TOPOLOGICAL PHASE TRANSITIONS IN A PERTURBED STOKES FLOW

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Abstract

1 Introduction

In a striking display of insight, Moffatt synthesized an example of a 3 dimensional flow that exhibits those features of stretching, twisting and folding (STF for short) of vector lines which are thought to be essential parts of the "fast dynamo" problem [Moffatt 1985]. At that time he suggested that the solution might exhibit chaotic behavior. In 1989 at the Cambridge IUTAM conference on Topology in Fluid Mechanics, Bayer, Moffatt and Nez presented a numerical study of a version of this "STF" flow that indicated that Lagrangian turbulence, or chaos, of a variety "different from that of the ABC flow" was contained within a bounded solution set of flow trajectories [Bayer 1989]. A later publication developed more detailed features of this flow as deduced from numerical analysis [Moffatt 1990].

At the 1989 IUTAM conference it was realized by the present author that the STF flow was a modification of one of a set of analytic closed form solutions to the Navier-Stokes equations in a rotating frame of reference found by the present author while studying the effects of topological torsion in evolutionary systems. [Kiehn 1988]. The particular solution of interest to the present article could be put in correspondence with those vector fields that generate a saddle-node Hopf bifurcation, as classified by Langford [Langford 1987]. In otherwords, it was demonstrated that there exist closed form vector field solution to the Navier-Stokes equations which will exhibit a Hopf bifurcation. The existence of such a bifurction has been subsumed previously on the basis of numercial analysis of truncated orthogonal expansions for the vector field solutions to the Navier-Stokes equations.

At the same conference, the present author also offered the thesis that a coherent structure in a fluid should be defined as a connected compact deformable domain of constant topological properties [Kiehn 1989b]. Herein that idea is extended to display a topological phase change in a compact domain of dynamical orbits. A brief report of this work was presented at the Permb-Moscow conference in 1990. In this article, more detail about the analytic method is presented. The method not only replicates the important features generated by the numerical analysis of Moffatt, but also permits an analysis to be made of the topological phase transformation associated with the Hopf bifurcation, a result hard to extract from the numerical analysis.

The Saddle Node Hopf, compressible, STF flow to be studied herein is given by the velocity field, $\mathbf{V} = [u, v, w]$, where

$$\begin{aligned} u &= (C - R)\ x\ z - \Omega y \qquad (1) \\ v &= C\ y\ z + \Omega x \\ w &= -F + D\ z^2 + R\ T\ x^2/(D - C) - T\ (x^2 + y^2). \end{aligned}$$

The perturbation parameter is R, and Ω represents the ambient rotation about the z axis. The perturbation parameter is a measure of the phase asymmetry of the rotational motion, and leads to phase folding or retrograde behavior. The mean flow parameter is F. The vector field given by (1) is a perturbation of one element of an equivalence class of solutions to the Navier-Stokes equations which have bounded domains of trajectories that may be embedded in environmental flows of different topology. As the parameters, F, of the flow (1) are changed, torsional defects and compact domains of re-entrant flow can be created within the once unidirectional flow [Kiehn, 1990b]. Herein, a topological phase transition is described within the defect domain due to a parametric pertubation, R.

2 Features of the perturbed saddle node Hopf bifurcation.

Ordinarily for appropriate choices of initial conditions and flow parameters, the solution trajectories to (1) extend to infinity. The entire domain consists of open trajectories. However for certain values of the initial conditions and parameters, defect domains of bounded topology will exist. The solution trajectories have a bound if a convex function, Φ,can be found such that

$$\begin{aligned} \mathbf{V} \circ\ grad\ \Phi &= \Lambda\ \Phi, \qquad (2) \\ \mathbf{V} \times grad\ \Phi \circ \{grad\ (\mathbf{V} \circ \mathbf{V})/2) \times \mathbf{V} + (\mathbf{V} \circ \mathbf{V})\ curl\ \mathbf{V} &< 0 \qquad (3) \end{aligned}$$

The first condition insures that the streamlines are tangential to the bounding surface, $\Phi = 0$, and the second condition insures that the streamline is reflected by the bounding surface. The surface is defined mod fixed points. It should be noted that Equation (2) indicates that the bounding set is a conformal invariant of the flow.

