Nanometer Vortices

R. M. Kiehn

Emeritus, Phys Dept., Univ. Houston
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http://www.cartan.pair.com
rkiehn2352@aol.com

Abstract

An exact complex mapping is used to show the close relationship between the Schroedinger equation for a charged particle in an external field, the Landau-Ginsburg equation for the order parameter in a superconductor, and the equation for the evolution of vorticity in a Navier-Stokes compressible, viscous, fluid. The absolute square of the wave function, or complex order paramter, is exactly equal to the vorticity distribution (including topological vorticity defects) in a fluid with a viscosity coefficient defined as $\nu = h/2\pi m_0$. This cohomological, but classical, interpretation of the wave function, offers an alternative to the Copenhagen dogma, and has application to the understanding of what recently have been described as superconducting vortices. However, the objects of experimental interest are better described as domains of quantized circulation, not vorticity. The vorticity distribution will contain topological circulation defects of null vorticity, for which the circulation integral around a closed path is quantized by deRham's theorems.

1. Introduction

Starting with Madelung [1], the physics literature contains many examples of the similarities between Schroedinger's quantum mechanics and fluid dynamics. The search for an interpretation of the square of the wave function, which satisfies a hydrodynamic-like equation of continuity, motivated many of the historical hydrodynamic interpretations and investigations. However, as the Schroedinger equation presumably is to represent reversible Hermitean operators, almost all historical investigations considered only the "ideal" incompressible fluid case without dissipation. That is, the historical work dealt only with those cases that could be interpreted in terms of Eulerian flows. With the advent and dominance of the Copenhagen-Born probability interpretation of the square of the wave function, the early hydrodynamic interpretations of the Schroedinger theory have been treated only as curiosities.

However, about mid 20th century, Landau and Ginsburg developed the theory of the order parameter, with the linearized version exacly equal in format to the Schroedinger equation for a charge particle in an external field. The complex wave funtion had a modulus which was interpreted as an order parameter distribution, not a Copenhagen probability distribution. Other subtle differences involved the boundary conditions on finite domains, such that in the Landau-Ginsburg formalism the coherence function boundary condition is dominated by the gradient of complex "wave funtiion" where in the Schroedinger problem the function is dominated by the value of the "wave function".

Herein, the point of departure is due partially to a serendipity event (1976), whereby during the study of problems in topological evolution, it was discovered that a certain complex mapping would convert the two dimensional time dependent Schroedinger equation (and therefore the linearized Landau-Ginzburg equation) exactly into the format of a *viscous* compressible Navier-Stokes fluid. In this new representation, the square of the wave function has an explicit and novel interpretation as the *distribution of vorticity in a viscous fluid*. The viscosity coefficient was identified as the ratio of Planck's constant to the effective mass: $\nu = h/2\pi m_0$. This point of view, based on a dissipative considerations, is completely different from the Born-Copenhagen hypothesis which interprets the absolute square of the wave funtion as the position probability of the electron.. At that time, the Bohm theory of Quantum Mechanics was unknown to the present author. The Bohm theory also transforms the Schroedinger equation by using the polar representaiton of a complex number for the wave function. The new Bohm feature is the introduction of the quantum potential.

The original application by this author of these ideas was to the problems of generating topological defects and disclinations in a fluid [2], using a "quantum picture" to generate a raison d'etre for the onset of turbulence in terms of "vortex pair production". In recent times, the focus of attention on high Tc superconducting films and nano structures has reversed the situation to where again a hydrodynamic perspective is being used to model quantum phenomena. The key feature in the application to nanometer superconductivity is that the *distribution of vorticity* corresponds to the penetration of the **B** field into internal regions of the superconductor. In hydrodynamics, the vorticity is defined as the curl of the velocity field. In electromagnetism, the magnetic intensity is defined as the curl of the vector potential. The vorticity distribution can have topological defect structures which are domains where the vorticity (or magnetic field) vanishes, but the velocity field (or vector potential) is finite. It is remarkable that these defect regions can support topological structures such that the closed circulation integral is not zero, and yet the vorticity is exactly zero. The defect regions are where Stokes theorem fails.

In a planar system these topological defects are "holes" in the otherwise perfect superconductor, and are described by deRham period integrals representing quantized flux [3]. The original conjecture is that the technology and experience gained from almost seventy five years of quantum mechanics should be applicable to the hydrodynamic problem of viscous compressible, and perhaps turbulent flow, and visa versa, the experience of viscous compressible rotational hydrodynamics should be applicable to quantum problems, because the transformation between the two systems of partial differential equations is exact. The recent interest in nanometer superconducting structures [18] revitalizes the Navier-Stokes interpretation of the Landau-Ginsburg equation.

1.1. Hydrodynamic considerations

At first, the present author was led to speculate that this exact correspondence between hydrodynamics and quantum mechanics is more than accidental, and that it would be meaningful to apply simple quantum theory results and concepts to the problems of hydrodynamics, and conversely, certain hydrodynamic results might have exploitation in quantum theory. In the hydrodynamic point of view presented below, an elemental vortex takes over the role of the elemental charge. It follows that questions of elemental vortex pair production, vortex effective mass, vortex "Compton" size in a fluid can be answered relying on experience with quantum theory results. Some of these results are presented below. In particular, it is to be noted that the theory predicts in a fluid the existence of an elemental hydrodynamic vortex mass of order of $\hbar/\nu$, where $\hbar = h/2\pi$ is Planck's constant and ν is the kinematic viscosity of the fluid. On the otherhand, the theory indicates that in a quantum system there is an effective viscosity coefficient ν to be measured in terms of $\hbar/m$ where h is Planck's constant and m is the effective inertia or mass of the flux quantum hole.

