```
> restart: with (linalg):with(liesymm):with(difforms):
> setup(x,y,z,t):defform(x=0,y=0,z=0,t=0,Vx=0,Vy=0,Vz=0,D1=0,D2=0,D3=0,Ax=0,Ay=0,A
  z=0,C=0,Phi=0,phi=0,theta=0,r=0,tau=0,a=const,b=const,c=const,aa=const,bb=const,
  M=const,cc=const,ee=const,Lx=0,Ly=0,Lz=0,vx=const,vy=const,vz=const);
Warning, new definition for norm
Warning, new definition for trace
Warning, new definition for close
Warning, new definition for `&^`
Warning, new definition for d
Warning, new definition for mixpar
Warning, new definition for wdegree
```

Metric and Torsion Perturbations in 4D and the Principle of Equivalence

A Shwarzschild metric solution with Frame based Torsion component

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INTRODUCTION

A coordinate transformation from an initial domain of variables R = {r,theta,phi,tau} will be given to a final range of {x,y,z,t}. An unperturbed metric will be prescribed on the final state as [eta] = {-1,-1,-1,+1}. The Jacobian matrix of the map will be used as the unperturbed (invertible) Frame matrix of functions with arguments on the initial state.   The Frame matrix will be used to compute the right Cartan matrix of 1-forms [C] on the initial state. The pullback metric [g], induced on the initial state by the compatibility condition

[g] = [Ftranspose][eta][F],

will be used to compute the Christoffel connection [Gamma] in terms of the metric on the initial state.

The two connections will be compared and will be identical: [C] = [Gamma]. The curvatures induced by the two connections are also identical and equal to zero.

Then two types of perturbations will be applied. The first perturbation will be to the metric on the final state. Again the Cartan Frame based connection [C] and the Christoffel metrical based connection [Gamma] will be computed on the initial state. The connections are no longer the same. The Cartan connection can be decomposed into two parts, a metrical part [Gamma] and a Ricci rotation part [T]:

[C] = [Gamma] - [T].

The Cartan matrix of curvature 2-forms will be computed by the formula

[Theta] = [dC]+[C]^[C].

As the Frame manifold has an inverse [G], the Cartan matrix of curvature 2-forms vanishes: [Theta] = 0. Such spaces are called A4 spaces, and are "flat". However, the total curvature formula can be written as

[Theta] = [dC]+[C]^[C] = {[dGamma]+[C]^[Gamma]} - {[dT]+[C]^[T]}.

The first term enclosed by curly brackets will be defined as metric curvature relative to the Cartan connection [C], and the second curly bracket term will be defined as inertial curvature relative to the Cartan connection [C].

As the total curvature [Theta] is zero, it is apparent that the metric curvature is balanced by the inertial curvature. It is conjectured that this result is in effect the proper definition of what intuitively is known as the principle of equivalence.

The second type of perturbation will be on the Frame matrix, and will introduce torsion to the Cartan connection. There are two species of Torsion (besides left and right handedness). One species of Torsion will be due to "vorticity" around a space-space axis, and the other will be due to a "vorticity" around a space time axis. The Cartan connection again will be compared to the Christoffel connection and the Ricci rotation terms [T] are not zero. The conclusion holds even though the perturbation of the final state metric is zero. Again, the metrical curvature is balanced by the inertial curvature to produce a total zero Cartan curvature.

When both types of perturbations are present simultaneously interactions between the two pertubations can occur, but the fundamental result of metric - inertial equivalence still holds.

{[dGamma]+[C]^[Gamma]} = {[dT]+[C]^[T]}.

{[metric curvature]} = {[inertial curvature]}.

The examples will be so constructed that the metric on the initial state will be the Scharzschild metric in isotropic coordinates, and the Frame matrix will have torsion, also influencing the pull back metric on the initial state.

The examples will start with the classic map of spherical coordinates mapped into the Cartesian coordiantes {x,y,z,t}. The Jacobian matrix of partial derivatives of the mapping function will serve as the primitive definition of a Basis Frame, [F]. The Right Cartan connection matrix for any basis frame is defined as [C] = - [dG][F] = + [G][dF] , where [G] is the inverse of [F]. Realize that the Right Cartan matrix is defined entirely on the initial state in terms of initial state independent variables. The curvature and the Affine Torsion of the initial state based upon an integrable set of mapping functions are both zero.

Initially it will be assumed that the metric on the final state is a set of constants equal to the identity matrix. Then a perturbation of this finalstate metric will be defined and its effect on curvatures and torsion will be examined. Then a pertubation will be made on the original Frame Field, such that the perturbed Frame Field will admit Affine Torsion. The effect of the combined perturbations on curvature and Torsion will be comput

The classic map from spherical to Cartesian Coordinates is given by the expressions:

```
> x:=r*sin(theta)*cos(phi);y:=r*sin(theta)*sin(phi);z:=r*cos(theta);t:=tau;
>
```

$$x := r\sin(\theta)\cos(\phi)$$
$$y := r\sin(\theta)\sin(\phi)$$
$$z := r\cos(\theta)$$
$$t := \tau$$

The induced 1-forms are exact differentials and have expressions in terms of the variables of the initial state:

```
> sigmax_or_dx:=d(x);
```

$$sigmax\_or\_dx := \sin(\theta)\cos(\phi)\,d(r) + r\cos(\phi)\cos(\theta)\,d(\theta) - r\sin(\theta)\sin(\phi)\,d(\phi)$$

```
> sigmay_or_dy:=d(y);
```

$$sigmay\_or\_dy := \sin(\theta)\sin(\phi)\,d(r) + r\sin(\phi)\cos(\theta)\,d(\theta) + r\sin(\theta)\cos(\phi)\,d(\phi)$$

```
> sigmaz_or_dz:=d(z);
```

$$sigmaz\_or\_dz := \cos(\theta)\,d(r) - r\sin(\theta)\,d(\theta)$$

```
> sigmat_or_dtau:=d(t);
```

$$sigmat\_or\_dtau := d(\tau)$$

The three 1-forms induced by the coordinate mapping can be deduced by multiplying a Frame matrix times a vector of differentials on the initial state. The Frame matrix elements are now computed for the assumed coordinate mapping. The frame matrix is equivalent to the JAcobian matrix of the mapping functions relative to the variables of the initial state. The components are as follows:

```
>
> FF11:=getcoeff(d(x)&^d(theta)&^d(phi)&^d(tau));FF12:=getcoeff(d(x)&^d(phi)&^d(ta
```

```
u)&^d(r));FF13:=getcoeff(d(x)&^d(tau)&^d(r)&^d(theta));FF14:=getcoeff(d(x)&^d(r)
&^d(theta)&^d(phi));
```

$$FF11 := \sin(\theta)\cos(\phi)$$
$$FF12 := r\cos(\phi)\cos(\theta)$$
$$FF13 := -r\sin(\theta)\sin(\phi)$$
$$FF14 := 0$$

```
>
> FF21:=getcoeff(d(y)&^d(theta)&^d(phi)&^d(t));FF22:=getcoeff(d(y)&^d(phi)&^d(t)&^
d(r));FF23:=getcoeff(d(y)&^d(t)&^d(r)&^d(theta));FF24:=getcoeff(d(y)&^d(r)&^d(th
eta)&^d(phi));
```

$$FF21 := \sin(\theta)\sin(\phi)$$
$$FF22 := r\sin(\phi)\cos(\theta)$$
$$FF23 := r\sin(\theta)\cos(\phi)$$
$$FF24 := 0$$

At this point the Frame matrix will be modified to make the third 1-form sigmaz not exact and not closed. A constant coefficient aa will scale the perturbation of the FRame. The associated Frame matrix elements are then:

```
> sigmaz:=wcollect(d(z)+aa*(y*d(x)-x*d(y)));dzz:=sigmaz;
> FF31:=getcoeff(dzz&^d(theta)&^d(phi)&^d(tau));FF32:=getcoeff(dzz&^d(phi)&^d(tau)
&^d(r));FF33:=factor(simplify(getcoeff(dzz&^d(tau)&^d(r)&^d(theta))));FF34:=getc
oeff(dzz&^d(r)&^d(theta)&^d(phi));
>
```

$$sigmaz := -r\sin(\theta)\,d(\theta) + aa\,(-r^2\sin(\theta)^2\sin(\phi)^2 - r^2\sin(\theta)^2\cos(\phi)^2)\,d(\phi) + \cos(\theta)\,d(r)$$
$$dzz := -r\sin(\theta)\,d(\theta) + aa\,(-r^2\sin(\theta)^2\sin(\phi)^2 - r^2\sin(\theta)^2\cos(\phi)^2)\,d(\phi) + \cos(\theta)\,d(r)$$
$$FF31 := \cos(\theta)$$
$$FF32 := -r\sin(\theta)$$
$$FF33 := aa\,r^2(\cos(\theta)-1)(\cos(\theta)+1)$$
$$FF34 := 0$$

```
>
> FF41:=getcoeff(d(t)&^d(theta)&^d(phi)&^d(tau));FF42:=getcoeff(d(t)&^d(phi)&^d(ta
u)&^d(r));FF43:=getcoeff(d(t)&^d(tau)&^d(r)&^d(theta));FF44:=getcoeff(d(t)&^d(r)
&^d(theta)&^d(phi));
```

$$FF41 := 0$$
$$FF42 := 0$$
$$FF43 := 0$$
$$FF44 := 1$$

```
>
```

These matrix elements will be put into a Frame matrix format as:

```
> R:=[x,y,z,t]:FF:=array([[FF11,FF12,FF13,FF14],[FF21,FF22,FF23,FF24],[FF31,FF32,F
F33,FF34],[FF41,FF42,FF43,FF44]]);
>
```

$$FF := \begin{bmatrix} \sin(\theta)\cos(\phi) & r\cos(\phi)\cos(\theta) & -r\sin(\theta)\sin(\phi) & 0 \\ \sin(\theta)\sin(\phi) & r\sin(\phi)\cos(\theta) & r\sin(\theta)\cos(\phi) & 0 \\ \cos(\theta) & -r\sin(\theta) & aa\, r^2 (\cos(\theta)-1)(\cos(\theta)+1) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Setting aa to zero reproduces the unperturbed JAcobian matrix.
The inverse to the Frame is :

```
> GG:=simplify(evalm(inverse(FF)));DETFF:=simplify(det(FF));
```

$GG :=$

$$[-\sin(\theta)(r\sin(\phi)\cos(\theta)\,aa-\cos(\phi)),\ \sin(\theta)(r\cos(\phi)\cos(\theta)\,aa+\sin(\phi)),\ \cos(\theta),\ 0]$$

$$\left[-\frac{\sin(\phi)\,aa\,r\cos(\theta)^2-r\sin(\phi)\,aa-\cos(\theta)\cos(\phi)}{r},\ \frac{\cos(\phi)\,aa\,r\cos(\theta)^2-r\cos(\phi)\,aa+\cos(\theta)\sin(\phi)}{r},\ -\frac{\sin(\theta)}{r},\ 0\right]$$

$$\left[-\frac{\sin(\phi)}{r\sin(\theta)},\ \frac{\cos(\phi)}{r\sin(\theta)},\ 0,\ 0\right]$$

$$[0,\ 0,\ 0,\ 1]$$

$$DETFF := \sin(\theta)\, r^2$$

Note that the Frame matrix has a singularity at values of theta equal to multiples of pi, and at r=0. The determinant of the Frame matrix does not depend upon the perturbation. It will be shown below that the pertubation influences the components of Affine Torsion.

