cos(\theta)^2)\,d(r)}{-1+\cos(\theta)^2} + 4\,\frac{(r+M)\cos(\theta)\,r\,aa\,d(\theta)}{2\,r+M}$$

$$+\,\frac{(2\,r - 2\,M\cos(\theta)^2 + 3\,M)\,r^2\,aa^2\cos(\theta)\sin(\theta)\,d(\phi)}{2\,r+M}\,,$$

$$-\frac{(2\,M+2\,r^3\,aa^2\cos(\theta)^2+3\,r^2\,M\,aa^2\cos(\theta)^2-2\,r^2\,M\,aa^2\cos(\theta)^4-2\,r^3\,aa^2-r^2\,M\,aa^2)\,\mathrm{d}(r)}{r\,(2\,r+M)}$$

$$+\frac{(2\,r-2\,M\cos(\theta)^2+3\,M)\,r^2\,aa^2\cos(\theta)\sin(\theta)\,\mathrm{d}(\theta)}{2\,r+M}$$

$$+2\,\frac{\cos(\theta)\,r\,aa\,M\,(r^2\cos(\theta)^4\,aa^2-2\,r^2\cos(\theta)^2\,aa^2+r^2\,aa^2+1-\cos(\theta)^2)\,\mathrm{d}(\phi)}{2\,r+M}\,,\,0\Bigg]$$

$$[\,0\,,\,0\,,\,0\,,\,0\,]$$

```
> OmegaD:=simpform(d(CartanT));
> OMEGAT:=simpform(simplify(array([[OmegaD[1,1]+OmegaCC11,OmegaD[1,2]+OmegaCC12,Om
  egaD[1,3]+OmegaCC13,OmegaD[1,4]+OmegaCC14],[OmegaD[2,1]+OmegaCC21,OmegaD[2,2]+Om
  egaCC22,OmegaD[2,3]+OmegaCC23,OmegaD[2,4]+OmegaCC24],[OmegaD[3,1]+OmegaCC31,Omeg
  aD[3,2]+OmegaCC32,OmegaD[3,3]+OmegaCC33,OmegaD[3,4]+OmegaCC34],[OmegaD[4,1]+Omeg
  aCC41,OmegaD[4,2]+OmegaCC42,OmegaD[4,3]+OmegaCC43,OmegaD[4,4]+OmegaCC44]])));
```

$OmegaD :=$

$$\Bigg[-r\sin(\theta)\,aa\,(-2\,M+6\,M\cos(\theta)^2+2\,r^3\,aa^2-12\,r^3\,aa^2\cos(\theta)^2+10\,r^3\,aa^2\cos(\theta)^4+25\,r^2\,M\,aa^2\cos(\theta)^4$$

$$-14\,r^2\,aa^2\cos(\theta)^6\,M+r^2\,M\,aa^2-12\,r^2\,M\,aa^2\cos(\theta)^2)\,(\mathrm{d}(\theta)\,\&^\wedge\,\mathrm{d}(\phi))\,/\,(2\,r+M)+aa\cos(\theta)\,(2\,M^2\cos(\theta)^2$$

$$-6\,r^2\,M^2\,aa^2\cos(\theta)^6-8\cos(\theta)^6\,r^3\,aa^2\,M-2\,M^2+3\,r^2\,M^2\,aa^2+12\,r^3\,aa^2\,M+12\,r^4\,aa^2+15\,r^2\,M^2\,aa^2\cos(\theta)^4$$

$$+28\,r^3\,aa^2\cos(\theta)^4\,M-32\,r^3\,aa^2\cos(\theta)^2\,M-12\,r^2\,M^2\,aa^2\cos(\theta)^2-24\,r^4\,aa^2\cos(\theta)^2+12\,r^4\,aa^2\cos(\theta)^4)$$

$$(\mathrm{d}(r)\,\&^\wedge\,\mathrm{d}(\phi))\,\Big/\,(2\,r+M)^2-2\,\frac{\cos(\theta)\,r\sin(\theta)\,aa^2\,(-2\,r-2\,M\cos(\theta)^2+M+4\,r\cos(\theta)^2)\,(\mathrm{d}(\theta)\,\&^\wedge\,\mathrm{d}(r))}{2\,r+M}\,,\,r^2$$

$$aa\cos(\theta)\,(-2\,M\cos(\theta)^2+2\,r^3\,aa^2\cos(\theta)^2-2\,r^3\,aa^2\cos(\theta)^4+3\,r^2\,M\,aa^2\cos(\theta)^2-5\,r^2\,M\,aa^2\cos(\theta)^4$$

$$+2\,r^2\,aa^2\cos(\theta)^6\,M-2\,r+M+4\sin(\theta)^2\,M-4\sin(\theta)^2\,aa^2\,r^3+8\sin(\theta)^2\,aa^2\,r^3\cos(\theta)^2-6\sin(\theta)^2\,aa^2\,r^2\,M$$

$$+20\sin(\theta)^2\,aa^2\,r^2\cos(\theta)^2\,M-12\sin(\theta)^2\,aa^2\,r^2\cos(\theta)^4\,M)\,(\mathrm{d}(\theta)\,\&^\wedge\,\mathrm{d}(\phi))\,/\,(2\,r+M)-2\,r\sin(\theta)\,aa\,($$

$$2\,M^2\cos(\theta)^2+2\,r\cos(\theta)^2\,M-4\,r^2\,M^2\,aa^2\cos(\theta)^6-6\cos(\theta)^6\,r^3\,aa^2\,M-M^2+2\,r\,M+4\,r^2+10\,r^2\,M^2\,aa^2\cos(\theta)^4$$

$$+20\,r^3\,aa^2\cos(\theta)^4\,M-14\,r^3\,aa^2\cos(\theta)^2\,M-6\,r^2\,M^2\,aa^2\cos(\theta)^2-8\,r^4\,aa^2\cos(\theta)^2+8\,r^4\,aa^2\cos(\theta)^4)$$

$$(\mathrm{d}(r)\,\&^\wedge\,\mathrm{d}(\phi))\,\Big/\,(2\,r+M)^2+((-4\,r^4\sin(\theta)^2\,aa^2-4\,r^3\sin(\theta)^2\,aa^2\,M-r^2\sin(\theta)^2\,aa^2\,M^2-20\,r^4\,aa^2\cos(\theta)^2$$

$$-28\,r^3\,aa^2\cos(\theta)^2\,M-11\,r^2\,M^2\,aa^2\cos(\theta)^2+24\,r^4\sin(\theta)^2\,aa^2\cos(\theta)^2+24\,r^3\sin(\theta)^2\,aa^2\cos(\theta)^2\,M$$