The dual question arises: What observational and theoretical results of hydrodynamics that can be exploited to yield new perspectives about the quantum theory, especially the quantum theory of superconductivity, and what aspects of quantum mechanics can be used to give an insight into the problems of hydrodynamcs? The most dramatic concept is that stated above: The square of the wave function in quantum mechanics has a novel interpretation as the vorticity distribution in a *compressible viscous* fluid. It seems almost paradoxical that a viscous compressible approach would have meaning in a "non-dissipative" quantum system, such as a superconductor. But this is not the only concept that may have application leading to a better understanding of the quantum phenomena of superconductivity.

For example, consider the almost 200 year old observation of coherent solitary waves in hydrodynamic flows. One of the most important applications of hydrodynamic theory in explaining this phenomena was due to Betchov and Hasimoto, who created hydrodynamic string soliton solutions from a self interacting vortex lines. The remarkable result is that these soliton solutions can also be derived from the non-linear Schroedinger equation [4]. The string solitons may be viewed as the topological equivalent of flux penetrations into a type II superconductor. In hydrodynamic experiments, Hopfiinger has observed the propagation of solitary waves along these topological discontinuity threads in a turbulent flow [5]! Such vortex strings present in a turbulent flow is an experimental research area actively being studied by Coudy [6]. More recently, it has been established that helical threads of vorticity are artifacts of irreversible turbulent flow [7]. Coherent wave packets have been of considerable interest and value in quantum mechanics, but they were predated in terms of hydrodynamic discoveries by more than 100 years.

The second and more recent experimental hydrodynamic observation is that of the Falaco vertex pair [7], which is a rotation-induced soliton that is easily produced in the density discontinuity surface, or free surface, of a fluid. The Falaco vertex is a topological defect in a viscous fluid, but due to its coherence it can form a long-lived metastable state in which two opposite "spins" are paired together. The observational evidence implies that a pair of two dimensional rotational defects are created in the discontinuity free surface of a fluid, and these two dimensional topological surface defects are connected by a string, or a one dimensional topological defect, to form a globally stabilized stationary state. See Figure 1. The visual evidence indicates that such an object is the topological equivalent of a macroscopic hadron, an idea which is based on the fact that a mechanical disruption of the (normally invisible) connecting string causes the observational surface defects (the quarks on the end of a string) to disappear abruptly, and in a non-diffusive manner. The fluid defect exhibits the problems of quark confinement! However, as the effect is a topological effect, it should appear at all scales. The possible practical application to quantum theory is that the Falaco vertex pair is another spin pairing mechanism (distinct from the BCS mechanism) that can occur in any discontinuity surface, including the Fermi surface. Hydrodynamically, it is observed that the "communication" of the coherent endcaps of the Falaco vertex is via transverse, not longitudinal, waves guided by the circular arc that forms the connecting string. The observation is similar to the propagation of electrons injected into the earths magnetic field, which spiral from one magnetic focus point to another, guided and trapped by the magnetic field. As the equations of a viscous compressible fluid and the Schroedinger equation are equivalent under a complex mapping, then these hydrodynamic comparisons should have direct application in the arena of quantum mechanics.

Other observational evidence concerning the Falaco soliton indicates that it is associated with a minimal surface defect, and although locally unstable, the Falaco soliton, like a minimal surface soap-film, is globally stabilized to yield a finite lifetime. The analogue between the Fermi surface and triply periodic minimal surfaces [8] suggests that the discontinuity surface in a superconductor is the Fermi surface.

The presentation below consists of several sections. The next section describes the analytic details of the transformation that maps the Schroedinger equation to the Navier-Stokes equation, and a subsidiary equation related to differential concomitants of the differential geometry of a surface. The third section discusses the key experimental properties of the Falaco soliton, which motivated much of the discussion presented herein. The fourth section uses simple quantum results of pair production to give a raison d'etre for what is known as the drag crisis in fluid flow. The fifth section applies the methods to the problem,s of nanometer vortex structures in superconductors. The sixth section exploits the recent results of minimal surface theory, its relationship to a variational principle on Finsler spaces and fractal sets, and the search for tangential discontinuities in the solutions to the Schroedinger equation.