The next equation checks to see that the specified frame produces the desired differential structures: |V>=[FF]|dR>. The 1-forms computed from the inverse Frame are also evaluated as |A> = [GG]|dR>

```
> sigmaf:=simplify(evalm(innerprod(FF,[d(r),d(theta),d(phi),d(tau)]))):vx_or_sigmax:=sigmaf[1];vy_or_sigmay:=sigmaf[2];vz_or_sigmaz:=sigmaf[3];sigmag:=simplify(evalm(innerprod(GG,[d(r),d(theta),d(phi),d(tau)]))):ax:=wcollect(sigmag[1]);ay:=wcollect(sigmag[2]);az:=sigmag[3];
>
```

$$vx\_or\_sigmax := \sin(\theta)\cos(\phi)\,d(r) + r\cos(\phi)\cos(\theta)\,d(\theta) - r\sin(\theta)\sin(\phi)\,d(\phi)$$

$$vy\_or\_sigmay := \sin(\theta)\sin(\phi)\,d(r) + r\sin(\phi)\cos(\theta)\,d(\theta) + r\sin(\theta)\cos(\phi)\,d(\phi)$$

$$vz\_or\_sigmaz := \cos(\theta)\,d(r) - r\sin(\theta)\,d(\theta) + aa\,r^2\,d(\phi)\cos(\theta)^2 - aa\,r^2\,d(\phi)$$

$$ax := (\sin(\theta)\,r\cos(\phi)\cos(\theta)\,aa + \sin(\theta)\sin(\phi))\,d(\theta) + \cos(\theta)\,d(\phi) + (-\sin(\theta)\,r\sin(\phi)\cos(\theta)\,aa + \sin(\theta)\cos(\phi))\,d(r)$$

$$ay := -\frac{(-\cos(\phi)\,aa\,r\cos(\theta)^2 + r\cos(\phi)\,aa - \cos(\theta)\sin(\phi))\,d(\theta)}{r} - \frac{\sin(\theta)\,d(\phi)}{r} - \frac{(\sin(\phi)\,aa\,r\cos(\theta)^2 - r\sin(\phi)\,aa - \cos(\theta)\cos(\phi))\,d(r)}{r}$$

$$az := -\frac{\sin(\phi)\,d(r) - \cos(\phi)\,d(\theta)}{\sin(\theta)\,r}$$

A test is made to see if the 1-forms |V> are closed. For the perturbed Frame example the construction indicates that the third 1-form sigmaz is not closed unless aa = 0. The third 1-form vz indeed exhibits topological torsion, which is a prerequisite to produce AFFINE TORSION, and also indicates that the system of perturbed 1-forms is not integrable.

```
> closure_vx:=d(vx);closure_vy:=d(vy);closure_vz:=wcollect(d(vz));TopTorsion_vz:=vz&^d(vz);
```

$$closure\_vx := 0$$

$$closure\_vy := 0$$

$$closure\_vz := 0$$

$$TopTorsion\_vz := 0$$

>

>

Now the conventional metric on {x,y,z} will be modified to have non - constant coefficients. The metric on the final state induces a metric on the initial state via the congruent mapping [pullbackmetric] = [Frame_transpose][finalmetric][Frame]. For the example, the coefficients Sch1 and Sch2 can be used to perturb the final state metric to the form of the Scharzschild metric in isotropic coordinates. In the following metric perturbation, the mass M is the perturbation parameter.

```
> Sch1:=(1+M/(2*r))^4;Sch2:=((1-M/(2*r))/(1+M/(2*r)))^2;
```

$$Sch1 := \left(1+\frac{1}{2}\frac{M}{r}\right)^4$$

$$Sch2 := \frac{\left(1-\frac{1}{2}\frac{M}{r}\right)^2}{\left(1+\frac{1}{2}\frac{M}{r}\right)^2}$$

```
> finalmetric := array([[-Sch1, 0, 0,0], [0,-Sch1, 0,0], [0,
  0,-Sch1,0],[0,0,0,Sch2]]);
```

$$finalmetric := \begin{bmatrix} -\left(1+\frac{1}{2}\frac{M}{r}\right)^4 & 0 & 0 & 0 \\ 0 & -\left(1+\frac{1}{2}\frac{M}{r}\right)^4 & 0 & 0 \\ 0 & 0 & -\left(1+\frac{1}{2}\frac{M}{r}\right)^4 & 0 \\ 0 & 0 & 0 & \frac{\left(1-\frac{1}{2}\frac{M}{r}\right)^2}{\left(1+\frac{1}{2}\frac{M}{r}\right)^2} \end{bmatrix}$$

>

```
> pullbackmetric:=simplify(innerprod(transpose(FF),finalmetric,FF),trig):g11:=fact
  or(pullbackmetric[1,1]);g12:=factor(pullbackmetric[1,2]);g13:=factor(pullbackmet
  ric[1,3]);g14:=factor(pullbackmetric[1,4]);g21:=factor(pullbackmetric[2,1]);g22:
  =factor(pullbackmetric[2,2]);g23:=factor(pullbackmetric[2,3]);g24:=factor(pullba
  ckmetric[2,4]);g31:=factor(pullbackmetric[3,1]);g32:=factor(pullbackmetric[3,2])
  ;g33:=factor(pullbackmetric[3,3]);g34:=factor(pullbackmetric[3,4]);g41:=factor(p
  ullbackmetric[4,1]);g42:=factor(pullbackmetric[4,2]);g43:=factor(pullbackmetric[
  4,3]);g44:=factor(pullbackmetric[4,4]);
```

$$g11 := -\frac{1}{16}\frac{(2r+M)^4}{r^4}$$

$$g12 := 0$$

$$g13 := -\frac{1}{16}\frac{\cos(\theta)(2r+M)^4 aa(\cos(\theta)-1)(\cos(\theta)+1)}{r^2}$$

$$g14 := 0$$

$$g21 := 0$$

$$g22 := -\frac{1}{16}\frac{(2\,r+M)^4}{r^2}$$

$$g23 := \frac{1}{16}\frac{\sin(\theta)\,(2\,r+M)^4\,aa\,(\cos(\theta)-1)\,(\cos(\theta)+1)}{r}$$

$$g24 := 0$$

$$g31 := -\frac{1}{16}\frac{\cos(\theta)\,(2\,r+M)^4\,aa\,(\cos(\theta)-1)\,(\cos(\theta)+1)}{r^2}$$

$$g32 := \frac{1}{16}\frac{\sin(\theta)\,(2\,r+M)^4\,aa\,(\cos(\theta)-1)\,(\cos(\theta)+1)}{r}$$

$$g33 := -\frac{1}{16}\frac{(2\,r+M)^4\,(\cos(\theta)-1)\,(\cos(\theta)+1)\,(aa^2\,r^2\cos(\theta)^2-aa^2\,r^2-1)}{r^2}$$

$$g34 := 0$$

$$g41 := 0$$

$$g42 := 0$$

$$g43 := 0$$

$$g44 := \frac{(-2\,r+M)^2}{(2\,r+M)^2}$$

```
> metric(Schwarzschild_with_torsion):=array(subs(M=M,aa=aa,([[g11,g12,g13,g14],[g21,g22,g23,g24],[g31,g32,g33,g34],[g41,g42,g43,g44]])));
```

metric(*Schwarzschild_with_torsion*) :=

$$\left[-\frac{1}{16}\frac{(2\,r+M)^4}{r^4},\,0\,,-\frac{1}{16}\frac{\cos(\theta)\,(2\,r+M)^4\,aa\,(\cos(\theta)-1)\,(\cos(\theta)+1)}{r^2},\,0\right]$$

$$\left[0\,,-\frac{1}{16}\frac{(2\,r+M)^4}{r^2},\,\frac{1}{16}\frac{\sin(\theta)\,(2\,r+M)^4\,aa\,(\cos(\theta)-1)\,(\cos(\theta)+1)}{r},\,0\right]$$

$$\left[-\frac{1}{16}\frac{\cos(\theta)\,(2\,r+M)^4\,aa\,(\cos(\theta)-1)\,(\cos(\theta)+1)}{r^2},\,\frac{1}{16}\frac{\sin(\theta)\,(2\,r+M)^4\,aa\,(\cos(\theta)-1)\,(\cos(\theta)+1)}{r},\right.$$
$$\left.-\frac{1}{16}\frac{(2\,r+M)^4\,(\cos(\theta)-1)\,(\cos(\theta)+1)\,(aa^2\,r^2\cos(\theta)^2-aa^2\,r^2-1)}{r^2},\,0\right]$$

$$\left[0\,,0\,,0\,,\frac{(-2\,r+M)^2}{(2\,r+M)^2}\right]$$

```
> metric(Schwarzschild_noTorsion):=array(subs(M=M,aa=0,([[g11,g12,g13,g14],[g21,g22,g23,g24],[g31,g32,g33,g34],[g41,g42,g43,g44]])));
```

metric(*Schwarzschild_noTorsion*) :=

$$\begin{bmatrix} -\frac{1}{16}\frac{(2\,r+M)^4}{r^4} & 0 & 0 & 0 \\ 0 & -\frac{1}{16}\frac{(2\,r+M)^4}{r^2} & 0 & 0 \\ 0 & 0 & \frac{1}{16}\frac{(2\,r+M)^4\,(\cos(\theta)-1)\,(\cos(\theta)+1)}{r^2} & 0 \\ 0 & 0 & 0 & \frac{(-2\,r+M)^2}{(2\,r+M)^2} \end{bmatrix}$$

```
> metric(Torsion_but_no_M):=array(subs(M=0,aa=aa,([[g11,g12,g13,g14],[g21,g22,g23,g24],[g31,g32,g33,g34],[g41,g42,g43,g44]])));
```

metric(*Torsion_but_no_M*) :=

$[-1, 0, -\cos(\theta)\, r^2\, aa\, (\cos(\theta)-1)\,(\cos(\theta)+1), 0]$

$[0, -r^2, r^3 \sin(\theta)\, aa\, (\cos(\theta)-1)\,(\cos(\theta)+1), 0]$

$[-\cos(\theta)\, r^2\, aa\, (\cos(\theta)-1)\,(\cos(\theta)+1), r^3 \sin(\theta)\, aa\, (\cos(\theta)-1)\,(\cos(\theta)+1),$
$-r^2\,(\cos(\theta)-1)\,(\cos(\theta)+1)\,(aa^2\, r^2 \cos(\theta)^2 - aa^2\, r^2 - 1), 0]$