$$+6\,r^2\sin(\theta)^2\,aa^2\cos(\theta)^2\,M^2+16\,r^4\,aa^2\cos(\theta)^4+24\,r^3\,aa^2\cos(\theta)^4\,M+10\,r^2\,M^2\,aa^2\cos(\theta)^4-2\,M^2)$$

$$(\mathrm{d}(\theta)\,\&^\wedge\,\mathrm{d}(r)))\,\Big/\,(2\,r+M)^2\,,\,4\,r\cos(\theta)\sin(\theta)\,(M-9\,r^4\cos(\theta)^4\,aa^4\,M+4\,r^4\cos(\theta)^6\,aa^4\,M-r^4\,aa^4\,M$$

$$+6\,r^4\cos(\theta)^2\,aa^4\,M-3\,r^2\,M\,aa^2\cos(\theta)^4-r^2\,M\,aa^2+4\,r^3\,aa^2\cos(\theta)^2-4\,r^3\,aa^2+4\,r^2\,M\,aa^2\cos(\theta)^2)$$

$$(\mathrm{d}(\theta)\,\&^\wedge\,\mathrm{d}(\phi))\,/\,(2\,r+M)-2\,((M^2\cos(\theta)^2-3\,r^2\,M^2\,aa^2\cos(\theta)^6-4\cos(\theta)^6\,r^3\,aa^2\,M-M^2+8\,r^3\,aa^2\,M$$

$$+24\,r^5\,aa^4\cos(\theta)^4\,M-5\,r^4\,M^2\,aa^4\cos(\theta)^2-8\,r^5\,aa^4\cos(\theta)^2\,M+5\,r^4\,M^2\,aa^4\cos(\theta)^8+8\,r^5\,aa^4\cos(\theta)^8\,M$$

$$-15\,r^4\,M^2\,aa^4\cos(\theta)^6+12\,r^4\,aa^2-24\,r^5\,aa^4\cos(\theta)^6\,M+15\,r^4\,M^2\,aa^4\cos(\theta)^4+6\,r^2\,M^2\,aa^2\cos(\theta)^4$$

$$+16\,r^3\,aa^2\cos(\theta)^4\,M-20\,r^3\,aa^2\cos(\theta)^2\,M-3\,r^2\,M^2\,aa^2\cos(\theta)^2-24\,r^4\,aa^2\cos(\theta)^2+12\,r^4\,aa^2\cos(\theta)^4)$$

$$(\mathrm{d}(r)\,\&^\wedge\,\mathrm{d}(\phi)))\,\Big/\,(2\,r+M)^2-r\sin(\theta)\,aa\,(-6\,r^2\,M^2\,aa^2\cos(\theta)^6-16\cos(\theta)^6\,r^3\,aa^2\,M+2\,M^2+r^2\,M^2\,aa^2$$

$$+4\,r^3\,aa^2\,M+4\,r^4\,aa^2-8\,r^2\cos(\theta)^2+5\,r^2\,M^2\,aa^2\cos(\theta)^4+20\,r^3\,aa^2\cos(\theta)^4\,M-8\,r^3\,aa^2\cos(\theta)^2\,M$$

$$-8\,r^4\,aa^2\cos(\theta)^2+4\,r^4\,aa^2\cos(\theta)^4)\,(\mathrm{d}(\theta)\,\&^\wedge\,\mathrm{d}(r))\,\Big/\,(2\,r+M)^2\,,\,0\Bigg]$$

$$\Bigg[\,aa\cos(\theta)\,(2\,r+M+4\,r^3\,aa^2\cos(\theta)^2-2\,r^3\,aa^2-2\,r^3\,aa^2\cos(\theta)^4+2\,r^2\,aa^2\cos(\theta)^6\,M-r^2\,M\,aa^2$$

$$-5\,r^2\,M\,aa^2\cos(\theta)^4+4\,r^2\,M\,aa^2\cos(\theta)^2-8\sin(\theta)^2\,aa^2\,r^3+8\sin(\theta)^2\,aa^2\,r^3\cos(\theta)^2-12\sin(\theta)^2\,aa^2\,r^2\cos(\theta)^4\,M$$

$+ 20 \sin(\theta)^2 \, aa^2 \, r^2 \cos(\theta)^2 \, M - 8 \sin(\theta)^2 \, aa^2 \, r^2 \, M) \, (d(\theta) \, \&^\wedge \, d(\phi)) \, / \, (2\,r+M) - 2 \, aa^3 \sin(\theta) \, r \, (-8 \, r^2 \cos(\theta)^2$

$- 10 \, r \cos(\theta)^2 \, M + 4 \, r^2 + 4 \, r \, M + 4 \cos(\theta)^4 \, r^2 + 8 \cos(\theta)^4 \, r \, M - 2 \, r \cos(\theta)^6 \, M - 2 \cos(\theta)^6 \, M^2 + M^2 + 5 \cos(\theta)^4 \, M^2$

$- 4 \, M^2 \cos(\theta)^2) \, (d(r) \, \&^\wedge \, d(\phi)) \, / \, (2\,r+M)^2 + ((24 \, r^4 \sin(\theta)^2 \, aa^2 \cos(\theta)^2 + 12 \, r^3 \sin(\theta)^2 \, aa^2 \cos(\theta)^2 \, M$

$- 8 \, r^4 \sin(\theta)^2 \, aa^2 - 4 \, r^3 \sin(\theta)^2 \, aa^2 \, M - 4 \, r^4 \, aa^2 \cos(\theta)^2 - 8 \, r^3 \, aa^2 \cos(\theta)^2 \, M + 4 \, r^3 \, aa^2 \cos(\theta)^4 \, M - 2 \, M^2$

$+ r^2 \, M^2 \, aa^2 + 4 \, r^3 \, aa^2 \, M + 4 \, r^4 \, aa^2 - 8 \, r \, M + 2 \, r^2 \, M^2 \, aa^2 \cos(\theta)^4 - 3 \, r^2 \, M^2 \, aa^2 \cos(\theta)^2) \, (d(\theta) \, \&^\wedge \, d(r))) \, \Big/ \, ($

$(2\,r+M)^2 \, r^2) , \, r^3 \, aa^3 \sin(\theta)$

$(2\,r - 12 \, r \cos(\theta)^2 + 10 \, r \cos(\theta)^4 + 3 \, M - 24 \, M \cos(\theta)^2 + 35 \cos(\theta)^4 \, M - 14 \cos(\theta)^6 \, M) \, (d(\theta) \, \&^\wedge \, d(\phi)) \, / \, ($