For the differential system governed by (1), a trial bounding surface will be given by the ellipsoid function,

$$\Phi = ax^2 + by^2 + cz^2 - f, \tag{4}$$

where for simplicity, a constraint of rotational symmetry is assumed, such that $a = b$. Direct computation reveals that the bounding ellipsoid is independent from the perturbation parameter, R, if the oblateness ratio is constrained by the equation,

$$(a/c) = T/(D - C). \tag{5}$$

It follows that for the vector field given by (1) that

$$a = TD/(D - C), \qquad c = D, \qquad f = F. \tag{6}$$

The function,

$$\Phi(x, y, z) = TD/(D - C)(x^2 + y^2) + D\ z^2 - F = 0, \tag{7}$$

is an envelope solution to the re-entrant domain of trajectories, and $\Phi(x, y, z) = 0$ is a bounding surface function which is a first integral of the flow. Specialization of the constants can force a spherical envelope. A typical numerical solution to equation (1) which demonstrates the bounding ellipsoid and the torsional orbits is presented in Figure 1. These bounded orbits can be embedded in an irrotational flow.

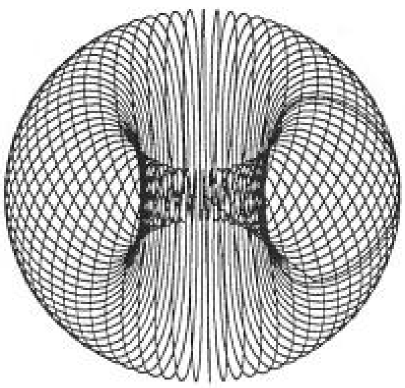

Figure 1: **0 < R < 2D Toriodal phase**

Figure 1 displays a typical numerical solution to the differential system. The steady flow is from left to right and swirls about the torsional ellipsoid. A single streamline is displayed for an initial condition that displays the limits of the bounding ellipsoid. This torsional defect can be embedded in irrotational flow with swirl. The flow parameters have been adjusted such that a bounded reentrant domain exists.

A sketch of the bounding ellipsoid is given in Figure 2.

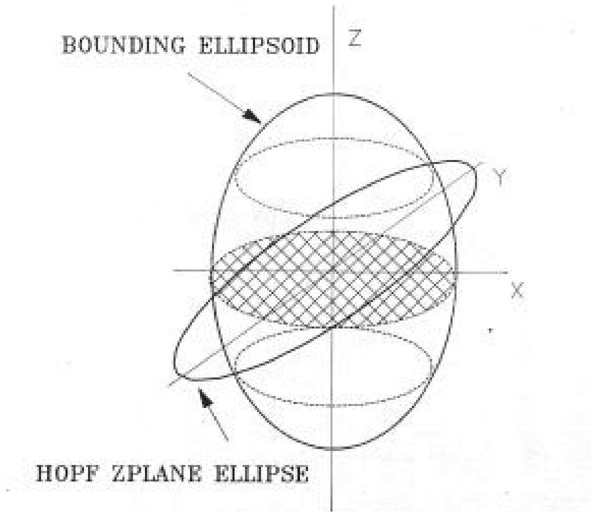

Figure 2: **Bounding Ellipsoid and Hopf zplane Ellipse**

Figure 2 displays the bounding ellipsoid $\Phi(x, y, z) = TD/(D-C)(x^2+y^2)+ D\ z^2-F = 0$, and the z=0 plane ellipse $\{-F+T(x^2+y^2)+(RT/(D-C))x^2\} = 0$ for the saddle-node Hopf STF flow given by equation (1). The shape of the bounding ellipsoid can be made oblate, spherical or prolate by appropriate choices of the constants. The domain of bounded solution exists when the flow parameters are such that (7) has real solutions. It is apparent that when the argument of $xmax$ becomes negative, the bounding ellipsoid vanishes. The value of xmax determines the "order parameter", or the coherence length of the topological defect. It will be assumed for simplicity that D, F, and T are positive. Hence, the enveloping surface vanishes when $D - C$ goes to zero.

When the additional constraint of zero divergence is also imposed on the flow given by (1), then

$$C = R/2 - D, \tag{8}$$

and it follows that the maximum value of R is equal to $4D$. The solution trajectories become unbounded for all initial conditions when $R > 4D$. The streamlines of the saddle-node Hopf flow, for certain values of the parameters, will converge to a set of (3-volume) measure zero, but such a set need not be of dimension zero. It is remarkable that when $divV = 0$, the streamlines for the perturbed saddle-node Hopf flow form a set of 3-measure zero for $R < 2D$, and lie on a toriodal surface with the z-axis forming the polar axis of the torus. The fixed points of the flow are at $z^2 = (F/D)$, and an ellipse of Hopf periodicity exists in the $z = 0$ plane. The Hopf ellipse is a point set of zero helicity density. The flow trajectories use the Hopf ellipse as a guiding center for their helical orbits. The major and minor radii of the z-plane = 0 Hopf ellipse are given by the equations,