$[0, 0, 0, 1]$

```
> metric_without_M_and_aa(Jacobian):=array(subs(M=0,aa=0,([[g11,g12,g13,g14],[g21,g22,g23,g24],[g31,g32,g33,g34],[g41,g42,g43,g44]])));
```

$$\text{metric\_without\_M\_and\_aa}(Jacobian) := \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -r^2 & 0 & 0 \\ 0 & 0 & r^2\,(\cos(\theta)-1)\,(\cos(\theta)+1) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

```
>
```

The pullback metric depends upon both the torsion perturbation coefficient aa and the metric perturbation coefficient M

```
>
> d(FF);
```

$[\cos(\phi)\cos(\theta)\, d(\theta) - \sin(\theta)\sin(\phi)\, d(\phi), d(r)\cos(\theta)\cos(\phi) - r\cos(\theta)\sin(\phi)\, d(\phi) - \sin(\theta)\, r\cos(\phi)\, d(\theta),$
$-\sin(\theta)\sin(\phi)\, d(r) - r\sin(\phi)\cos(\theta)\, d(\theta) - r\sin(\theta)\cos(\phi)\, d(\phi), 0]$

$[d(\theta)\cos(\theta)\sin(\phi) + \sin(\theta)\cos(\phi)\, d(\phi), \sin(\phi)\cos(\theta)\, d(r) + r\cos(\theta)\cos(\phi)\, d(\phi) - r\sin(\phi)\sin(\theta)\, d(\theta),$
$\sin(\theta)\cos(\phi)\, d(r) + r\cos(\phi)\cos(\theta)\, d(\theta) - r\sin(\theta)\sin(\phi)\, d(\phi), 0]$

$[-\sin(\theta)\, d(\theta), -\sin(\theta)\, d(r) - r\cos(\theta)\, d(\theta),$
$-2\, aa\, r^2 \sin(\theta)\cos(\theta)\, d(\theta) + 2\, aa\, r\,(\cos(\theta)-1)\,(\cos(\theta)+1)\, d(r), 0]$

$[0, 0, 0, 0]$

Now Compute the Right Cartan Matrix [CR]

```
> cartan:=simplify(innerprod(inverse(FF),d(FF)));
```

cartan :=

$[-\cos(\theta)^3\, r\, aa\, d(\phi) + \cos(\theta)\, r\, aa\, d(\phi), (\cos(\theta)^2\, r\, aa\sin(\theta)\, d(\phi) - d(\theta))\, r,$
$(aa\, d(r)\cos(\theta)^3 + \cos(\theta)^2\, d(\phi) - \sin(\theta)\, r\cos(\theta)^2\, aa\, d(\theta) - aa\, d(r)\cos(\theta) - d(\phi))\, r, 0]$

$$\left[\frac{\cos(\theta)^2\, r\, aa\sin(\theta)\, d(\phi) - r\, aa\sin(\theta)\, d(\phi) + d(\theta)}{r}, \frac{aa\, r^2\, d(\phi)\cos(\theta)^3 - aa\, r^2\, d(\phi)\cos(\theta) + d(r)}{r},\right.$$
$$\left.-aa\, r\, d(\theta)\cos(\theta)^3 - aa\sin(\theta)\, d(r)\cos(\theta)^2 + \cos(\theta)\, aa\, r\, d(\theta) - \cos(\theta)\sin(\theta)\, d(\phi) + aa\sin(\theta)\, d(r), 0\right]$$

$$\left[\frac{d(\phi)}{r}, \frac{\cos(\theta)\, d(\phi)}{\sin(\theta)}, \frac{\sin(\theta)\, d(r) + r\cos(\theta)\, d(\theta)}{\sin(\theta)\, r}, 0\right]$$

$[0, 0, 0, 0]$

The matrix elements of the Right Cartan connection matrix (indices and values on the initial non-cartesian state) using the matrix methods:

```
> Gamma11:=wcollect(cartan[1,1]);Gamma12:=wcollect(cartan[1,2]);Gamma13:=wcollect(cartan[1,3]);Gamma14:=wcollect(cartan[1,4]);
```

$$\Gamma 11 := (-\cos(\theta)^3\, r\, aa + aa\cos(\theta)\, r)\, d(\phi)$$

$$\Gamma 12 := -r\, d(\theta) + \sin(\theta)\, r^2 \cos(\theta)^2\, aa\, d(\phi)$$

$$\Gamma 13 := -\sin(\theta)\, r^2 \cos(\theta)^2\, aa\, d(\theta) + r\,(\cos(\theta)^2 - 1)\, d(\phi) + (aa \cos(\theta)^3 - aa \cos(\theta))\, r\, d(r)$$

$$\Gamma 14 := 0$$

```
> Gamma21:=wcollect(cartan[2,1]);Gamma22:=wcollect(cartan[2,2]);Gamma23:=wcollect(
cartan[2,3]);Gamma24:=wcollect(cartan[2,4]);
```

$$\Gamma 21 := \frac{d(\theta)}{r} + \frac{(\cos(\theta)^2\, r\, aa \sin(\theta) - \sin(\theta)\, r\, aa)\, d(\phi)}{r}$$

$$\Gamma 22 := \frac{(aa\, r^2 \cos(\theta)^3 - aa\, r^2 \cos(\theta))\, d(\phi)}{r} + \frac{d(r)}{r}$$

$$\Gamma 23 := (-\cos(\theta)^3\, r\, aa + aa \cos(\theta)\, r)\, d(\theta) - \cos(\theta) \sin(\theta)\, d(\phi) + (aa \sin(\theta) - aa \sin(\theta) \cos(\theta)^2)\, d(r)$$

$$\Gamma 24 := 0$$

```
> Gamma31:=wcollect(cartan[3,1]);Gamma32:=wcollect(cartan[3,2]);Gamma33:=wcollect(
cartan[3,3]);Gamma34:=wcollect(cartan[3,4]);
```

$$\Gamma 31 := \frac{d(\phi)}{r}$$

$$\Gamma 32 := \frac{\cos(\theta)\, d(\phi)}{\sin(\theta)}$$

$$\Gamma 33 := \frac{\cos(\theta)\, d(\theta)}{\sin(\theta)} + \frac{d(r)}{r}$$

$$\Gamma 34 := 0$$

```
>
> Gamma41:=wcollect(cartan[4,1]);Gamma42:=wcollect(cartan[4,2]);Gamma43:=wcollect(
cartan[4,3]);Gamma44:=wcollect(cartan[4,4]);
```

$$\Gamma 41 := 0$$

$$\Gamma 42 := 0$$

$$\Gamma 43 := 0$$

$$\Gamma 44 := 0$$

The Trace of the connection will be determined and in this case is equal to a closed 1-form.

```
>
>
> TRACEGAMMA:=simplify(wcollect(Gamma11+Gamma22+Gamma33+Gamma44));TRACECURV:=d(TRA
CEGAMMA);
```

$$TRACEGAMMA := \frac{r \cos(\theta)\, d(\theta) + 2 \sin(\theta)\, d(r)}{\sin(\theta)\, r}$$

$$TRACECURV := 0$$

The Affine Torsion coefficients are computed, and they depend upon the Frame pertubation coefficient aa.:

```
> AFFINE1:=wcollect(simplify(Gamma11&^d(r)+Gamma12&^d(theta)+Gamma13&^d(phi)+Gamma
14&^d(tau)));AFFINE2:=wcollect(simplify(Gamma21&^d(r)+Gamma22&^d(theta)+Gamma23&
^d(phi)+Gamma24&^d(tau)));AFFINE3:=wcollect(simplify(Gamma31&^d(r)+Gamma32&^d(th
eta)+Gamma33&^d(phi)+Gamma34&^d(tau)));AFFINE4:=wcollect(simplify(Gamma41&^d(r)+
Gamma42&^d(theta)+Gamma43&^d(phi)+Gamma44&^d(tau)));
```

$$AFFINE1 := \cos(\theta)^2\, r^2\, aa \sin(\theta)\, (d(\phi) \,\&\wedge\, d(\theta)) - \cos(\theta)^2\, r^2\, aa \sin(\theta)\, (d(\theta) \,\&\wedge\, d(\phi)) + (\cos(\theta)^3\, r\, aa - aa \cos(\theta)\, r)\, (d(r) \,\&\wedge\, d(\phi)) + (-\cos(\theta)^3\, r\, aa + aa \cos(\theta)\, r)\, (d(\phi) \,\&\wedge\, d(r))$$

$$AFFINE2 := \frac{(aa\, r^2 \cos(\theta)^3 - aa\, r^2 \cos(\theta))\, (d(\phi) \&\wedge d(\theta))}{r} + \frac{d(r) \&\wedge d(\theta)}{r} + \frac{(aa\, r^2 \cos(\theta) - aa\, r^2 \cos(\theta)^3)\, (d(\theta) \&\wedge d(\phi))}{r} + \frac{(-\cos(\theta)^2\, r\, aa \sin(\theta) + \sin(\theta)\, r\, aa)\, (d(r) \&\wedge d(\phi))}{r} + \frac{(\cos(\theta)^2\, r\, aa \sin(\theta) - \sin(\theta)\, r\, aa)\, (d(\phi) \&\wedge d(r))}{r} + \frac{d(\theta) \&\wedge d(r)}{r}$$

$$AFFINE3 := \frac{d(\phi) \&\wedge d(r)}{r} + \frac{\cos(\theta)\, (d(\phi) \&\wedge d(\theta))}{\sin(\theta)} + \frac{\cos(\theta)\, (d(\theta) \&\wedge d(\phi))}{\sin(\theta)} + \frac{d(r) \&\wedge d(\phi)}{r}$$

$$AFFINE4 := 0$$

Now the components of the right Cartan matrix will be computed by the tensor method, as a check

```
> dim:=4;coord:=[r,theta,phi,tau];GG:=simplify(inverse(FF));
```

$$dim := 4$$

$$coord := [r, \theta, \phi, \tau]$$

$GG :=$

$$\begin{bmatrix} -\sin(\theta)\,(r \sin(\phi) \cos(\theta)\, aa - \cos(\phi)) & \sin(\theta)\,(r \cos(\phi) \cos(\theta)\, aa + \sin(\phi)) & \cos(\theta) & 0 \\ -\frac{\sin(\phi)\, aa\, r \cos(\theta)^2 - r \sin(\phi)\, aa - \cos(\theta) \cos(\phi)}{r} & \frac{\cos(\phi)\, aa\, r \cos(\theta)^2 - r \cos(\phi)\, aa + \cos(\theta) \sin(\phi)}{r} & -\frac{\sin(\theta)}{r} & 0 \\ -\frac{\sin(\phi)}{r \sin(\theta)} & \frac{\cos(\phi)}{r \sin(\theta)} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