$2\,r+M) - aa^3 \cos(\theta) \, r^2 \, (12 \, r^2 + 20 \, r \, M - 24 \, r^2 \cos(\theta)^2 - 48 \, r \cos(\theta)^2 \, M + 12 \cos(\theta)^4 \, r^2 + 36 \cos(\theta)^4 \, r \, M + 9 \, M^2$

$- 24 \, M^2 \cos(\theta)^2 + 21 \cos(\theta)^4 \, M^2 - 8 \, r \cos(\theta)^6 \, M - 6 \cos(\theta)^6 \, M^2) \, (d(r) \, \&^\wedge \, d(\phi)) \, \Big/ \, (2\,r+M)^2$

$+ 2 \, \dfrac{\cos(\theta) \, r \sin(\theta) \, aa^2 \, (-4 \, r^2 - 2 \, r \, M + M^2 + 8 \, r^2 \cos(\theta)^2 + 6 \, r \cos(\theta)^2 \, M) \, (d(\theta) \, \&^\wedge \, d(r))}{(2\,r+M)^2} , \, -2 \, r^2 \, aa^2 \, ($

$-\cos(\theta)^8 \, r^2 \, aa^2 \, M + r^2 \, M \, aa^2 \cos(\theta)^2 + 3 \, r^2 \, aa^2 \cos(\theta)^6 \, M - 3 \, r^2 \, M \, aa^2 \cos(\theta)^4 - 2 \, r \cos(\theta)^4 + 2 \, r \cos(\theta)^2$

$+ 2 \, M \cos(\theta)^2 + \cos(\theta)^6 \, M - 3 \cos(\theta)^4 \, M + 7 \sin(\theta)^2 \, r^2 \, aa^2 \cos(\theta)^6 \, M - \sin(\theta)^2 \, aa^2 \, r^2 \, M$

$- 15 \sin(\theta)^2 \, aa^2 \, r^2 \cos(\theta)^4 \, M + 9 \sin(\theta)^2 \, aa^2 \, r^2 \cos(\theta)^2 \, M + 6 \cos(\theta)^2 \sin(\theta)^2 \, r - 2 \sin(\theta)^2 \, r - 2 \sin(\theta)^2 \, M$

$- 5 \sin(\theta)^2 \cos(\theta)^4 \, M + 9 \cos(\theta)^2 \sin(\theta)^2 \, M) \, (d(\theta) \, \&^\wedge \, d(\phi)) \, / \, (2\,r+M) + 4 \cos(\theta) \sin(\theta) \, r \, aa^2 \, (-3 \, r^3 \, aa^2 \, M$

$- 4 \, r^2 - 2 \, r^2 \, M^2 \, aa^2 + 6 \, r^2 \, M^2 \, aa^2 \cos(\theta)^2 + 2 \, r^2 \, M^2 \, aa^2 \cos(\theta)^6 - 6 \, r^2 \, M^2 \, aa^2 \cos(\theta)^4 - 5 \, r \, M - 2 \, M^2$

$+ 9 \, r^3 \, aa^2 \cos(\theta)^2 \, M + 3 \cos(\theta)^6 \, r^3 \, aa^2 \, M - 9 \, r^3 \, aa^2 \cos(\theta)^4 \, M - \cos(\theta)^4 \, M^2 + 3 \, M^2 \cos(\theta)^2 + 4 \, r^2 \cos(\theta)^2$

$+ 6 \, r \cos(\theta)^2 \, M - \cos(\theta)^4 \, r \, M) \, (d(r) \, \&^\wedge \, d(\phi)) \, \Big/ \, (2\,r+M)^2 + aa \cos(\theta) \, (M^2 - 16 \, r^4 \, aa^2 \cos(\theta)^2$

$- 20 \, r^2 \, M^2 \, aa^2 \cos(\theta)^2 + 16 \, r^2 \, M^2 \, aa^2 \cos(\theta)^4 + 8 \, r^2 \, M^2 \, aa^2 + 16 \, r^3 \, aa^2 \, M - 4 \, r^2 \, M^2 \, aa^2 \cos(\theta)^6$

$- 36 \, r^3 \, aa^2 \cos(\theta)^2 \, M + 8 \, r^4 \, aa^2 \cos(\theta)^4 + 24 \, r^3 \, aa^2 \cos(\theta)^4 \, M - 4 \cos(\theta)^6 \, r^3 \, aa^2 \, M - 12 \, r^2 \sin(\theta)^2 \, aa^2 \cos(\theta)^4 \, M^2$

$+ 20 \, r^2 \sin(\theta)^2 \, aa^2 \cos(\theta)^2 \, M^2 - 8 \, r^2 \sin(\theta)^2 \, aa^2 \, M^2 + 16 \, r^4 \sin(\theta)^2 \, aa^2 \cos(\theta)^2 - 16 \sin(\theta)^2 \, r \, M + 8 \, r^4 \, aa^2 + 4 \, r \, M$

$+ 4 \, r^2 - 4 \sin(\theta)^2 \, M^2 - 16 \sin(\theta)^2 \, r^2 - 16 \, r^4 \sin(\theta)^2 \, aa^2 + 48 \, r^3 \sin(\theta)^2 \, aa^2 \cos(\theta)^2 \, M - 24 \, r^3 \sin(\theta)^2 \, aa^2 \cos(\theta)^4 \, M$

$- 24 \, r^3 \sin(\theta)^2 \, aa^2 \, M) \, (d(\theta) \, \&^\wedge \, d(r)) \, \Big/ \, (2\,r+M)^2 , \, 0 \Big]$

$\Big[4 \, \dfrac{aa \sin(\theta) \, (d(\theta) \, \&^\wedge \, d(r))}{2\,r+M} + 2 \, \dfrac{r \, aa^2 \cos(\theta) \sin(\theta) \, (2\,r + 3 \, M - 4 \, M \cos(\theta)^2) \, (d(\theta) \, \&^\wedge \, d(\phi))}{2\,r+M} - ((($

$4 \, r^4 \, aa^2 \cos(\theta)^2 + 4 \, r^3 \, aa^2 \cos(\theta)^2 \, M + 3 \, r^2 \, M^2 \, aa^2 \cos(\theta)^2 - 2 \, r^2 \, M^2 \, aa^2 \cos(\theta)^4 - 4 \, r^4 \, aa^2 - 4 \, r^3 \, aa^2 \, M - r^2 \, M^2 \, aa^2$

$- 8 \, r \, M - 2 \, M^2) \, (d(r) \, \&^\wedge \, d(\phi))) \, \Big/ \, ((2\,r+M)^2 \, r^2) , \, - r^2 \, aa^2$