2. The Transformation (Two spatial dimensions)

Consider the two dimensional Schroedinger equation for a charged particle of mass m, charge q, interacting with an external electromagnetic field represented by a vector potential, $\mathbf{A}$, and a scalar potential, ϕ . [9]

$$-(\hbar/i)\partial\Psi/\partial t = \{(\hbar/i)\nabla - q\mathbf{A}\}\{(\hbar/i)\nabla - q\mathbf{A}\}\Psi/2m + q\phi\Psi \tag{2.1}$$

Consider the complex transformation given by the expression,

$$\Psi \;= exp\{(1/2 - im\nu/\hbar)\ln\zeta\} = exp\{(1/2 - i\nu/\nu_0)\ln\zeta\} \tag{2.2}$$

where m is the electron mass, h is Planck's constant, ν is an arbitrary constant later to be identified as the kinematic viscosity, and $\zeta \equiv \Psi^*\Psi$ is a function of $\{x, y, t\}$. Note that by using the measured kinematic viscosity an elemental mass $\mathrm{m}_0 = \hbar/\nu$ can be determined. Alternately, using the mass of the electron, there is a minimum kinematic viscosity of the order of $\nu_0 = \hbar/m$.

Direct substitution of this wave function into the Schroedinger equation, followed by separation into real and imaginary parts yields two sets of partial differential equations. The imaginary set yields the equation,

$$\begin{aligned} & \partial\zeta/\partial t + u\partial\zeta/\partial x + v\partial\zeta/\partial y + \zeta(\partial u/\partial x + \partial v/\partial y) \\ = \;& \nu(\partial^2\zeta/\partial x^2 + \partial^2\zeta/\partial y^2), \end{aligned} \tag{2.3}$$

where the components of the vector velocity $\mathbf{V} = [u(x, y, t), v(x, y, t), 0]$ are identified with the components of the vector potential by means of the equations,

$$\mathbf{V} = [u, v, 0] = -q\mathbf{A}/m = (-q/m)[A_x, A_y, 0]. \tag{2.4}$$

Now consider the two dimensional Navier-Stokes vorticity equation for a compressible viscous barotropic fluid. Identify the single single component of vorticity as $curl\mathbf{V} = [0, 0, \zeta(x, y, t)]$, and substitute into the full Navier-Stokes equation

$$\begin{aligned} & \partial\boldsymbol{\omega}/\partial t + \mathbf{V}\circ\nabla\boldsymbol{\omega} - \boldsymbol{\omega}\circ\nabla\mathbf{V} + \boldsymbol{\omega}\, div\, \mathbf{V} = \\ = \;& \nu(\partial^2\boldsymbol{\omega}/\partial x^2 + \partial^2\boldsymbol{\omega}/\partial y^2 + \partial^2\boldsymbol{\omega}/\partial z^2). \end{aligned} \tag{2.5}$$

The result is equation (3) above. Note that

$$curl\mathbf{V}\circ curl\mathbf{V} = \zeta^2 = \Psi^*\Psi\cdot\Psi^*\Psi, \tag{2.6}$$

which establishes the relationship between the enstrophy of the fluid flow and the norm of the wave function. The two dimensional result is especially simple for the vortex stretching term of fluid mechanics vanishes, $\boldsymbol{\omega}\circ\nabla\mathbf{V}\Rightarrow 0$.

In this section, and in most of this article, a restriction to the two dimensional case will be subsumed. (However see the section below in three spatial dimensions). Although the equation (3) is expressed in terms of the two spatial variables, x and y, these coordinates are to be considered the abstract coordinates of a surface which may have curvature and topological defects of holes and self-intersections. The solutions to these Navier-Stokes equations are not necessarily planar, but correspond to those domains where the vector flow field generates a Pfaffian field of class (or Pfaff dimension) 3. This means that such time dependent vector fields can be mapped to a space of three functions, preserving most of their topological features. Such flows are not necessarily Frobenius integrable [10], and they need not be exact. (That is, they can support Topological Torsion)

The real parts of the substitution of (2) into (1) yield the equations

$$(\hbar/m^*)^2[+\nabla^2\ln\zeta + (1/2)(\nabla\ln\zeta)^2] = 1/2(\mathbf{V}\circ\mathbf{V}) + q\phi/m + \nu\, div\mathbf{V}, \tag{2.7}$$

As mentioned above, equation (3) is to be recognized as the exact Navier-Stokes-Helmholtz equation for the z-component of vorticity, ζ, of a compressible viscous fluid. Equation (7) is less transparent, but remarkably this equation represents (on the LHS) a differential geometry constraint between the extrinsic curvature invariants of a surface given by the Monge function,

$$z = \text{constant } ln\zeta(x,y,t) \tag{2.8}$$

and (on the RHS) the energy per unit mass of the fluid, similar in principle to the geometric relation expressed by the mean curvature equation of differential geometry [11]. By forming the shape matrix, Σ, of the surface and the induced metric, g, the LHS of (5) is found to be proportional to (Trace Σ)(det g)$^{1/2}$ - 1/2(det g -1). Recall that Trace Σ is related to the mean curvature of the surface, a result that will be utilized in the last section of this article which discusses the Schroedinger theory and minimal surfaces. The idea that there is a geometrical connection to energy density is the theme of General Relativity, but there the Einstein Ansatz, $G^\mu_\nu = T^\mu_\nu$, is an assumption. Under the mapping described above the curvature-energy Ansatz is derived as a consequence, not injected as an axiom, of the theory.