First compute the differentials of the inverse matrix [GG]

```
> for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do d1GG[i,j,k]
  := (diff(GG[i,j],coord[k])) od od od:
```

Compute the elements of the matrix product of - d[G][F]

```
> for b from 1 to dim do for a from 1 to dim do for k from 1 to dim do  ss:=0;for
  m from 1 to dim do ss := ss+(d1GG[a,m,k]*FF[m,b]); CC[a,b,k]:=simplify(-ss) od
  od od od ;
> ;
> for b from 1 to dim do for a from 1 to dim do for k from 1 to dim do if
  CC[a,b,k]=0 then else print(`Cartan_RIGHT`(a,b,k)=factor(CC[a,b,k])) fi od od od
  ;
```

THE non zero CARTAN RIGHT CONNECTION coefficients.

CC(abk) index (1,-1,-1))

$$\text{Cartan\_RIGHT}(1, 1, 3) = -\cos(\theta)\, r\, aa\, (\cos(\theta) - 1)\, (\cos(\theta) + 1)$$

$$\text{Cartan\_RIGHT}(2, 1, 2) = \frac{1}{r}$$

$$\text{Cartan\_RIGHT}(2, 1, 3) = \sin(\theta)\, aa\, (\cos(\theta) - 1)\, (\cos(\theta) + 1)$$

$$\text{Cartan\_RIGHT}(3, 1, 3) = \frac{1}{r}$$

$$\text{Cartan\_RIGHT}(1, 2, 2) = -r$$

$$Cartan\_RIGHT(1,2,3) = \cos(\theta)^2 r^2 aa \sin(\theta)$$

$$Cartan\_RIGHT(2,2,1) = \frac{1}{r}$$

$$Cartan\_RIGHT(2,2,3) = \cos(\theta)\, r\, aa\, (\cos(\theta)-1)\,(\cos(\theta)+1)$$

$$Cartan\_RIGHT(3,2,3) = \frac{\cos(\theta)}{\sin(\theta)}$$

$$Cartan\_RIGHT(1,3,1) = \cos(\theta)\, r\, aa\, (\cos(\theta)-1)\,(\cos(\theta)+1)$$

$$Cartan\_RIGHT(1,3,2) = -\cos(\theta)^2 r^2 aa \sin(\theta)$$

$$Cartan\_RIGHT(1,3,3) = r\,(\cos(\theta)-1)\,(\cos(\theta)+1)$$

$$Cartan\_RIGHT(2,3,1) = -\sin(\theta)\, aa\, (\cos(\theta)-1)\,(\cos(\theta)+1)$$

$$Cartan\_RIGHT(2,3,2) = -\cos(\theta)\, r\, aa\, (\cos(\theta)-1)\,(\cos(\theta)+1)$$

$$Cartan\_RIGHT(2,3,3) = -\cos(\theta)\sin(\theta)$$

$$Cartan\_RIGHT(3,3,1) = \frac{1}{r}$$

$$Cartan\_RIGHT(3,3,2) = \frac{\cos(\theta)}{\sin(\theta)}$$

These results agree with matrix method.
Next check for Affine Torsion using the tensor methods:

```
> for j from 1 to dim do for i from 1 to dim do for k from 1 to dim do  ss :=
  (CC[i,j,k]-CC[i,k,j])/2; CCTTS[i,j,k]:=ss od od od  ;
>
> for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do if
  CCTTS[i,j,k]=0 then else print(`RIGHT_AffineTorsion`(i,k,j)=CCTTS[i,k,j]) fi od
  od od ;
```

$$RIGHT\_AffineTorsion(1,3,1) = \cos(\theta)^3 r\, aa - aa \cos(\theta)\, r$$

$$RIGHT\_AffineTorsion(1,3,2) = -\cos(\theta)^2 r^2 aa \sin(\theta)$$

$$RIGHT\_AffineTorsion(1,1,3) = -\cos(\theta)^3 r\, aa + aa \cos(\theta)\, r$$

$$RIGHT\_AffineTorsion(1,2,3) = \cos(\theta)^2 r^2 aa \sin(\theta)$$

$$RIGHT\_AffineTorsion(2,3,1) = aa \sin(\theta) - aa \sin(\theta) \cos(\theta)^2$$

$$RIGHT\_AffineTorsion(2,3,2) = -\cos(\theta)^3 r\, aa + aa \cos(\theta)\, r$$

$$RIGHT\_AffineTorsion(2,1,3) = aa \sin(\theta) \cos(\theta)^2 - aa \sin(\theta)$$

$$RIGHT\_AffineTorsion(2,2,3) = \cos(\theta)^3 r\, aa - aa \cos(\theta)\, r$$

IF NO ENTRIES APPEAR ABOVE, THE AFFINE TORSION IS ZERO

Note for the torsion example, there is no affine torsion of the connection coefficients if aa = 0. Torsion does not depend upon Metric perturbation M, but only upon the perturbation aa for the example displayed.

Christoffel Connection coefficients from the induced metric

```
>
> metric:= simplify(subs(evalm(pullbackmetric)));
```

$metric :=$

$$\left[-\frac{1}{16}\frac{16 r^4+32 r^3 M+24 r^2 M^2+8 r M^3+M^4}{r^4}, 0, -\frac{1}{16}\cos(\theta)\, aa\, (16 r^4 \cos(\theta)^2-16 r^4+32 r^3 M \cos(\theta)^2 -32 r^3 M+24 r^2 M^2 \cos(\theta)^2-24 r^2 M^2+8 r M^3 \cos(\theta)^2-8 r M^3+M^4 \cos(\theta)^2-M^4) \Big/ r^2, 0\right]$$

$$\left[0, -\frac{1}{16}\frac{16 r^4+32 r^3 M+24 r^2 M^2+8 r M^3+M^4}{r^2}, \frac{1}{16}\sin(\theta)\, aa\, (16 r^4 \cos(\theta)^2-16 r^4+32 r^3 M \cos(\theta)^2-32 r^3 M +24 r^2 M^2 \cos(\theta)^2-24 r^2 M^2+8 r M^3 \cos(\theta)^2-8 r M^3+M^4 \cos(\theta)^2-M^4) / r, 0\right]$$

$$\left[-\frac{1}{16}\cos(\theta)\, aa\, (16 r^4 \cos(\theta)^2-16 r^4+32 r^3 M \cos(\theta)^2-32 r^3 M+24 r^2 M^2 \cos(\theta)^2-24 r^2 M^2+8 r M^3 \cos(\theta)^2 -8 r M^3+M^4 \cos(\theta)^2-M^4) \Big/ r^2, \frac{1}{16}\sin(\theta)\, aa\, (16 r^4 \cos(\theta)^2-16 r^4+32 r^3 M \cos(\theta)^2-32 r^3 M +24 r^2 M^2 \cos(\theta)^2-24 r^2 M^2+8 r M^3 \cos(\theta)^2-8 r M^3+M^4 \cos(\theta)^2-M^4) / r, -\frac{1}{16}(-8 r M^3 \cos(\theta)^2+32 r^3 M +24 r^2 M^2+8 r M^3+M^4-24 r^2 M^2 \cos(\theta)^2-32 r^3 M \cos(\theta)^2+16 r^4-16 r^4 \cos(\theta)^2-M^4 \cos(\theta)^2 +8 r^3 M^3 aa^2 \cos(\theta)^4-48 r^4 M^2 aa^2 \cos(\theta)^2+32 r^5 M aa^2 \cos(\theta)^4-16 r^3 M^3 aa^2 \cos(\theta)^2+M^4 aa^2 r^2 \cos(\theta)^4 -2 M^4 aa^2 r^2 \cos(\theta)^2-64 r^5 M aa^2 \cos(\theta)^2+24 r^4 M^2 aa^2 \cos(\theta)^4+32 r^5 M aa^2+24 r^4 M^2 aa^2+8 r^3 M^3 aa^2 +M^4 aa^2 r^2+16 r^6 aa^2 \cos(\theta)^4-32 r^6 aa^2 \cos(\theta)^2+16 r^6 aa^2) \Big/ r^2, 0\right]$$

$$\left[0, 0, 0, \frac{(-2 r+M)^2}{(2 r+M)^2}\right]$$

```
> metricinverse:=inverse(metric):
> for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do
  d1gun[i,j,k] := (diff(metric[i,j],coord[k])) od od od:
> #for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do if
  d1gun[i,j,k]=0 then  else print(`dgun`(i,j,k)=d1gun[i,j,k]) fi od od od;
> for i from 1 to dim do for j from i to dim do for k from 1 to dim do C1S[i,j,k]
  := 0  od od od; for i from 1 to dim do for j from 1 to dim do for k from 1 to
  dim do C1S[i,j,k] := 1/2*d1gun[i,k,j]+1/2*d1gun[j,k,i]-1/2*d1gun[i,j,k]  od od
  od;
>  for k from 1 to dim do for i from 1 to dim do for j from 1 to dim do ss := 0;
  for m to dim do ss := ss+metricinverse[k,m]*C1S[i,j,m] od; C2S[k,i,j] :=
  simplify(factor(ss),trig) od od od;
> for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do if
  C2S[i,j,k]=0 then  else print(`Christoffel_Gamma2`(i,j,k)=simplify(C2S[i,j,k]))
  fi od od od;
```

The non zero Christoffel Connection coefficients 2nd kind   on the initial space (domain)