$(-6 \cos(\theta)^2 \sin(\theta)^2 \, M + 2 \sin(\theta)^2 \, r + 3 \sin(\theta)^2 \, M - 2 \, r \cos(\theta)^2 + 2 \cos(\theta)^4 \, M - 3 \, M \cos(\theta)^2) \, (d(\theta) \, \&^\wedge \, d(\phi)) \, / \, ($

$2\,r+M) + 2 \, \dfrac{r \, aa^2 \cos(\theta) \sin(\theta) \, (4 \, r^2 + 6 \, r \, M - 2 \, r \cos(\theta)^2 \, M - 2 \, M^2 \cos(\theta)^2 + 3 \, M^2) \, (d(r) \, \&^\wedge \, d(\phi))}{(2\,r+M)^2} + aa \cos(\theta)$

$(-4 \, r^2 - 4 \, r \, M - 3 \, M^2 + 4 \, r^2 \cos(\theta)^2 + 4 \, r \cos(\theta)^2 \, M + 5 \, M^2 \cos(\theta)^2 - 2 \cos(\theta)^4 \, M^2 + 8 \sin(\theta)^2 \, r^2 + 8 \sin(\theta)^2 \, r \, M$

$+ 2 \sin(\theta)^2 \, M^2) \, (d(\theta) \, \&^\wedge \, d(r)) \, \Big/ \, ((-1 + \cos(\theta)^2)^2 \, (2\,r+M)^2) ,$

$-2 \, \dfrac{r \, aa \, M \sin(\theta) \, (5 \, r^2 \cos(\theta)^4 \, aa^2 - 6 \, r^2 \cos(\theta)^2 \, aa^2 + r^2 \, aa^2 + 1 - 3 \cos(\theta)^2) \, (d(\theta) \, \&^\wedge \, d(\phi))}{2\,r+M} + 2 \, M \, aa \cos(\theta) \, ($

$4 \, r^3 \, aa^2 \cos(\theta)^4 + 3 \, r^2 \, M \, aa^2 \cos(\theta)^4 - 8 \, r^3 \, aa^2 \cos(\theta)^2 - 6 \, r^2 \, M \, aa^2 \cos(\theta)^2 + 4 \, r^3 \, aa^2 + 3 \, r^2 \, M \, aa^2 + M - M \cos(\theta)^2$

$) \, (d(r) \, \&^\wedge \, d(\phi)) \, \Big/ \, (2\,r+M)^2 - 4 \, \dfrac{\cos(\theta) \, r \sin(\theta) \, aa^2 \, M \, (-r + 3 \, r \cos(\theta)^2 + M \cos(\theta)^2) \, (d(\theta) \, \&^\wedge \, d(r))}{(2\,r+M)^2} , \, 0 \Big]$

$[0,0,0,0]$

$OMEGAT :=$

$$\left[\frac{r \sin(\theta)\, aa^2 \cos(\theta)\, (-2\,r - 2\,M \cos(\theta)^2 + M + 4\,r \cos(\theta)^2)\, (d(r)\, \&^\wedge\, d(\theta))}{2\,r + M} + ((-6\,r^3\, aa^2 \cos(\theta)^4 \right.$$

$$+ 6\,r^3\, aa^2 \cos(\theta)^2 + 7\,r^2\, M\, aa^2 \cos(\theta)^2 - 17\,r^2\, M\, aa^2 \cos(\theta)^4 + 10\,r^2\, aa^2 \cos(\theta)^6\, M - 4\,r \cos(\theta)^2 + 2\,M$$

$$- 4\,M \cos(\theta)^2)\, aa \sin(\theta)\, r\, (d(\theta)\, \&^\wedge\, d(\phi))) / (2\,r + M) - 2\,aa \cos(\theta)\, (4\,r^2 - 4\,r^2 \cos(\theta)^2 - M^2 + M^2 \cos(\theta)^2$$

$$- 8\,r^3\, aa^2 \cos(\theta)^4\, M - 4\,r^4\, aa^2 + 2 \cos(\theta)^6\, r^3\, aa^2\, M - 4\,r^4\, aa^2 \cos(\theta)^4 + 2\,r^2\, M^2\, aa^2 \cos(\theta)^6 + 8\,r^4\, aa^2 \cos(\theta)^2$$

$$- 5\,r^2\, M^2\, aa^2 \cos(\theta)^4 - 4\,r^3\, aa^2\, M + 10\,r^3\, aa^2 \cos(\theta)^2\, M + 4\,r^2\, M^2\, aa^2 \cos(\theta)^2 - r^2\, M^2\, aa^2 - 2\,r\,M + 2\,r \cos(\theta)^2\, M)$$

$$(d(r)\, \&^\wedge\, d(\phi)) \Big/ (4\,r^2 + 4\,r\,M + M^2),\ ((6\,r^3\, aa^2 \cos(\theta)^4 - 8\,r^3\, aa^2 \cos(\theta)^2 + 2\,r^3\, aa^2 + 3\,r^2\, M\, aa^2$$

$$+ 25\,r^2\, M\, aa^2 \cos(\theta)^4 - 18\,r^2\, M\, aa^2 \cos(\theta)^2 - 10\,r^2\, aa^2 \cos(\theta)^6\, M + 4\,r \cos(\theta)^2 - 2\,r + 8\,M \cos(\theta)^2 - 7\,M)\, aa$$

$$\cos(\theta)\, r^2\, (d(\phi)\, \&^\wedge\, d(\theta))) / (2\,r + M) + aa \sin(\theta)\, r\, (-20\,r^3\, aa^2 \cos(\theta)^2\, M - 6\,r^2\, M^2\, aa^2 \cos(\theta)^6$$

$$+ 15\,r^2\, M^2\, aa^2 \cos(\theta)^4 + 28\,r^3\, aa^2 \cos(\theta)^4\, M - 8 \cos(\theta)^6\, r^3\, aa^2\, M - 12\,r^4\, aa^2 \cos(\theta)^2 + 12\,r^4\, aa^2 \cos(\theta)^4$$

$$+ 8\,r \cos(\theta)^2\, M - 9\,r^2\, M^2\, aa^2 \cos(\theta)^2 + 8\,r^2 \cos(\theta)^2 + 4\,r^2 + 4\,M^2 \cos(\theta)^2 - 3\,M^2)\, (d(\phi)\, \&^\wedge\, d(r)) \Big/ ($$