The elemental vortex mass, m^*, defined such that $(\hbar/m^*)^2 = (\nu^2/4 + \nu_0^2)$, appears in the theory in a manner similar to the effective "hole" mass in the theory of semiconductors. Along with Popov [12] it is argued that the system described is that of a "molecule" with elementary vortex excitations (spin up and spin down) interacting through a wave field with a characteristic velocity of propagation. The idea that m^* is the equivalent "inertial" mass of the elementary vortex is strengthened by the realization that $\zeta \equiv \Psi^*\Psi$ is the "vorticity" distribution in the fluid, exactly. Representative values for m^* are given in Table 1 for a gas, two fluids, and a liquid metal, based on the assumption that the characteristic speed is the speed of sound, Cs. Again using the experience of quantum mechanics, an estimate may be made for the effective (core) size of the elemental vortex: it should be of the order of the Compton wavelength based upon the elemental mass and the characteristic limiting velocity, $= (h/(2m^*Cs))$. This "sonic" Compton wavelength for several fluids is also estimated in Table 1. It appears that the size of the elemental vortex cores in classic fluids is of the order of a few Angstroms!

3. The Falaco Soliton

During the summer of 1986, while visiting an old friend in Rio de Janeiro, Brazil, the present author became aware of a significant topological experiment that could be observed in a hydrodynamic system. This observation greatly stimulated the author to further research in applied topology, and led to many of the ideas presented in this article. To replicate the experiment, inject kinetic energy and angular momentum into a stratified fluid with a free surface (a swimming pool) by stroking a half submerged, flat, circular plate in a direction parallel to its oblate axis. Remove the plate at the end of the stroke to produce, initially, a pair of ordinary Rankine vortices in the surface of the density discontinuity. These Rankine vortices cause the initially flat surface of discontinuity to form a pair of parabolic concave indentations, indicative of the "rigid body" rotation of a pair of contra-rotating vortex cores of uniform vorticity. However, in a matter of a few seconds each concave shape will decay into the metastable soliton configuration of an inverted hyperbolic convex dimple of negative Gaussian curvature. The dimple depression is usually of the order of a few millimeters, but the circulation zone typically extends over a disc of some 10 to 15 centimeters or more, depending on the plate diameter. This configuration, or coherent structure, has been defined as the Falaco Soliton. See Figure 1.

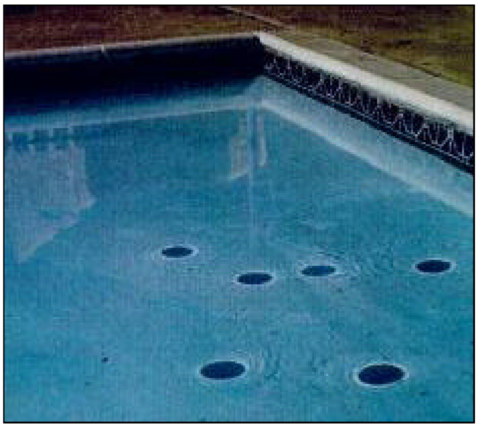


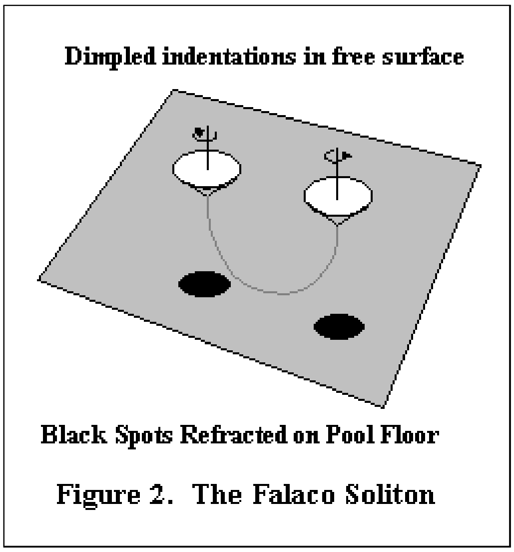
Figure 3.1: **Falaco Solitons**

For purposes of illustration, the vertical depression has been greatly exaggerated. The Falaco Solitons will persist for many minutes in a still pool of water.

The effect is easily observed, for in strong sunlight the convex hyperbolic indentation will cause an intensely black circular disk (or absence of light) to be imaged on the bottom of the pool. A bright ring of focused light will surround the black disk, emphasizing the contrast. The optics of the problem are completely described by Snell refraction from a surface of revolution that has negative Gauss curvature. See Figure 3. This effect has been reported upon elsewhere, but the figures are replicated herein for clarity [13].

Dye injection near an axis of rotation during the formative stages indicates that there is a unseen thread, or 1-dimensional string singularity, in the form of a circular arc that connects the two 2-dimensional surface singularities or dimples. Transverse waves can be observed to propagate from one dimple vertex to the other dimple vertex, guided by the “string“ singularity. If the string is “severed“, the confined, two dimensional endcap singularities do not diffuse away, but instead disappear almost explosively. It is as if the Falaco soliton is the macroscopic topological equivalent of the illusive hadron in elementary particle theory, where two 2-dimensional surface defects (the quarks) are bound together by a string of confinement.