Gamma2(i,j,k) index (1,-1,-1)

$$\text{Christoffel\_Gamma2}(1, 1, 1) = 2\,\frac{-M - 2\,r^3\,aa^2\cos(\theta)^2 + 2\,r^3\,aa^2\cos(\theta)^4}{r\,(2\,r + M)}$$

$$\text{Christoffel\_Gamma2}(1, 1, 2) = -2\cos(\theta)^3\,aa^2\,r^2\sin(\theta) + \cos(\theta)\,aa^2\,r^2\sin(\theta)$$

$$\text{Christoffel\_Gamma2}(1, 1, 3) = -r\,aa\cos(\theta)\,(M - M\,aa^2\,r^2 + 4\,M\,aa^2\,r^2\cos(\theta)^2 - 5\,M\,aa^2\,r^2\cos(\theta)^4 - M\cos(\theta)^2 + 2\,r^2\,aa^2\cos(\theta)^6\,M + 4\,r^3\,aa^2\cos(\theta)^2 - 2\,r^3\,aa^2\cos(\theta)^4 - 2\,r^3\,aa^2 - 2\,r + 2\,r\cos(\theta)^2)\,/\,(2\,r + M)$$

$$\text{Christoffel\_Gamma2}(1, 2, 1) = -2\cos(\theta)^3\,aa^2\,r^2\sin(\theta) + \cos(\theta)\,aa^2\,r^2\sin(\theta)$$

$$\text{Christoffel\_Gamma2}(1, 2, 2) = -\frac{r\,(4\,M\,aa^2\,r^2\cos(\theta)^4 - 4\,M\,aa^2\,r^2\cos(\theta)^2 - M + 4\,r^3\,aa^2\cos(\theta)^4 - 4\,r^3\,aa^2\cos(\theta)^2 + 2\,r)}{2\,r + M}$$

$$\text{Christoffel\_Gamma2}(1, 2, 3) = \sin(\theta)\,aa\,r^2\,(M + 3\,M\,aa^2\,r^2\cos(\theta)^2 - 5\,M\,aa^2\,r^2\cos(\theta)^4 - M\cos(\theta)^2 + 2\,r^2\,aa^2\cos(\theta)^6\,M - 2\,r^3\,aa^2\cos(\theta)^4 + 2\,r^3\,aa^2\cos(\theta)^2 - 2\,r + 2\,r\cos(\theta)^2)\,/\,(2\,r + M)$$

$$\text{Christoffel\_Gamma2}(1, 3, 1) = -r\,aa\cos(\theta)\,(M - M\,aa^2\,r^2 + 4\,M\,aa^2\,r^2\cos(\theta)^2 - 5\,M\,aa^2\,r^2\cos(\theta)^4 - M\cos(\theta)^2 + 2\,r^2\,aa^2\cos(\theta)^6\,M + 4\,r^3\,aa^2\cos(\theta)^2 - 2\,r^3\,aa^2\cos(\theta)^4 - 2\,r^3\,aa^2 - 2\,r + 2\,r\cos(\theta)^2)\,/\,(2\,r + M)$$

$$\text{Christoffel\_Gamma2}(1, 3, 2) = \sin(\theta)\,aa\,r^2\,(M + 3\,M\,aa^2\,r^2\cos(\theta)^2 - 5\,M\,aa^2\,r^2\cos(\theta)^4 - M\cos(\theta)^2 + 2\,r^2\,aa^2\cos(\theta)^6\,M - 2\,r^3\,aa^2\cos(\theta)^4 + 2\,r^3\,aa^2\cos(\theta)^2 - 2\,r + 2\,r\cos(\theta)^2)\,/\,(2\,r + M)$$

$$\text{Christoffel\_Gamma2}(1, 3, 3) = -r\,(4\,M\,aa^2\,r^2\cos(\theta)^4 - 2\,M\,aa^2\,r^2\cos(\theta)^2 - M - 8\,r^3\,aa^2\cos(\theta)^2 + 4\,r^3\,aa^2\cos(\theta)^4 - 2\,M\,aa^4\,r^4\cos(\theta)^2 + 6\,M\,aa^4\,r^4\cos(\theta)^4 - 6\,M\,aa^4\,r^4\cos(\theta)^6 + 2\,M\,aa^4\,r^4\cos(\theta)^8 - 2\,r^2\,aa^2\cos(\theta)^6\,M + 2\,r + M\cos(\theta)^2 - 2\,r\cos(\theta)^2 + 4\,r^3\,aa^2)\,/\,(2\,r + M)$$

$$\text{Christoffel\_Gamma2}(1, 4, 4) = 64\,\frac{r^4\,M\,(2\,r^3\,aa^2\cos(\theta)^2 - M\,aa^2\,r^2\cos(\theta)^2 - 2\,r^3\,aa^2\cos(\theta)^4 + M\,aa^2\,r^2\cos(\theta)^4 + 2\,r - M)}{128\,r^7 + 448\,M\,r^6 + 672\,M^2\,r^5 + 560\,M^3\,r^4 + 280\,M^4\,r^3 + 84\,M^5\,r^2 + 14\,r\,M^6 + M^7}$$

$$\text{Christoffel\_Gamma2}(2, 1, 1) = -4\,\frac{\sin(\theta)\cos(\theta)\,aa^2\,r\,(\cos(\theta)^2 - 1)}{2\,r + M}$$

$$\text{Christoffel\_Gamma2}(2, 1, 2) = -\frac{M - 3\,M\,aa^2\,r^2\cos(\theta)^2 + 2\,M\,aa^2\,r^2\cos(\theta)^4 + M\,aa^2\,r^2 + 2\,r^3\,aa^2 - 6\,r^3\,aa^2\cos(\theta)^2 + 4\,r^3\,aa^2\cos(\theta)^4 - 2\,r}{r\,(2\,r + M)}$$

$$\text{Christoffel\_Gamma2}(2, 1, 3) = \sin(\theta)\,aa\,(4\,M\,aa^2\,r^2\cos(\theta)^2 - 5\,M\,aa^2\,r^2\cos(\theta)^4 + M\cos(\theta)^2 - M\,aa^2\,r^2 + 2\,r^2\,aa^2\cos(\theta)^6\,M + 4\,r^3\,aa^2\cos(\theta)^2 + 2\,r\cos(\theta)^2 - 2\,r^3\,aa^2 - 2\,r^3\,aa^2\cos(\theta)^4)\,/\,(2\,r + M)$$

$$\text{Christoffel\_Gamma2}(2, 2, 1) = -\frac{M - 3\,M\,aa^2\,r^2\cos(\theta)^2 + 2\,M\,aa^2\,r^2\cos(\theta)^4 + M\,aa^2\,r^2 + 2\,r^3\,aa^2 - 6\,r^3\,aa^2\cos(\theta)^2 + 4\,r^3\,aa^2\cos(\theta)^4 - 2\,r}{r\,(2\,r + M)}$$

$$\text{Christoffel\_Gamma2}(2, 2, 2) = 4\,\frac{\sin(\theta)\,aa^2\,r^2\cos(\theta)\,(-M + M\cos(\theta)^2 - r + r\cos(\theta)^2)}{2\,r + M}$$

$$\text{Christoffel\_Gamma2}(2, 2, 3) = aa\,r\cos(\theta)\,(8\,M\,aa^2\,r^2\cos(\theta)^2 - 7\,M\,aa^2\,r^2\cos(\theta)^4 - M + M\cos(\theta)^2 - 3\,M\,aa^2\,r^2 + 2\,r^2\,aa^2\cos(\theta)^6\,M - 2\,r^3\,aa^2 + 4\,r^3\,aa^2\cos(\theta)^2 - 2\,r + 2\,r\cos(\theta)^2 - 2\,r^3\,aa^2\cos(\theta)^4)\,/\,(2\,r + M)$$

$$\text{Christoffel\_Gamma2}(2, 3, 1) = \sin(\theta)\,aa\,(4\,M\,aa^2\,r^2\cos(\theta)^2 - 5\,M\,aa^2\,r^2\cos(\theta)^4 + M\cos(\theta)^2 - M\,aa^2\,r^2 + 2\,r^2\,aa^2\cos(\theta)^6\,M + 4\,r^3\,aa^2\cos(\theta)^2 + 2\,r\cos(\theta)^2 - 2\,r^3\,aa^2 - 2\,r^3\,aa^2\cos(\theta)^4)\,/\,(2\,r + M)$$

$$\text{Christoffel\_Gamma2}(2, 3, 2) = aa\,r\cos(\theta)\,(8\,M\,aa^2\,r^2\cos(\theta)^2 - 7\,M\,aa^2\,r^2\cos(\theta)^4 - M + M\cos(\theta)^2 - 3\,M\,aa^2\,r^2 + 2\,r^2\,aa^2\cos(\theta)^6\,M - 2\,r^3\,aa^2 + 4\,r^3\,aa^2\cos(\theta)^2 - 2\,r + 2\,r\cos(\theta)^2 - 2\,r^3\,aa^2\cos(\theta)^4)\,/\,(2\,r + M)$$

$$\text{Christoffel\_Gamma2}(2, 3, 3) = \sin(\theta)\cos(\theta)\,(2\,M\,aa^4\,r^4\cos(\theta)^6 + 6\,M\,aa^4\,r^4\cos(\theta)^2 + 6\,M\,aa^2\,r^2\cos(\theta)^2 - M$$

$$- 2 M aa^2 r^2 \cos(\theta)^4 - 2 M aa^4 r^4 - 4 M aa^2 r^2 - 6 M aa^4 r^4 \cos(\theta)^4 - 2 r + 4 r^3 aa^2 \cos(\theta)^2 - 4 r^3 aa^2) / (2 r + M)$$

Christoffel_Gamma2(2, 4, 4) =

$$-64 \frac{r^5 M aa^2 \cos(\theta) \sin(\theta) (-2 r \cos(\theta)^2 + M \cos(\theta)^2 + 2 r - M)}{128 r^7 + 448 M r^6 + 672 M^2 r^5 + 560 M^3 r^4 + 280 M^4 r^3 + 84 M^5 r^2 + 14 r M^6 + M^7}$$

$$\text{Christoffel\_Gamma2}(3, 1, 1) = -4 \frac{aa \cos(\theta)}{2 r + M}$$

$$\text{Christoffel\_Gamma2}(3, 1, 2) = \frac{(2 \cos(\theta)^2 - 1) \sin(\theta) aa}{\cos(\theta)^2 - 1}$$