$$4\,r^2 + 4\,r\,M + M^2)$$

$$+ 4\,\frac{r\,M\, (r\,M\, aa^2 \cos(\theta)^4 + 1 - r\,M\, aa^2 \cos(\theta)^2 + r^2 \cos(\theta)^4\, aa^2 - r^2 \cos(\theta)^2\, aa^2)\, (d(\theta)\, \&^\wedge\, d(r))}{4\,r^2 + 4\,r\,M + M^2},\ 2\,(($$

$$8\,r^4 \cos(\theta)^2\, aa^4\, M - 13\,r^4 \cos(\theta)^4\, aa^4\, M + 6\,r^4 \cos(\theta)^6\, aa^4\, M - r^4\, aa^4\, M - 4\,r^3\, aa^2 + 4\,r^3\, aa^2 \cos(\theta)^2 + r^2\, M\, aa^2$$

$$- 4\,r^2\, M\, aa^2 \cos(\theta)^4 + 3\,r^2\, M\, aa^2 \cos(\theta)^2 + 2\,M)\, r \cos(\theta) \sin(\theta)\, (d(\theta)\, \&^\wedge\, d(\phi))) / (2\,r + M) + r\,($$

$$-24\,r^3\, M^2\, aa^4 \cos(\theta)^6 + 20\,r^3\, aa^2 - aa^2\, r \cos(\theta)^2\, M^2 - 8\,r^3\, M^2\, aa^4 \cos(\theta)^2 - 4\,aa^2\, r \cos(\theta)^6\, M^2$$

$$+ 24\,r^3\, M^2\, aa^4 \cos(\theta)^4 + 6\,aa^2\, r \cos(\theta)^4\, M^2 - 36\,r^3\, aa^2 \cos(\theta)^2 - aa^2\, r\, M^2 + 12\,r^2\, M\, aa^2 + 16\,r^3\, aa^2 \cos(\theta)^4$$

$$+ 12\,r^4 \cos(\theta)^8\, aa^4\, M + 8\,r^3\, M^2\, aa^4 \cos(\theta)^8 - 12\,r^4 \cos(\theta)^2\, aa^4\, M + 36\,r^4 \cos(\theta)^4\, aa^4\, M - 36\,r^4 \cos(\theta)^6\, aa^4\, M$$

$$- 4\,r^2\, aa^2 \cos(\theta)^6\, M - 24\,r^2\, M\, aa^2 \cos(\theta)^2 + 4\,M + 16\,r^2\, M\, aa^2 \cos(\theta)^4 - 4\,M \cos(\theta)^2)\, (d(\phi)\, \&^\wedge\, d(r)) \Big/ ($$

$$4\,r^2 + 4\,r\,M + M^2) + aa \sin(\theta)\, r\, (-12\,r^3\, aa^2 \cos(\theta)^4\, M + 4\,r^2\, M^2\, aa^2 \cos(\theta)^6 - 2\,r^2\, M^2\, aa^2 \cos(\theta)^4$$

$$+ 12 \cos(\theta)^6\, r^3\, aa^2\, M - 4\,r \cos(\theta)^2\, M - 2\,r^2\, M^2\, aa^2 \cos(\theta)^2 + 4\,r^2 - M^2 + 4\,r\,M)\, (d(\theta)\, \&^\wedge\, d(r)) \Big/ ($$

$$\left. 4\,r^2 + 4\,r\,M + M^2),\ 0 \right]$$

$$\left[((6\,r^3\, aa^2 + 6\,r^3\, aa^2 \cos(\theta)^4 - 12\,r^3\, aa^2 \cos(\theta)^2 + 7\,r^2\, M\, aa^2 + 27\,r^2\, M\, aa^2 \cos(\theta)^4 - 10\,r^2\, aa^2 \cos(\theta)^6\, M \right.$$

$$- 24\,r^2\, M\, aa^2 \cos(\theta)^2 + 4\,r \cos(\theta)^2 - 4\,r - 4\,M \cos(\theta)^2 + 4\,M)\, aa \cos(\theta)\, (d(\phi)\, \&^\wedge\, d(\theta))) / (2\,r + M) - \sin(\theta)$$

$$aa\, (-4\,r^2\, M^2\, aa^2 \cos(\theta)^6 + 16\,r^3\, aa^2 \cos(\theta)^4\, M - 4 \cos(\theta)^6\, r^3\, aa^2\, M + 8\,r^4\, aa^2 \cos(\theta)^4 - 16\,r^4\, aa^2 \cos(\theta)^2 + 8\,r^4\, aa^2$$

$$+ 8\,r^3\, aa^2\, M + 2\,r^2\, M^2\, aa^2 - 20\,r^3\, aa^2 \cos(\theta)^2\, M + 10\,r^2\, M^2\, aa^2 \cos(\theta)^4 - 4\,r^2 - 8\,r^2\, M^2\, aa^2 \cos(\theta)^2 - 4\,r \cos(\theta)^2\, M$$

$$+ 4\,r\,M - 4\,M^2 \cos(\theta)^2 + 8\,r^2 \cos(\theta)^2 + 3\,M^2)\, (d(r)\, \&^\wedge\, d(\phi)) \Big/ (r\,(4\,r^2 + 4\,r\,M + M^2)) - (($$

$$8\,r\,M + 2\,M^2 + 4\,r^4\, aa^2 - 2\,r^2\, M^2\, aa^2 \cos(\theta)^4 - 12\,r^4\, aa^2 \cos(\theta)^2 + 8\,r^4\, aa^2 \cos(\theta)^4 + 3\,r^2\, M^2\, aa^2 \cos(\theta)^2 - r^2\, M^2\, aa^2$$

$$)\, (d(\theta)\, \&^\wedge\, d(r))) \Big/ ((4\,r^2 + 4\,r\,M + M^2)\, r^2),$$

$$4\,\frac{aa^2 \sin(\theta)\, r \cos(\theta)\, M\, (-r + r \cos(\theta)^2 + M \cos(\theta)^2 - M)\, (d(r)\, \&^\wedge\, d(\theta))}{4\,r^2 + 4\,r\,M + M^2} - ((8\,r^3\, aa^2 \cos(\theta)^2 - 2\,r^3\, aa^2$$

$$- 6\,r^3\, aa^2 \cos(\theta)^4 - 3\,r^2\, M\, aa^2 - 25\,r^2\, M\, aa^2 \cos(\theta)^4 + 10\,r^2\, aa^2 \cos(\theta)^6\, M + 18\,r^2\, M\, aa^2 \cos(\theta)^2 - 4\,r \cos(\theta)^2$$