As the phenomena appears to be the result of a topological defect, it follows that, as a topological property of hydrodynamic evolution, it should appear in any


Dimpled indentations in free surface
Black Spots Refracted on Pool Floor
Figure 2. The Falaco Soliton

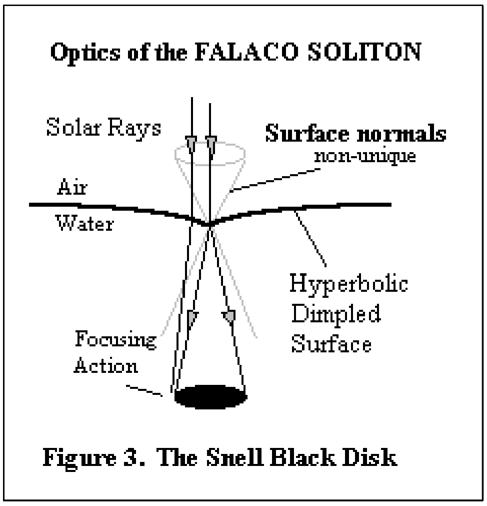

Optics of the FALACO SOLITON
Solar Rays
Surface normals
non-unique
Air
Water
Hyperbolic
Dimpled
Surface
Focusing
Action
Figure 3. The Snell Black Disk

Figure 3.2:

density discontinuity, at any scale. This spin pairing mechanism, as a topological phenomenon independent from size and shape, could occur at both the microscopic and the cosmic scales. In fact, during the formative stages of the Falaco vertex pair, the decaying Rankine vortices exhibit spiral arms easily visible as caustics emanating from each vortex core. To this author the importance of the Falaco Solitons is that they offer the first clean experimental evidence of topological defects taking place in a fluid. Moreover, the experiments are easily replicated by anyone with access to a swimming pool. They certainly are among the most easily produced solitons. Interactions with a pole placed vertically to the bottom of the pool experimentally emulate the coherent scattering of solitons observed in computer simulations. As the drifting soliton pair interacts with pole, the Snell black discs shimmer and disappear, only to coherently reappear after the soliton pair has passed beyond the interaction zone. For hydrodynamics, the observation firmly cements the idea that these objects are truly coherent structures.

Early on, it was recognized that the Snell refraction on the bottom of the pool produces a circular disk, more or less independent from the angle of solar incidence. This observation, as well as the negative Gauss curvature of the surface, lends further credence to the idea (an idea exploited later on in this article) that the topological surface distortion is a minimal surface. The argument is that only spheres and minimal surfaces have a conformal Gauss map (Snell projection). It should be noted that if chalk dust is sprinkled on the surface of the pool during the formative stages of the Falaco soliton, then the topological signature of the familiar Mushroom Spiral pattern is exposed.

Recall that a Rankine vortex consists of two topologically distinct domains in which a compact rigid body rotation with vorticity is embedded in an exterior irrotational flow with circulation. In the classic analysis [14], the velocity components of a Rankine vortex are matched on a bounding surface of radius R. No estimate of the size of R was given by Rankine, but it is to be noted that for single Rankine vortex the pressure at infinity in an unbounded fluid is not zero [15]. The size of the vortex core, as given by Milne-Thompson, depends upon the circulation and the pressure at infinity. When the Rankine vortex is formed in the free surface of water, the water surface associated with the interior, rotational, domain of the Rankine vortex initially has a concave shape of positive Gaussian curvature. In the exterior domain, which is irrotational, the free surface of the Rankine vortex is of negative Gaussian curvature. It is a result of observation that for certain initial conditions of kinetic energy and angular momentum, the Rankine vortex pair will decay until its rotational residue is almost entirely irrotational, and the free surface is almost entirely of negative Gaussian curvature. A pair of cusp-like singularities form in the free surface. This soliton state is remarkably long lived, and easily observed through its striking Snell refraction of overhead sunlight that produces a dark disk, or absence of light, in its image. It is not clear experimentally if there exists a residual elemental domain of vorticity, an elemental vortex whose solenoidal core dimension is very small. If the estimates of Table 1 are true, then this elemental core size could be well below the limits of visual sensitivity.

These discrete sets of potential vortex pairs are globally stabilized by a "spin-pairing" mechanism, producing the equivalent of a macroscopic hadron in a surface of discontinuity in a real fluid. The coherent solitary wave states are easily produced in the discontinuity surface of a fluid in contact with the atmosphere. As Voropayev has shown very recently [16], these "mushroom dipoles" exhibit quantized behavior in the decay of a turbulent fluid. The result is not surprising when it is realized that these Falaco vortices are topological singularities, and therefore can appear at all scales. Turning the argument around, the concept also permits the development of another mechanism for producing spin-pairing of electrons in the discontinuity of the Fermi surface. This method depends upon transverse wave, not longitudinal sound waves as in the BCS theory.

The moral of this story is that as there are experimental evidences of topological (hence quantized) states in the discontinuity sets of viscous, compressible hydrodynamics, it should be expected that similar phenomena should be observable in quantum mechanics, for the partial differential equations of both evolutionary systems are transformable into one another. The question arises, where are the tangential discontinuities in the solution set to the Schroedinger equation?

4. The Drag Crisis and Nanometer Vortex Pair Production

Again following the experience of quantum theory, it would be expected that elemental vortex pairs would be produced at some level of energy density, in analog with charge pair production in quantum physics. An estimate can be made in terms of the "mass" equivalent energy of the elemental vortex pair. That is, it would be expected that when the "field" energy density over a domain of the size of the Compton wavelength exceeds twice the mass equivalent energy of the elemental vortex pair, then a pair of vortices could be created. The mass equivalent energy is to be computed relative to the limiting speed, which in this case is the (longitudinal) speed of sound. The field energy density is the energy associated with the square of the vorticity, or the enstrophy, of the flow.