Christoffel_Gamma2(3, 1, 3) =

$$\frac{-3 M aa^2 r^2 \cos(\theta)^2 + 2 M aa^2 r^2 \cos(\theta)^4 - 2 r^3 aa^2 \cos(\theta)^2 + M aa^2 r^2 - M + 2 r^3 aa^2 + 2 r}{r (2 r + M)}$$

$$\text{Christoffel\_Gamma2}(3, 2, 1) = \frac{(2 \cos(\theta)^2 - 1) \sin(\theta) aa}{\cos(\theta)^2 - 1}$$

$$\text{Christoffel\_Gamma2}(3, 2, 2) = 4 \frac{(M + r) aa r \cos(\theta)}{2 r + M}$$

Christoffel_Gamma2(3, 2, 3) =

$$-\frac{(-5 M aa^2 r^2 \cos(\theta)^2 + 2 M aa^2 r^2 \cos(\theta)^4 + 3 M aa^2 r^2 + M - 2 r^3 aa^2 \cos(\theta)^2 + 2 r + 2 r^3 aa^2) \cos(\theta) \sin(\theta)}{M \cos(\theta)^2 - M - 2 r + 2 r \cos(\theta)^2}$$

Christoffel_Gamma2(3, 3, 1) =

$$\frac{-3 M aa^2 r^2 \cos(\theta)^2 + 2 M aa^2 r^2 \cos(\theta)^4 - 2 r^3 aa^2 \cos(\theta)^2 + M aa^2 r^2 - M + 2 r^3 aa^2 + 2 r}{r (2 r + M)}$$

Christoffel_Gamma2(3, 3, 2) =

$$-\frac{(-5 M aa^2 r^2 \cos(\theta)^2 + 2 M aa^2 r^2 \cos(\theta)^4 + 3 M aa^2 r^2 + M - 2 r^3 aa^2 \cos(\theta)^2 + 2 r + 2 r^3 aa^2) \cos(\theta) \sin(\theta)}{M \cos(\theta)^2 - M - 2 r + 2 r \cos(\theta)^2}$$

$$\text{Christoffel\_Gamma2}(3, 3, 3) = 2 \frac{aa r \cos(\theta) M (-2 aa^2 r^2 \cos(\theta)^2 + aa^2 r^2 \cos(\theta)^4 + 1 - \cos(\theta)^2 + aa^2 r^2)}{2 r + M}$$

Christoffel_Gamma2(3, 4, 4) =

$$-64 \frac{r^4 aa \cos(\theta) (-2 r + M) M}{128 r^7 + 448 M r^6 + 672 M^2 r^5 + 560 M^3 r^4 + 280 M^4 r^3 + 84 M^5 r^2 + 14 r M^6 + M^7}$$

$$\text{Christoffel\_Gamma2}(4, 1, 4) = -4 \frac{M}{(-2 r + M) (2 r + M)}$$

$$\text{Christoffel\_Gamma2}(4, 4, 1) = -4 \frac{M}{(-2 r + M) (2 r + M)}$$

If no entries appear above the Christoffel symbols on the domain space vanish

However, for the distorted example it is evident that there are contributions to Christoffel symbols (induced metric) which depend both on the torsion aa and the distortion bb.

The Right Cartan matrix is often defined as the sum of Christoffel Symbols and Rotation coefficients, T(i,j,k)

CartanRight(ijk) = ChristoffelGamma(ijk) - T(ijk)

Compute the T(i,j,k): These coefficients if no-zero indicate the effects of the pertubations on the metric and Basis. If there is no difference between the Christoffel symbols and the Cartan connection symbols, then the T(i,j,k) are zero.

```
> for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do ss:=0; ss
  := (CC[i,j,k]-C2S[i,j,k]); SHIPTR[i,j,k]:=simplify(ss)  od od od   ;
>
>
> for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do if
  C2S[i,j,k]=0 and CC[i,j,k]=0 then else
  print(`T`(i,j,k)=simplify(subs(SHIPTR[i,j,k]))) fi od od od ;
```

T(ijk) index (1,-1,-1)

$$\mathrm{T}(1,1,1) = -2\,\frac{-M - 2\,r^3\,aa^2\cos(\theta)^2 + 2\,r^3\,aa^2\cos(\theta)^4}{r\,(2\,r+M)}$$

$$\mathrm{T}(1,1,2) = 2\cos(\theta)^3\,aa^2\,r^2\sin(\theta) - \cos(\theta)\,aa^2\,r^2\sin(\theta)$$

$$\mathrm{T}(1,1,3) = \cos(\theta)\,r\,aa\,(-2\,M\cos(\theta)^2 + 2\,M - M\,aa^2\,r^2 + 4\,M\,aa^2\,r^2\cos(\theta)^2 - 5\,M\,aa^2\,r^2\cos(\theta)^4 + 2\,r^2\,aa^2\cos(\theta)^6\,M + 4\,r^3\,aa^2\cos(\theta)^2 - 2\,r^3\,aa^2\cos(\theta)^4 - 2\,r^3\,aa^2)\,/\,(2\,r+M)$$

$$\mathrm{T}(1,2,1) = 2\cos(\theta)^3\,aa^2\,r^2\sin(\theta) - \cos(\theta)\,aa^2\,r^2\sin(\theta)$$

$$\mathrm{T}(1,2,2) = 2\,\frac{r\,(-M + 2\,M\,aa^2\,r^2\cos(\theta)^4 - 2\,M\,aa^2\,r^2\cos(\theta)^2 + 2\,r^3\,aa^2\cos(\theta)^4 - 2\,r^3\,aa^2\cos(\theta)^2)}{2\,r+M}$$

$$\mathrm{T}(1,2,3) = -\,r^2\,aa\sin(\theta)\,(-2\,M\cos(\theta)^2 + M + 3\,M\,aa^2\,r^2\cos(\theta)^2 - 5\,M\,aa^2\,r^2\cos(\theta)^4 + 2\,r^2\,aa^2\cos(\theta)^6\,M - 2\,r^3\,aa^2\cos(\theta)^4 + 2\,r^3\,aa^2\cos(\theta)^2 - 2\,r)\,/\,(2\,r+M)$$

$$\mathrm{T}(1,3,1) = \cos(\theta)\,r^2\,aa\,(4\cos(\theta)^2 - 4 - M\,aa^2\,r + 4\,M\,aa^2\,r\cos(\theta)^2 - 5\,M\,aa^2\,r\cos(\theta)^4 + 2\,r\,aa^2\cos(\theta)^6\,M + 4\,aa^2\,r^2\cos(\theta)^2 - 2\,aa^2\,r^2\cos(\theta)^4 - 2\,aa^2\,r^2)\,/\,(2\,r+M)$$

$$\mathrm{T}(1,3,2) = -\,r^2\,aa\sin(\theta)\,(4\,r\cos(\theta)^2 + M + 3\,M\,aa^2\,r^2\cos(\theta)^2 - 5\,M\,aa^2\,r^2\cos(\theta)^4 + 2\,r^2\,aa^2\cos(\theta)^6\,M - 2\,r^3\,aa^2\cos(\theta)^4 + 2\,r^3\,aa^2\cos(\theta)^2 - 2\,r)\,/\,(2\,r+M)$$

$$\mathrm{T}(1,3,3) = 2\,r\,(-M + M\cos(\theta)^2 + 2\,M\,aa^2\,r^2\cos(\theta)^4 - M\,aa^2\,r^2\cos(\theta)^2 - 4\,r^3\,aa^2\cos(\theta)^2 + 2\,r^3\,aa^2\cos(\theta)^4 - M\,aa^4\,r^4\cos(\theta)^2 + 3\,M\,aa^4\,r^4\cos(\theta)^4 - 3\,M\,aa^4\,r^4\cos(\theta)^6 + M\,aa^4\,r^4\cos(\theta)^8 - r^2\,aa^2\cos(\theta)^6\,M + 2\,r^3\,aa^2)\,/\,(2\,r+M)$$

$$\mathrm{T}(1,4,4) = -64\,\frac{r^4\,M\,(2\,r^3\,aa^2\cos(\theta)^2 - M\,aa^2\,r^2\cos(\theta)^2 - 2\,r^3\,aa^2\cos(\theta)^4 + M\,aa^2\,r^2\cos(\theta)^4 + 2\,r - M)}{128\,r^7 + 448\,M\,r^6 + 672\,M^2\,r^5 + 560\,M^3\,r^4 + 280\,M^4\,r^3 + 84\,M^5\,r^2 + 14\,r\,M^6 + M^7}$$

$$\mathrm{T}(2,1,1) = 4\,\frac{\sin(\theta)\cos(\theta)\,aa^2\,r\,(\cos(\theta)^2 - 1)}{2\,r+M}$$

$$\mathrm{T}(2,1,2) = \frac{2\,M - 3\,M\,aa^2\,r^2\cos(\theta)^2 + 2\,M\,aa^2\,r^2\cos(\theta)^4 + M\,aa^2\,r^2 + 2\,r^3\,aa^2 - 6\,r^3\,aa^2\cos(\theta)^2 + 4\,r^3\,aa^2\cos(\theta)^4}{r\,(2\,r+M)}$$

$$\mathrm{T}(2,1,3) = -\,aa\sin(\theta)\,(2\,r + M + 4\,M\,aa^2\,r^2\cos(\theta)^2 - 5\,M\,aa^2\,r^2\cos(\theta)^4 - M\,aa^2\,r^2 + 2\,r^2\,aa^2\cos(\theta)^6\,M + 4\,r^3\,aa^2\cos(\theta)^2 - 2\,r^3\,aa^2 - 2\,r^3\,aa^2\cos(\theta)^4)\,/\,(2\,r+M)$$