$$+ 2\,r - 3\,M)\, aa \sin(\theta)\, r\, (d(\theta)\, \&^\wedge\, d(\phi))) / (2\,r + M) + aa \cos(\theta)\, (4\,r^2 - 8\,r^2 \cos(\theta)^2 - M^2 - 36\,r^3\, aa^2 \cos(\theta)^4\, M$$

$$- 12\,r^4\, aa^2 + 8 \cos(\theta)^6\, r^3\, aa^2\, M - 12\,r^4\, aa^2 \cos(\theta)^4 + 6\,r^2\, M^2\, aa^2 \cos(\theta)^6 + 24\,r^4\, aa^2 \cos(\theta)^2 - 21\,r^2\, M^2\, aa^2 \cos(\theta)^4$$

$$- 20\,r^3\, aa^2\, M + 48\,r^3\, aa^2 \cos(\theta)^2\, M + 24\,r^2\, M^2\, aa^2 \cos(\theta)^2 - 9\,r^2\, M^2\, aa^2 - 4\,r \cos(\theta)^2\, M)\, (d(r)\, \&^\wedge\, d(\phi)) \Big/ ($$

$$4\,r^2 + 4\,r\,M + M^2),\ ((2\,r^4\, aa^4\, M - 18\,r^4 \cos(\theta)^2\, aa^4\, M + 42\,r^4 \cos(\theta)^4\, aa^4\, M + 12\,r^4 \cos(\theta)^8\, aa^4\, M$$

$$- 38\,r^4 \cos(\theta)^6\, aa^4\, M + 8\,r^3\, aa^2 \cos(\theta)^4 - 2\,r^3\, aa^2 - 6\,r^3\, aa^2 \cos(\theta)^2 + 5\,r^2\, M\, aa^2 + 26\,r^2\, M\, aa^2 \cos(\theta)^4$$

$$- 8\,r^2\, aa^2 \cos(\theta)^6\, M - 23\,r^2\, M\, aa^2 \cos(\theta)^2 - 2\,M \cos(\theta)^2 + 2\,M)\, (d(\theta)\, \&^\wedge\, d(\phi))) / (2\,r + M) + 2 \sin(\theta) \cos(\theta)\,($$

$$-12\,r^4\,M^2\,aa^4\cos(\theta)^4 - 5\,r^2\,M^2\,aa^2 + 7\,r^2\,M^2\,aa^2\cos(\theta)^2 + 12\,r^4\,M^2\,aa^4\cos(\theta)^2 + 14\,r^3\,aa^2\cos(\theta)^2\,M$$

$$+\,4\,r^4\,M^2\,aa^4\cos(\theta)^6 - 12\,r^3\,aa^2\,M - M^2 - 4\,r^4\,M^2\,aa^4 + 18\,r^5\,aa^4\cos(\theta)^2\,M + 6\,r^5\,aa^4\cos(\theta)^6\,M - 8\,r^4\,aa^2$$

$$+\,8\,r^4\,aa^2\cos(\theta)^2 - 18\,r^5\,aa^4\cos(\theta)^4\,M - 2\,r^3\,aa^2\cos(\theta)^4\,M - 6\,r^5\,M\,aa^4 - 2\,r^2\,M^2\,aa^2\cos(\theta)^4 - 2\,r\,M)$$

$$(d(r)\,\&^\wedge\,d(\phi)) \Big/ (r\,(4\,r^2 + 4\,r\,M + M^2)) + \cos(\theta)\,aa\,(-4\,r\,M + 4\,r^2\,M^2\,aa^2\cos(\theta)^6 + 12\cos(\theta)^6\,r^3\,aa^2\,M$$

$$-\,24\,r^3\,aa^2\cos(\theta)^4\,M + 4\,M^2\cos(\theta)^2 + 12\,r^3\,aa^2\cos(\theta)^2\,M - 3\,M^2 + 4\,r^2 - 6\,r^2\,M^2\,aa^2\cos(\theta)^4 + 2\,r^2\,M^2\,aa^2$$

$$+\,8\,r\cos(\theta)^2\,M)\,(d(\theta)\,\&^\wedge\,d(r)) \Big/ (4\,r^2 + 4\,r\,M + M^2)\,,\,0\Big]$$

$$\Big[\frac{(-2\,r - 2\,M\cos(\theta)^2 + M + 4\,r\cos(\theta)^2)\,aa\,(d(r)\,\&^\wedge\,d(\theta))}{(2\,r + M)\sin(\theta)\,r}$$

$$+\,\frac{(-6\,r\cos(\theta)^2 + 6\,r + 7\,M + 10\cos(\theta)^4\,M - 17\,M\cos(\theta)^2)\cos(\theta)\,r\,aa^2\,(d(\theta)\,\&^\wedge\,d(\phi))}{\sin(\theta)\,(2\,r + M)} + 2\,((4\,r^4\,aa^2$$

$$-\,6\,r^3\,aa^2\cos(\theta)^2\,M - 3\,r^2\,M^2\,aa^2\cos(\theta)^2 + 2\,r^2\,M^2\,aa^2\cos(\theta)^4 + 4\,r\,M - 4\,r^4\,aa^2\cos(\theta)^2 + 2\,r^3\,aa^2\cos(\theta)^4\,M$$

$$+\,4\,r^3\,aa^2\,M + r^2\,M^2\,aa^2 + M^2)\,(d(r)\,\&^\wedge\,d(\phi))) \Big/ ((4\,r^2 + 4\,r\,M + M^2)\,r^2)\,,$$

$$-\,\frac{(10\,r^2\,M\,aa^2\cos(\theta)^4 - 6\,r^3\,aa^2\cos(\theta)^2 - 15\,r^2\,M\,aa^2\cos(\theta)^2 + 2\,r^3\,aa^2 + 3\,r^2\,M\,aa^2 + 2\,M)\,(d(\theta)\,\&^\wedge\,d(\phi))}{2\,r + M}\,-$$

$$\cos(\theta)\sin(\theta)\,(6\,r^2\,M^2\,aa^2\cos(\theta)^4 + 2\,M^2 + 8\,r^3\,aa^2\cos(\theta)^4\,M + 4\,r\,M - 12\,r^4\,aa^2\cos(\theta)^2 - 28\,r^3\,aa^2\cos(\theta)^2\,M$$

$$+\,20\,r^3\,aa^2\,M + 12\,r^4\,aa^2 + 9\,r^2\,M^2\,aa^2 - 15\,r^2\,M^2\,aa^2\cos(\theta)^2)\,(d(r)\,\&^\wedge\,d(\phi)) \Big/ ($$