To summarize, elemental vortex pair production should spontaneously occur when

$$\text{Volume} \circ \text{field energy density} = 2(\text{mass energy})$$
$$\text{where mass energy} = m \cdot C_s^2$$
$$\text{volume} = \lambda_s^3 \ ,$$
$$\text{field energy density} = \text{density x enstrophy x length}^2 \ .$$

For Poiseuille flow, the maximum enstrophy occurs near the walls and the energy density is given by the expression

$$E(wall) = \zeta^2 \rho \pi b^2 / 16. \tag{4.1}$$

In terms of the Reynolds number (Rey) for the flow down a pipe, the critical expression for elemental vortex production is given by the equation

$$\operatorname{Re} y_c^2 (critical value) = (2b^2 m^* / \pi \rho)(C_s/\nu)^5. \tag{4.2}$$

Representative values of Rey_c for b = 1 cm appear in Table 1. The fact that these values are even crudely comparable to macroscopic measurement of the onset of the drag crisis [17] or turbulence is remarkable, and give credence to the idea that the onset of turbulence, or the perhaps the drag crisis, might be due to the production of elemental domains, or defects, of vorticity within an embedding environment of laminar flow of null helicity. The other aspect of this work is that it puts a lower bound on the size of filamentary vortex. As a vortex represents a rotational mass with angular momentum, and as the quantum hypothesis asserts that the elemental unit of angular momentum must be of the order of Planck's constant, it should not be unexpected to have the continuum theory of fluids fail at some small scale dependent upon some constant angular momentum unit. The analysis given herein gives an estimate of that size limit in terms of a sonic Compton wavelength.

Fluid	Cs	ρ	ν	m_0	λ_s	Rey_c
CO_2	0.26	0.002	0.0743	16	246	$1.47\ 10^2$
C_3H_6O	1.17	0.79	.0272	427	2.3	$6.74\ 10^3$
H_2O	1.49	1.0	.01	116	6.7	$1.81\ 10^5$
Hg	1.45	13.5	.0011	1054	0.8	$4.26\ 10^7$

Cs in m/sec; ρ in gm/cm^3; ν in cm^2/sec, m_0 in electron masses, core size λ_s in Angstroms.

Note that the effective (sonic) Compton wavelength is relative to the speed of sound.

5. Nanometer Vortex Defects in Superconductors

The linearized Landau-Ginzburg equation is of essentially the same differential format as the two dimensional Schroedinger equation. Hence the transformation described above implies that the square of the order parameter is the equivalent to the vorticity distribution in the two dimensional Navier-Stokes fluid. This hydrodynamic interpretation has led to interesting descriptions of nanometer superconductors where surface effects and B field penetrations cannot be ignored [18].

A topological perspective of superconductivity can be based upon two fundamental topological postulates, independent from metric and/or choice of gauge [21]. Without reference to quantum mechanics and without reference to Landau-Ginsburg theory, is is subsumed that an electromagnetic system can be characterized by an exterior differential system involving the differential forms $A, F, G,$ *and* J The basis of this exterior differential system is expressed as the two exterior differential equations

$$Maxwell - Faraday:\ \ F - dA = 0 \tag{5.1}$$

and

$$Maxwell - Ampere:\ \ J - dG = 0. \tag{5.2}$$

The exterior differential equations are naturally covariant in form with respect to all diffeomorphisms, and with respect to all continuous differentiable maps (which do not have to be homeomorphisms) in a retrodictive sense.

By exterior differentiation, these two equations translate into the standard Maxwell - Faraday partial differential equations of induction, in any dimension, and the Maxwell - Ampere partial differential equations which lead to the conservation of charge-currents. On a four dimensional space-time of independent variables, (x, y, z, t) the 1-form of Action (per unit charge) can be written in terms of conventional vector-potential notation as the 1-form

$$A = \Sigma_{k=1}^{3} A_k(x, y, z, t)dx^k - \phi(x, y, z, t)dt = \mathbf{A} \circ d\mathbf{r} - \phi dt. \tag{5.3}$$

Subject to the constraint of the exterior differential system, the 2-form of field intensities, F, becomes:

$$F = dA = \{\partial A_k/\partial x^j - \partial A_j/\partial x^k\} dx^j\hat{}dx^k = F_{jk}dx^j\hat{}dx^k = \mathbf{B}_z dx\hat{}dy ... + \mathbf{E}_x dx\hat{}dt ..., \tag{5.4}$$

where in usual engineering notation,

$$\mathbf{E} = -\partial \mathbf{A}/\partial t - grad\phi, \qquad \mathbf{B} = curl\ \mathbf{A} \equiv \partial A_k/\partial x^j - \partial A_j/\partial x^k. \tag{5.5}$$

The closure of the exterior differential system, $dF = 0$,

$$dF = ddA = \{curl\ \mathbf{E} + \partial \mathbf{B}/\partial t\}_x dy\hat{}dz\hat{}dt - .. + .. - div\,\mathbf{B} dx\hat{}dy\hat{}dz\} \Rightarrow 0, \tag{5.6}$$

generates the Maxwell-Faraday partial differential equations.:

$$\{curl\ \mathbf{E} + \partial \mathbf{B}/\partial t = 0, \qquad div\,\mathbf{B} = 0\}. \tag{5.7}$$

The component functions ($\mathbf{E}$ and $\mathbf{B}$) of the 2-form, F, transform as a covariant tensor of rank 2. The topological constraint that F is exact, implies that the domain of support for the field intensities cannot, in general, be compact without boundary. The integral of the 2-form F over any closed 2-manifold is a deformation (topological) invariant of any evolutionary process that can be described by a singly parameterized vector field.