$\mathrm{T}(2,2,1) =$

$$\frac{2\,M - 3\,M\,aa^2\,r^2\cos(\theta)^2 + 2\,M\,aa^2\,r^2\cos(\theta)^4 + M\,aa^2\,r^2 + 2\,r^3\,aa^2 - 6\,r^3\,aa^2\cos(\theta)^2 + 4\,r^3\,aa^2\cos(\theta)^4}{r\,(2\,r + M)}$$

$$\mathrm{T}(2, 2, 2) = -4\,\frac{\sin(\theta)\,aa^2\,r^2\cos(\theta)\,(-M + M\cos(\theta)^2 - r + r\cos(\theta)^2)}{2\,r + M}$$

$$\mathrm{T}(2, 2, 3) = -\frac{aa^3\cos(\theta)\,r^3\,(8\,M\cos(\theta)^2 - 7\,M\cos(\theta)^4 - 3\,M + 2\cos(\theta)^6\,M - 2\,r + 4\,r\cos(\theta)^2 - 2\,r\cos(\theta)^4)}{2\,r + M}$$

$$\mathrm{T}(2, 3, 1) = -\,aa\sin(\theta)\,(-2\,r - M + 4\,r\cos(\theta)^2 + 2\,M\cos(\theta)^2 + 4\,M\,aa^2\,r^2\cos(\theta)^2 - 5\,M\,aa^2\,r^2\cos(\theta)^4 - M\,aa^2\,r^2 + 2\,r^2\,aa^2\cos(\theta)^6\,M + 4\,r^3\,aa^2\cos(\theta)^2 - 2\,r^3\,aa^2 - 2\,r^3\,aa^2\cos(\theta)^4)\,/\,(2\,r + M)$$

$$\mathrm{T}(2, 3, 2) = -\,aa\cos(\theta)\,r\,(4\,r\cos(\theta)^2 + 2\,M\cos(\theta)^2 - 4\,r - 2\,M + 8\,M\,aa^2\,r^2\cos(\theta)^2 - 7\,M\,aa^2\,r^2\cos(\theta)^4 - 3\,M\,aa^2\,r^2 + 2\,r^2\,aa^2\cos(\theta)^6\,M - 2\,r^3\,aa^2 + 4\,r^3\,aa^2\cos(\theta)^2 - 2\,r^3\,aa^2\cos(\theta)^4)\,/\,(2\,r + M)$$

$$\mathrm{T}(2, 3, 3) = -2\cos(\theta)\sin(\theta)\,aa^2\,r^2\,(r^2\,aa^2\cos(\theta)^6\,M + 3\,M\,aa^2\,r^2\cos(\theta)^2 + 3\,M\cos(\theta)^2 - M\cos(\theta)^4 - M\,aa^2\,r^2 - 2\,M - 3\,M\,aa^2\,r^2\cos(\theta)^4 + 2\,r\cos(\theta)^2 - 2\,r)\,/\,(2\,r + M)$$

$$\mathrm{T}(2, 4, 4) = 64\,\frac{r^5\,M\,aa^2\cos(\theta)\sin(\theta)\,(-2\,r\cos(\theta)^2 + M\cos(\theta)^2 + 2\,r - M)}{128\,r^7 + 448\,M\,r^6 + 672\,M^2\,r^5 + 560\,M^3\,r^4 + 280\,M^4\,r^3 + 84\,M^5\,r^2 + 14\,r\,M^6 + M^7}$$

$$\mathrm{T}(3, 1, 1) = 4\,\frac{aa\cos(\theta)}{2\,r + M}$$

$$\mathrm{T}(3, 1, 2) = -\frac{(2\cos(\theta)^2 - 1)\sin(\theta)\,aa}{\cos(\theta)^2 - 1}$$

$$\mathrm{T}(3, 1, 3) = -\frac{-2\,M - 3\,M\,aa^2\,r^2\cos(\theta)^2 + 2\,M\,aa^2\,r^2\cos(\theta)^4 - 2\,r^3\,aa^2\cos(\theta)^2 + M\,aa^2\,r^2 + 2\,r^3\,aa^2}{r\,(2\,r + M)}$$

$$\mathrm{T}(3, 2, 1) = -\frac{(2\cos(\theta)^2 - 1)\sin(\theta)\,aa}{\cos(\theta)^2 - 1}$$

$$\mathrm{T}(3, 2, 2) = -4\,\frac{(M + r)\,aa\,r\cos(\theta)}{2\,r + M}$$

$$\mathrm{T}(3, 2, 3) = \frac{(2\,M\cos(\theta)^2 - 2\,r - 3\,M)\cos(\theta)\,r^2\,aa^2\sin(\theta)}{2\,r + M}$$

$$\mathrm{T}(3, 3, 1) = -\frac{-2\,M - 3\,M\,aa^2\,r^2\cos(\theta)^2 + 2\,M\,aa^2\,r^2\cos(\theta)^4 - 2\,r^3\,aa^2\cos(\theta)^2 + M\,aa^2\,r^2 + 2\,r^3\,aa^2}{r\,(2\,r + M)}$$

$$\mathrm{T}(3, 3, 2) = \frac{(2\,M\cos(\theta)^2 - 2\,r - 3\,M)\cos(\theta)\,r^2\,aa^2\sin(\theta)}{2\,r + M}$$

$$\mathrm{T}(3, 3, 3) = -2\,\frac{aa\,r\cos(\theta)\,M\,(-2\,aa^2\,r^2\cos(\theta)^2 + aa^2\,r^2\cos(\theta)^4 + 1 - \cos(\theta)^2 + aa^2\,r^2)}{2\,r + M}$$

$$\mathrm{T}(3, 4, 4) = 64\,\frac{r^4\,aa\cos(\theta)\,(-2\,r + M)\,M}{128\,r^7 + 448\,M\,r^6 + 672\,M^2\,r^5 + 560\,M^3\,r^4 + 280\,M^4\,r^3 + 84\,M^5\,r^2 + 14\,r\,M^6 + M^7}$$

$$\mathrm{T}(4, 1, 4) = 4\,\frac{M}{(-2\,r + M)\,(2\,r + M)}$$

$$\mathrm{T}(4, 4, 1) = 4\,\frac{M}{(-2\,r + M)\,(2\,r + M)}$$

>
>
>
>

Right Cartan(ijk) = ChristoffelGamma(ijk) - T(ijk)

In this example above the rotation coefficients on the domain space depend upon both the metric perturbation M and the Frame perturbation aa. When both are zero, the Ricci rotation coefficients vanish, and the Right Cartan matrix is exactly equal to the Christoffel symbols based upon the pullbackmetric.

NOW EXAMINE THE EFFECTS OF A METRIC PERTURBATION WITHOUT A TORSION PERTUBATION

```
> MetricMP:=evalm(subs(M=M,pullbackmetric));
```

MetricMP :=

$$\left[-\frac{1}{16}\frac{16\,r^4+32\,r^3\,M+24\,r^2\,M^2+8\,r\,M^3+M^4}{r^4},\,0\,,-\frac{1}{16}\cos(\theta)\,aa\,(16\,r^4\cos(\theta)^2-16\,r^4+32\,r^3\,M\cos(\theta)^2 -32\,r^3\,M+24\,r^2\,M^2\cos(\theta)^2-24\,r^2\,M^2+8\,r\,M^3\cos(\theta)^2-8\,r\,M^3+M^4\cos(\theta)^2-M^4)\Big/r^2,\,0\right]$$

$$\left[0\,,-\frac{1}{16}\frac{16\,r^4+32\,r^3\,M+24\,r^2\,M^2+8\,r\,M^3+M^4}{r^2},\,\frac{1}{16}\sin(\theta)\,aa\,(16\,r^4\cos(\theta)^2-16\,r^4+32\,r^3\,M\cos(\theta)^2-32\,r^3\,M +24\,r^2\,M^2\cos(\theta)^2-24\,r^2\,M^2+8\,r\,M^3\cos(\theta)^2-8\,r\,M^3+M^4\cos(\theta)^2-M^4)\,/\,r\,,\,0\right]$$

$$\left[-\frac{1}{16}\cos(\theta)\,aa\,(16\,r^4\cos(\theta)^2-16\,r^4+32\,r^3\,M\cos(\theta)^2-32\,r^3\,M+24\,r^2\,M^2\cos(\theta)^2-24\,r^2\,M^2+8\,r\,M^3\cos(\theta)^2 -8\,r\,M^3+M^4\cos(\theta)^2-M^4)\Big/r^2,\,\frac{1}{16}\sin(\theta)\,aa\,(16\,r^4\cos(\theta)^2-16\,r^4+32\,r^3\,M\cos(\theta)^2-32\,r^3\,M +24\,r^2\,M^2\cos(\theta)^2-24\,r^2\,M^2+8\,r\,M^3\cos(\theta)^2-8\,r\,M^3+M^4\cos(\theta)^2-M^4)\,/\,r\,,-\frac{1}{16}(-8\,r\,M^3\cos(\theta)^2+32\,r^3\,M +24\,r^2\,M^2+8\,r\,M^3+M^4-24\,r^2\,M^2\cos(\theta)^2-32\,r^3\,M\cos(\theta)^2+16\,r^4-16\,r^4\cos(\theta)^2-M^4\cos(\theta)^2 +8\,r^3\,M^3\,aa^2\cos(\theta)^4-48\,r^4\,M^2\,aa^2\cos(\theta)^2+32\,r^5\,M\,aa^2\cos(\theta)^4-16\,r^3\,M^3\,aa^2\cos(\theta)^2+M^4\,aa^2\,r^2\cos(\theta)^4 -2\,M^4\,aa^2\,r^2\cos(\theta)^2-64\,r^5\,M\,aa^2\cos(\theta)^2+24\,r^4\,M^2\,aa^2\cos(\theta)^4+32\,r^5\,M\,aa^2+24\,r^4\,M^2\,aa^2+8\,r^3\,M^3\,aa^2 +M^4\,aa^2\,r^2+16\,r^6\,aa^2\cos(\theta)^4-32\,r^6\,aa^2\cos(\theta)^2+16\,r^6\,aa^2)\Big/r^2,\,0\right]$$

$$\left[0\,,0\,,0\,,\frac{(-2\,r+M)^2}{(2\,r+M)^2}\right]$$

```
> for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do ss:=0; ss
  := (CC[i,j,k]-C2S[i,j,k]); SHIPTR[i,j,k]:=simplify(ss)  od od od   ;
>
>
> for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do if
  C2S[i,j,k]=0 and CC[i,j,k]=0 then else
  print(`T`(i,j,k)=simplify(subs(aa=0,SHIPTR[i,j,k]))) fi od od od ;
```

$$\mathrm{T}(1,\,1,\,1)=2\,\frac{M}{r\,(2\,r+M)}$$

$$T(1,1,2)=0$$
$$T(1,1,3)=0$$
$$T(1,2,1)=0$$
$$T(1,2,2)=-2\frac{r M}{2 r+M}$$
$$T(1,2,3)=0$$
$$T(1,3,1)=0$$
$$T(1,3,2)=0$$
$$T(1,3,3)=2\frac{r M (\cos(\theta)^2-1)}{2 r+M}$$
$$T(1,4,4)=64\frac{r^4 M (-2 r+M)}{128 r^7+448 M r^6+672 M^2 r^5+560 M^3 r^4+280 M^4 r^3+84 M^5 r^2+14 r M^6+M^7}$$
$$T(2,1,1)=0$$
$$T(2,1,2)=2\frac{M}{r (2 r+M)}$$
$$T(2,1,3)=0$$
$$T(2,2,1)=2\frac{M}{r (2 r+M)}$$
$$T(2,2,2)=0$$
$$T(2,2,3)=0$$
$$T(2,3,1)=0$$
$$T(2,3,2)=0$$
$$T(2,3,3)=0$$
$$T(2,4,4)=0$$
$$T(3,1,1)=0$$
$$T(3,1,2)=0$$
$$T(3,1,3)=2\frac{M}{r (2 r+M)}$$
$$T(3,2,1)=0$$
$$T(3,2,2)=0$$
$$T(3,2,3)=0$$
$$T(3,3,1)=2\frac{M}{r (2 r+M)}$$
$$T(3,3,2)=0$$
$$T(3,3,3)=0$$
$$T(3,4,4)=0$$
$$T(4,1,4)=4\frac{M}{(-2 r+M) (2 r+M)}$$
$$T(4,4,1)=4\frac{M}{(-2 r+M) (2 r+M)}$$