$$(4\,r^2\cos(\theta)^2 + 4\,r\cos(\theta)^2\,M + M^2\cos(\theta)^2 - 4\,r^2 - 4\,r\,M - M^2)\,r) - 4\,\frac{(r + M)\,aa\cos(\theta)\,M\,(d(\theta)\,\&^\wedge\,d(r))}{4\,r^2 + 4\,r\,M + M^2}\,,\,-2$$

$$M\cos(\theta)\,(-2\,r - M + 6\,r^3\,aa^2\cos(\theta)^4 - r^2\,M\,aa^2\cos(\theta)^2 + 2\,r^2\,M\,aa^2\cos(\theta)^4 - r^2\,M\,aa^2 - 6\,r^3\,aa^2\cos(\theta)^2)$$

$$(d(r)\,\&^\wedge\,d(\theta)) \Big/ (r\sin(\theta)\,(4\,r^2 + 4\,r\,M + M^2)) + ((-26\,r^2\,M\,aa^2\cos(\theta)^4 - 2\,r^2\,M\,aa^2 + 16\,r^2\,M\,aa^2\cos(\theta)^2$$

$$+\,12\,r^2\,aa^2\cos(\theta)^6\,M + 2\,r - 2\,r\cos(\theta)^2 - 10\cos(\theta)^4\,M + 13\,M\cos(\theta)^2 - 3\,M)\,r\,aa\,(d(\theta)\,\&^\wedge\,d(\phi))) / ($$

$$(2\,r + M)\sin(\theta)) + aa\cos(\theta)\,(-4\,r^2 + 5\,M^2 - 6\,M^2\cos(\theta)^2 + 12\,r^3\,aa^2\cos(\theta)^4\,M + 8\,r^2\,M^2\,aa^2\cos(\theta)^4$$

$$+\,12\,r^3\,aa^2\,M - 24\,r^3\,aa^2\cos(\theta)^2\,M - 16\,r^2\,M^2\,aa^2\cos(\theta)^2 + 8\,r^2\,M^2\,aa^2 + 4\,r\,M - 8\,r\cos(\theta)^2\,M)\,(d(r)\,\&^\wedge\,d(\phi))$$

$$\Big/ (4\,r^2 + 4\,r\,M + M^2)\,,\,0\Big]$$

$$[0\,,\,0\,,\,0\,,\,0]$$

NOw prove that the metric curvature = the inertial curvature.

```
> evalm(OMEGAM-OMEGAT);
```

$$[0\,,\,-\,((-6\,r^3\,aa^2\cos(\theta)^4 + 8\,r^3\,aa^2\cos(\theta)^2 - 2\,r^3\,aa^2 + 10\,r^2\,aa^2\cos(\theta)^6\,M - 3\,r^2\,M\,aa^2 + 18\,r^2\,M\,aa^2\cos(\theta)^2$$

$$-\,25\,r^2\,M\,aa^2\cos(\theta)^4 + 2\,r - 4\,r\cos(\theta)^2 - 8\,M\cos(\theta)^2 + 7\,M)\,aa\cos(\theta)\,r^2\,(d(\phi)\,\&^\wedge\,d(\theta))) / (2\,r + M) - aa$$

$$\sin(\theta)\,r\,(20\,r^3\,aa^2\cos(\theta)^2\,M + 6\,r^2\,M^2\,aa^2\cos(\theta)^6 - 15\,r^2\,M^2\,aa^2\cos(\theta)^4 - 28\,r^3\,aa^2\cos(\theta)^4\,M$$

$$+\,8\cos(\theta)^6\,r^3\,aa^2\,M - 4\,M^2\cos(\theta)^2 + 12\,r^4\,aa^2\cos(\theta)^2 - 12\,r^4\,aa^2\cos(\theta)^4 - 8\,r\cos(\theta)^2\,M + 9\,r^2\,M^2\,aa^2\cos(\theta)^2$$

$$-\,8\,r^2\cos(\theta)^2 - 4\,r^2 + 3\,M^2)\,(d(\phi)\,\&^\wedge\,d(r)) \Big/ (4\,r^2 + 4\,r\,M + M^2) - ((6\,r^3\,aa^2\cos(\theta)^4 - 8\,r^3\,aa^2\cos(\theta)^2$$

$$+\,2\,r^3\,aa^2 + 3\,r^2\,M\,aa^2 + 25\,r^2\,M\,aa^2\cos(\theta)^4 - 18\,r^2\,M\,aa^2\cos(\theta)^2 - 10\,r^2\,aa^2\cos(\theta)^6\,M + 4\,r\cos(\theta)^2 - 2\,r$$

$$+\,8\,M\cos(\theta)^2 - 7\,M)\,aa\cos(\theta)\,r^2\,(d(\phi)\,\&^\wedge\,d(\theta))) / (2\,r + M) - aa\sin(\theta)\,r\,(-20\,r^3\,aa^2\cos(\theta)^2\,M$$

$$-\,6\,r^2\,M^2\,aa^2\cos(\theta)^6 + 15\,r^2\,M^2\,aa^2\cos(\theta)^4 + 28\,r^3\,aa^2\cos(\theta)^4\,M - 8\cos(\theta)^6\,r^3\,aa^2\,M - 12\,r^4\,aa^2\cos(\theta)^2$$

$$+\,12\,r^4\,aa^2\cos(\theta)^4 + 8\,r\cos(\theta)^2\,M - 9\,r^2\,M^2\,aa^2\cos(\theta)^2 + 8\,r^2\cos(\theta)^2 + 4\,r^2 + 4\,M^2\cos(\theta)^2 - 3\,M^2)$$

$$(d(\phi)\,\&^\wedge\,d(r)) \Big/ (4\,r^2 + 4\,r\,M + M^2)\,,\,0\,,\,0]$$

$$[-\,((-6\,r^3\,aa^2 - 6\,r^3\,aa^2\cos(\theta)^4 + 12\,r^3\,aa^2\cos(\theta)^2 - 7\,r^2\,M\,aa^2 - 27\,r^2\,M\,aa^2\cos(\theta)^4 + 10\,r^2\,aa^2\cos(\theta)^6\,M$$

$$+\,24\,r^2\,M\,aa^2\cos(\theta)^2 - 4\,r\cos(\theta)^2 + 4\,r + 4\,M\cos(\theta)^2 - 4\,M)\cos(\theta)\,aa\,(d(\phi)\,\&^\wedge\,d(\theta))) / (2\,r + M) + \sin(\theta)$$