On a four dimensional domain of independent variables, the existence of a N-2 form density, G, is assumed to be given by the expression,

$$G = G^{34}(x, y, z, t)dx\hat{}dy ... + G^{12}(x, y, z, t)dz\hat{}dt ... = \mathbf{D}^z dx\hat{}dy ... \mathbf{H}^z dz\hat{}dt ... \tag{5.8}$$

Exterior differentiation produces an N-1 form,

$$J = \mathbf{J}^z(x,y,z,t)dx\hat{}dy\hat{}dt... - \rho(x,y,z,t)dx\hat{}dy\hat{}dz. \tag{5.9}$$

Matching the coefficients of the exterior expression $dG = J$ leads to the Maxwell-Ampere equations,

$$curl\,\mathbf{H} - \partial\mathbf{D}/\partial t = \mathbf{J} \quad and \quad div\,\mathbf{D} = \rho. \tag{5.10}$$

The fact that J is exact leads to the charge conservation law, $dJ = ddG = 0$, or

$$\partial\mathbf{J}^x/\partial x + \partial\mathbf{J}^y/\partial y + \partial\mathbf{J}^z/\partial z + \partial\rho/\partial t = 0. \tag{5.11}$$

The exterior differential system is a topological constraint for by Stokes theorem the support for G can be compact without boundary only if the domain is without charge-currents. The integral of J over a closed 3 dimensional domain is a relative integral invariant (a deformation invariant) of any process that can be described in terms of a singly parametrized vector field.

In addition to the basis exterior differential system given above, there are two additional induced exterior differential systems given by the equations that utilize the concepts of topological torsion, $A\hat{}F$, and topological spin, $A\hat{}G$. The two induced differential systems are:

$$F\hat{}F - d(A\hat{}F) = 0 \tag{5.12}$$

and

$$\{F\hat{}G - A\hat{}J\} - d(A\hat{}G) = 0. \tag{5.13}$$

The exact differential 4 forms, $\{F\hat{}G - A\hat{}J\}$ and $F\hat{}F$ are proportional to the first and second Poincare invariants, and are topological deformation integral invariants.

By direct evaluation of the exterior product, and on a domain of 4 independent variables, each 3-form will have 4 components that can be symbolized by the 4-vector arrays

$$Spin - Current : \mathbf{S}_4 = [\mathbf{A} \times \mathbf{H} + \mathbf{D}\phi, \mathbf{A} \circ \mathbf{D}] \equiv [\mathbf{S}, \sigma], \tag{5.14}$$

and

$$Torsion - vector : \mathbf{T}_4 = [\mathbf{E} \times \mathbf{A} + \mathbf{B}\phi, \mathbf{A} \circ \mathbf{B}] \equiv [\mathbf{T}, h], \tag{5.15}$$

which are to be compared with the charge current 4-vector density:

$$Ch\arg e - Current : \mathbf{J}_4 = [\mathbf{J}, \rho], \tag{5.16}$$

The 3-forms then can be defined by the equivalent contraction processes

$$\begin{aligned} & Topological\ \ Spin\ \ 3 - form \doteq\ A\hat{}G \\ = \ & i(\mathbf{S}_4)dx\hat{}dy\hat{}dz\hat{}dt = \mathbf{S}^x dy\hat{}dz\hat{}dt..... - \sigma dx\hat{}dy\hat{}dz \end{aligned} \tag{5.17}$$

and

$$\begin{aligned} & Topological\ Torsion - helicity\ \ 3 - form \doteq A\hat{}F \\ = \ & i(\mathbf{T}_4)dx\hat{}dy\hat{}dz\hat{}dt =\ \mathbf{T}^x dy\hat{}dz\hat{}dt..... - hdx\hat{}dy\hat{}dz. \end{aligned} \tag{5.18}$$