>

NOW EXAMINE THE EFFECTS OF A TORSION PERTURBATION WITHOUT

A METRIC PERTUBATION

```
> for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do ss:=0; ss
  := (CC[i,j,k]-C2S[i,j,k]); SHIPTR[i,j,k]:=simplify(ss)  od od od    ;
>
>
> for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do if
  C2S[i,j,k]=0 and CC[i,j,k]=0 then else
  print(`T`(i,j,k)=simplify(subs(M=0,SHIPTR[i,j,k]))) fi od od od ;
```

$$T(1, 1, 1) = 2\, aa^2\, r \cos(\theta)^2 - 2\, aa^2\, r \cos(\theta)^4$$

$$T(1, 1, 2) = 2 \cos(\theta)^3\, aa^2\, r^2 \sin(\theta) - \cos(\theta)\, aa^2\, r^2 \sin(\theta)$$

$$T(1, 1, 3) = -r^3 \cos(\theta)\, aa^3 + 2\, r^3 \cos(\theta)^3\, aa^3 - r^3 \cos(\theta)^5\, aa^3$$

$$T(1, 2, 1) = 2 \cos(\theta)^3\, aa^2\, r^2 \sin(\theta) - \cos(\theta)\, aa^2\, r^2 \sin(\theta)$$

$$T(1, 2, 2) = -2\, r^3\, aa^2 \cos(\theta)^2 + 2\, r^3\, aa^2 \cos(\theta)^4$$

$$T(1, 2, 3) = r^4\, aa^3 \sin(\theta) \cos(\theta)^4 - r^4\, aa^3 \sin(\theta) \cos(\theta)^2 + r^2\, aa \sin(\theta)$$

$$T(1, 3, 1) = 2 \cos(\theta)^3\, r\, aa - 2\, aa \cos(\theta)\, r + 2 \cos(\theta)^3\, r^3\, aa^3 - \cos(\theta)^5\, r^3\, aa^3 - \cos(\theta)\, r^3\, aa^3$$

$$T(1, 3, 2) = -2 \cos(\theta)^2\, r^2\, aa \sin(\theta) + r^4\, aa^3 \sin(\theta) \cos(\theta)^4 - r^4\, aa^3 \sin(\theta) \cos(\theta)^2 + r^2\, aa \sin(\theta)$$

$$T(1, 3, 3) = -4\, r^3\, aa^2 \cos(\theta)^2 + 2\, r^3\, aa^2 \cos(\theta)^4 + 2\, r^3\, aa^2$$

$$T(1, 4, 4) = 0$$

$$T(2, 1, 1) = 2 \sin(\theta) \cos(\theta)^3\, aa^2 - 2 \sin(\theta) \cos(\theta)\, aa^2$$

$$T(2, 1, 2) = r\, aa^2 - 3\, aa^2\, r \cos(\theta)^2 + 2\, aa^2\, r \cos(\theta)^4$$

$$T(2, 1, 3) = -aa \sin(\theta) - 2\, aa^3 \sin(\theta)\, r^2 \cos(\theta)^2 + aa^3 \sin(\theta)\, r^2 + aa^3 \sin(\theta)\, r^2 \cos(\theta)^4$$

$$T(2, 2, 1) = r\, aa^2 - 3\, aa^2\, r \cos(\theta)^2 + 2\, aa^2\, r \cos(\theta)^4$$

$$T(2, 2, 2) = -2 \cos(\theta)^3\, aa^2\, r^2 \sin(\theta) + 2 \cos(\theta)\, aa^2\, r^2 \sin(\theta)$$

$$T(2, 2, 3) = -2 \cos(\theta)^3\, r^3\, aa^3 + \cos(\theta)^5\, r^3\, aa^3 + \cos(\theta)\, r^3\, aa^3$$

$$T(2, 3, 1) = aa \sin(\theta) - 2\, aa \sin(\theta) \cos(\theta)^2 - 2\, aa^3 \sin(\theta)\, r^2 \cos(\theta)^2 + aa^3 \sin(\theta)\, r^2 + aa^3 \sin(\theta)\, r^2 \cos(\theta)^4$$

$$T(2, 3, 2) = -2 \cos(\theta)^3\, r\, aa + 2\, aa \cos(\theta)\, r - 2 \cos(\theta)^3\, r^3\, aa^3 + \cos(\theta)^5\, r^3\, aa^3 + \cos(\theta)\, r^3\, aa^3$$

$$T(2, 3, 3) = -2 \cos(\theta)^3\, aa^2\, r^2 \sin(\theta) + 2 \cos(\theta)\, aa^2\, r^2 \sin(\theta)$$

$$T(2, 4, 4) = 0$$

$$T(3, 1, 1) = 2 \frac{aa \cos(\theta)}{r}$$

$$T(3, 1, 2) = -\frac{(2 \cos(\theta)^2 - 1) \sin(\theta)\, aa}{\cos(\theta)^2 - 1}$$

$$T(3, 1, 3) = aa^2\, r \cos(\theta)^2 - r\, aa^2$$

$$T(3, 2, 1) = -\frac{(2 \cos(\theta)^2 - 1) \sin(\theta)\, aa}{\cos(\theta)^2 - 1}$$

$$T(3, 2, 2) = -2\, aa \cos(\theta)\, r$$

$$T(3, 2, 3) = -\cos(\theta)\, aa^2\, r^2 \sin(\theta)$$

$$T(3, 3, 1) = aa^2\, r \cos(\theta)^2 - r\, aa^2$$

$$T(3, 3, 2) = -\cos(\theta)\, aa^2\, r^2 \sin(\theta)$$

$$T(3, 3, 3) = 0$$

$$T(3, 4, 4) = 0$$

$$T(4, 1, 4) = 0$$

$$T(4, 4, 1) = 0$$

>

NOW EXAMINE THE EFFECTS OF WITHOUT METRIC PERTURBATIONS AND WITHOUT TORSION PERTUBATIONS

```
> for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do ss:=0; ss
  := (CC[i,j,k]-C2S[i,j,k]); SHIPTR[i,j,k]:=simplify(ss)  od od od   ;
>
>
> for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do if
  C2S[i,j,k]=0 and CC[i,j,k]=0 then else
  print(`T`(i,j,k)=simplify(subs(M=0,aa=0,SHIPTR[i,j,k]))) fi od od od ;
```

$$T(1, 1, 1) = 0$$

$$T(1, 1, 2) = 0$$

$$T(1, 1, 3) = 0$$

$$T(1, 2, 1) = 0$$

$$T(1, 2, 2) = 0$$

$$T(1, 2, 3) = 0$$

$$T(1, 3, 1) = 0$$

$$T(1, 3, 2) = 0$$

$$T(1, 3, 3) = 0$$

$$T(1, 4, 4) = 0$$

$$T(2, 1, 1) = 0$$

$$T(2, 1, 2) = 0$$

$$T(2, 1, 3) = 0$$

$$T(2, 2, 1) = 0$$

$$T(2, 2, 2) = 0$$

$$T(2, 2, 3) = 0$$

$$T(2, 3, 1) = 0$$

$$T(2, 3, 2) = 0$$

$$T(2, 3, 3) = 0$$

$$T(2, 4, 4) = 0$$

$$T(3, 1, 1) = 0$$

$$T(3, 1, 2) = 0$$

$$T(3, 1, 3) = 0$$

$$T(3, 2, 1) = 0$$

$$T(3, 2, 2) = 0$$

$$T(3, 2, 3) = 0$$

$$T(3, 3, 1) = 0$$

$$T(3, 3, 2) = 0$$

$$T(3, 3, 3) = 0$$

$$T(3, 4, 4) = 0$$

$$T(4, 1, 4) = 0$$

$$T(4, 4, 1) = 0$$

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CONCLUSION. The matrix elements of an integrable frame field [F] can be perturbed such that the projected 1-forms on the final state are not integrable. That is, the 1-forms created by the matrix operation [F] |dR> are not necessarily closed, and have exterior derivatives that do not vanish. When the pertubations are of the spatial vorticity type, Torsion coefficients will appear in the right Cartan connection. Perturbations of the metric type lead to the Schwarzshild metric in terms of the Mass parameter M. The combined Schwarzschild with torsion metric is given by the expression:

```
> Schwarzschild_with_torsion:=array(subs(M=M,aa=aa,([[g11,g12,g13,g14],[g21,g22,g23,g24],[g31,g32,g33,g34],[g41,g42,g43,g44]])));
```

Schwarzschild_with_torsion :=

$$\left[-\frac{1}{16}\frac{(2r+M)^4}{r^4}, 0, -\frac{1}{16}\frac{\cos(\theta)(2r+M)^4 aa(\cos(\theta)-1)(\cos(\theta)+1)}{r^2}, 0\right]$$

$$\left[0, -\frac{1}{16}\frac{(2r+M)^4}{r^2}, \frac{1}{16}\frac{\sin(\theta)(2r+M)^4 aa(\cos(\theta)-1)(\cos(\theta)+1)}{r}, 0\right]$$

$$\left[-\frac{1}{16}\frac{\cos(\theta)(2r+M)^4 aa(\cos(\theta)-1)(\cos(\theta)+1)}{r^2}, \frac{1}{16}\frac{\sin(\theta)(2r+M)^4 aa(\cos(\theta)-1)(\cos(\theta)+1)}{r},\right.$$

$$\left.-\frac{1}{16}\frac{(2r+M)^4(\cos(\theta)-1)(\cos(\theta)+1)(aa^2 r^2\cos(\theta)^2 - aa^2 r^2 - 1)}{r^2}, 0\right]$$

$$\left[0, 0, 0, \frac{(-2r+M)^2}{(2r+M)^2}\right]$$

Note that the Schwarzschild with torsion metric is not diagonalized.

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