$$aa \, (4 \, r^2 \, M^2 \, aa^2 \cos(\theta)^6 - 16 \, r^3 \, aa^2 \cos(\theta)^4 \, M + 4 \cos(\theta)^6 \, r^3 \, aa^2 \, M - 8 \, r^4 \, aa^2 \cos(\theta)^4 + 16 \, r^4 \, aa^2 \cos(\theta)^2 - 8 \, r^4 \, aa^2$$

$$- \, 8 \, r^3 \, aa^2 \, M - 2 \, r^2 \, M^2 \, aa^2 + 20 \, r^3 \, aa^2 \cos(\theta)^2 \, M - 10 \, r^2 \, M^2 \, aa^2 \cos(\theta)^4 + 4 \, r^2 + 8 \, r^2 \, M^2 \, aa^2 \cos(\theta)^2 + 4 \, r \cos(\theta)^2 \, M$$

$$- \, 4 \, r \, M + 4 \, M^2 \cos(\theta)^2 - 8 \, r^2 \cos(\theta)^2 - 3 \, M^2) \, (\mathrm{d}(r) \, \&^\wedge \, \mathrm{d}(\phi)) \Big/ (r \, (4 \, r^2 + 4 \, r \, M + M^2)) - ((6 \, r^3 \, aa^2$$

$$+ \, 6 \, r^3 \, aa^2 \cos(\theta)^4 - 12 \, r^3 \, aa^2 \cos(\theta)^2 + 7 \, r^2 \, M \, aa^2 + 27 \, r^2 \, M \, aa^2 \cos(\theta)^4 - 10 \, r^2 \, aa^2 \cos(\theta)^6 \, M$$

$$- \, 24 \, r^2 \, M \, aa^2 \cos(\theta)^2 + 4 \, r \cos(\theta)^2 - 4 \, r - 4 \, M \cos(\theta)^2 + 4 \, M) \, aa \cos(\theta) \, (\mathrm{d}(\phi) \, \&^\wedge \, \mathrm{d}(\theta))) \, / \, (2 \, r + M) + \sin(\theta)$$

$$aa \, (-4 \, r^2 \, M^2 \, aa^2 \cos(\theta)^6 + 16 \, r^3 \, aa^2 \cos(\theta)^4 \, M - 4 \cos(\theta)^6 \, r^3 \, aa^2 \, M + 8 \, r^4 \, aa^2 \cos(\theta)^4 - 16 \, r^4 \, aa^2 \cos(\theta)^2 + 8 \, r^4 \, aa^2$$

$$+ \, 8 \, r^3 \, aa^2 \, M + 2 \, r^2 \, M^2 \, aa^2 - 20 \, r^3 \, aa^2 \cos(\theta)^2 \, M + 10 \, r^2 \, M^2 \, aa^2 \cos(\theta)^4 - 4 \, r^2 - 8 \, r^2 \, M^2 \, aa^2 \cos(\theta)^2 - 4 \, r \cos(\theta)^2 \, M$$

$$+ \, 4 \, r \, M - 4 \, M^2 \cos(\theta)^2 + 8 \, r^2 \cos(\theta)^2 + 3 \, M^2) \, (\mathrm{d}(r) \, \&^\wedge \, \mathrm{d}(\phi)) \Big/ (r \, (4 \, r^2 + 4 \, r \, M + M^2)) \, , 0 \, , 0 \, , 0]$$

$$[0 \, , 0 \, , 0 \, , 0]$$

$$[0 \, , 0 \, , 0 \, , 0]$$

```
compute the trace of the inertial curvature.
> TRACEOMEGA_noTorsion_smallmass:=subs(aa=0,M^2=0,2*r+M=2*r,simpform(wcollect(OMEG
  AT[1,1]+OMEGAT[2,2]+OMEGAT[3,3]+OMEGAT[4,4])));JJ:=subs(d(TRACEOMEGA_noTorsion_s
  mallmass));
```

$$TRACEOMEGA\_noTorsion\_smallmass := \frac{M \cos(\theta) \, (\mathrm{d}(r) \, \&^\wedge \, \mathrm{d}(\theta))}{r^2 \sin(\theta)}$$

$$JJ := 0$$

```
> TRACEOMEGA_smallTorsion_nomass:=subs(aa^2=0,M=0,simpform(wcollect(OMEGAT[1,1]+OM
  EGAT[2,2]+OMEGAT[3,3]+OMEGAT[4,4])));JJ:=simplify(d(TRACEOMEGA_smallTorsion_noma
  ss));
```

$$TRACEOMEGA\_smallTorsion\_nomass :=$$

$$\frac{1}{2} \frac{aa \, (2 \, r - 2 \, r \cos(\theta)^2 - 2 \sin(\theta)^2 \, r) \, (\mathrm{d}(\theta) \, \&^\wedge \, \mathrm{d}(\phi))}{\sin(\theta)} - 2 \, aa \cos(\theta) \, (\mathrm{d}(r) \, \&^\wedge \, \mathrm{d}(\phi))$$

$$JJ := -2 \, aa \, \&^\wedge(\mathrm{d}(r), \mathrm{d}(\theta), \mathrm{d}(\phi)) \sin(\theta)$$

```
> TRACEOMEGA:=((subs(aa^2=0,M^2=0,2*r+M=2*r,simpform(wcollect(OMEGAT[1,1]+OMEGAT[2
  ,2]+OMEGAT[3,3]+OMEGAT[4,4])))));JJ:=simplify(subs(M^2=0,d(TRACEOMEGA)));
```

$$TRACEOMEGA := \frac{M \cos(\theta) \, (\mathrm{d}(r) \, \&^\wedge \, \mathrm{d}(\theta))}{r^2 \sin(\theta)} + \frac{1}{2} \, aa$$

$$(-10 \cos(\theta)^4 \, M + 13 \, M \cos(\theta)^2 + 2 \, r - 3 \, M - 2 \, r \cos(\theta)^2 - 2 \sin(\theta)^2 \, r + 5 \sin(\theta)^2 \, M - 4 \cos(\theta)^2 \sin(\theta)^2 \, M)$$

$$(\mathrm{d}(\theta) \, \&^\wedge \, \mathrm{d}(\phi)) \, / \sin(\theta) + \frac{1}{2} \frac{(-4 \, r - 8 \, M \cos(\theta)^2 + 6 \, M) \, aa \cos(\theta) \, (\mathrm{d}(r) \, \&^\wedge \, \mathrm{d}(\phi))}{r}$$

$$JJ := -\frac{\sin(\theta) \, (12 \, M \cos(\theta)^2 + 2 \, r - 3 \, M) \, aa \, \&^\wedge(\mathrm{d}(r), \mathrm{d}(\theta), \mathrm{d}(\phi))}{r}$$

```
> 
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