Guided by experience in hydrodynamics, it is possible to solve for evolutionary vector fields, V, such that the 1-form of virtual work, $W = i(V)dA$, is zero (leading to Eulerian-Hamiltonian flows, $W = 0$), exact (leading to Bernouilli flows, $W = d\Theta$), closed (leading to Stokes flows, $dW = 0$) or not closed (leading to Navier-Stokes flows, $dW \neq 0$). A 4-dimensional super current J_s will be defined in terms of a special 4-vector evolutionary direction field that produces zero interaction with the potentials in a Lagrangian sense. This definition includes the stringent constraints the the Lorentz force is zero, and that there is no dissipative power ($\mathbf{J}_s \cdot \mathbf{E} = 0$). There are many 4-vector direction fields that are orthogonal to the one form of Action, but there is a unique direction field such that the virtual work is proportional to the 1-form of Action (similar to the London approximation for the supercurrent). From the definition of virtual work for an evolutionary process it is presumed that supercurrent has a direction field such that

$$W = i(J_s)dA \Longrightarrow\ \alpha A. \tag{5.19}$$

Such a class of vector fields, J_s, guarantees that the Lagrangian interaction energy density vanishes, for

$$i(J_s)i(J_s)dA = i(J_s)A = \{A \cdot J_s - \rho_s\phi\} = 0. \tag{5.20}$$

It will be presumed that the supercurrent is of the class of vector direction fields that produces no interaction energy density and includes a possible non-closed component such that the virtual work is proportional to Acion 1-form. However this special component of the supercurrent direction field is such that the evolution of the Action is in the unique direction of the Torsion vector, $\mathbf{T}_4$. The space time constraint that the virtual work 1-form, W, be proportional to the Action 1-form, A, is equivalent to the additioinal restriction that this special, non-Hamiltonian component of the super current satisfies the equation:.

$$\mathbf{J}_4 = \beta\, \mathbf{T}_4 = \beta\, [\mathbf{E} \times \mathbf{A} + \mathbf{B}\phi, \mathbf{A} \circ \mathbf{B}]. \tag{5.21}$$

This statement is to be compared below to the London hypothesis, where the 3-vector current density is presumed to be proportional to the 3 vector potential, $\mathbf{J} = \lambda \mathbf{A}$.

THIS SECTION TO BE EXPANDED

6. Discontinuities in Hydrodynamics and Quantum theory

The original emphasis in this article was directed towards exploiting the connection between quantum mechanics and hydrodynamics to gain an insight into certain problems of hydrodynamics. In this last section the emphasis is reversed and several suggestions and conjectures are proposed. The basic question is: What fluid concepts can lead to insights into quantum mechanics?

The new interpretation of the square of the wave function as a vorticity distribution leads to the concept that quantum waves should have several measureable holographic features. That is, counter to current dogma, both phase and amplitude should be of importance to measurable quantum phenomena. The Bohm-Aharanov effect is indeed evidence that such phase effects lead to measurable differences, but note that this effect has only recently been measured, in a convincing closed geometry, by techniques built upon electron holography [19].

The usual representations of the "particle" momentum operator in quantum mechanics are by gradient operations, which induce zero curl components, hence no vorticity. When the particle momentum is augmented by the field momentum, $\mathbf{A}$, then curl components can survive, and are proportional to the $\mathbf{B}$ field. It is the vector potential that is the key. The non-zero values of div $\mathbf{A}$ and curl $\mathbf{A}$ that are to be associated with hydrodynamic compressibility and dissipation. From the hydrodynamic experience, it the $\mathbf{A}$ field is harmonic, such that $grad\, div\mathbf{A} + curl\, curl\mathbf{A} = 0$, then there is no dissipation, even in the presence of a viscosity coefficient. Those charge distributions and motions that lead to harmonic vector potentials somehow must be related to superconductivity phenomena. From Sophus Lie it is known that harmonic vector fields are generators of minimal surfaces, hence the argument presented above implies that superconductivity should be associated with minimal surfaces. In fluids, minimal surfaces are associated with tangential discontinuities. So the question arises, where are the discontinuity solutions, the wave front solutions, in quantum mechanics?

In elementary quantum mechanics, continuity of wave function value and slope are the dogmatic rules. However, continuity of probability current and the square of the real part of the wave function can be satisfied by other boundary conditions. Namely, if the slope of the wave function is continuous, but the amplitudes are of equal value but of opposite sign, the $\Psi^*\Psi$ is still continuous and so is $\Psi^* grad\Psi - \Psi grad\Psi^*$. These "equal value but of opposite sign" boundary conditions lead to discontinuity solutions and propagating wave fronts in quantum systems.

7. Reprise

The basic ideas in this article were first conceived about 1976, and then later modified under the title "An Interpretation of the Wave Function as a Cohomological Measure of Quantum Vorticity". (Phys Rev letters rejected the first draft. No one seemed interested in the idea that turbulence could be related to what now would be called nanometer vortices) The work, however, was accepted for presentation at the Helsinki conference on the Foundations of Quantum Mechanics, but due to other commitments I could not attend the Helsinki meeting. It was not until 1986, with the recognition of the Falaco Solitons, that the interesting mapping oddity took on a more important character. The basic ideas were presented later in Florida,1989, but all without the author's appreciation of Bohmian quantum mechanics. However, like Catastrophe theory, Bohmian QM is based upon gradient fields, such that the possibility of torsion fields, and other interesting topological features are excluded. Herein, the 1976 concepts are reinterpreted in terms of Bohmian like ideas, mostly to stimulate (and stimulated by) J. Sarfatti. The method has its limitations to 2D + time geometry, and 3D+time versions have not been found. A conjecture is that if the map was written in terms of Clifford methods (quaternions) then the full 3D Navier-Stokes equations would be found. Some of the new references are incomplete, as I do not have easy access to a library here in the Vaucluse area of southern France.

For more detail, see

http://www22.pair.com/csdc/car/carhomep.htm

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