$$xe\max = \{F(D-C)/T(D-C+R)\}^{1/2} = \{F(4D-R)/T(4D+R)\}^{1/2} \quad (9)$$

$$ye\max = (F/T)^{1/2} \quad (10)$$

The bounding ellipsoid will have principal radii of

$$zmax = (F/D)^{1/2} \qquad and \qquad xmax = ymax = (F(D-C)/DT)^{1/2}, \quad (11)$$

independent of the "perturbation" parameter, R. The second expression on the right presumes $divV = 0$. The z-plane Hopf ellipse is also displayed in Figure 3.

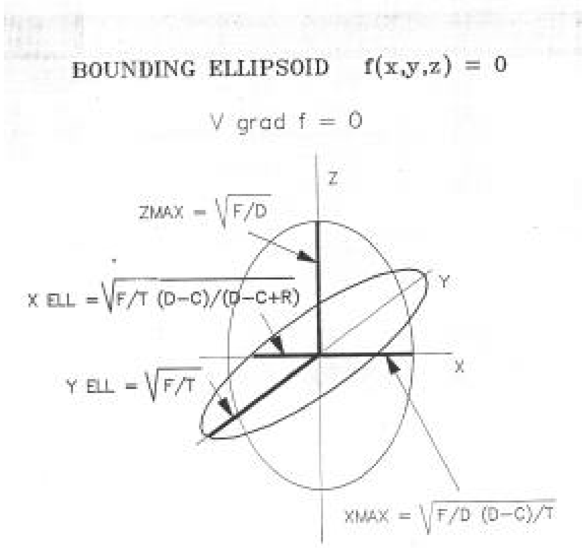

Figure 3: **Bounding ellipsoid and Hopf axes**

Depending on the parameters of the perturbed flow, the z-plane Hopf ellipse is either contained within the bounding ellipsoid, or penetrates the bounding ellipsoid. The ellipse is contained for $R < 2D$ and penetrates for $R > 2D$. For the divergence free case, a topological phase transition takes place at $R = 2D$. In addition, the flow becomes unbounded at $R = 4D$. For $0 < R < 2D$, the solution trajectories are confined to a torus of Euler characteristic 0. The solutions are not necessarily periodic, but exhibit Poincare sections typical of two periods of irrational ratio. Above $R > 2D$ and for $R < 4D$ the solution trajectories transiently appear to appear to reside on a 2-dimensional surface of Euler characteristic minus two. However the 2-dimensional transient behavior is illusory for the orbits are not confined to a two dimensional surface for $R > 2D$.

3 A topological phase transition

The parametric topological phase transition is displayed in Figure 4 in terms of the behavior of the order parameter as a function of the perturbation parameter R. The z-plane ellipse and its relationship to the bounding ellipsoid is also displayed in Figure 4.

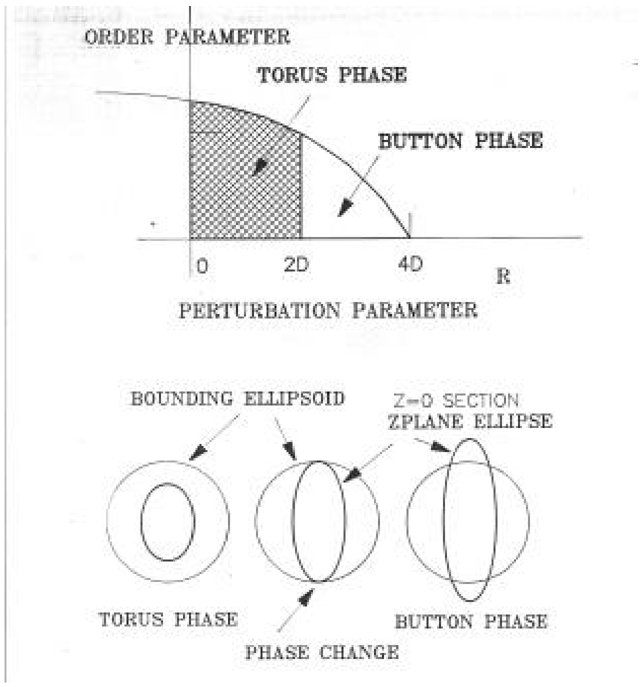

Figure 4: **Order parameter and phase transitions**

Figure 4 describes the parametric behavior of the order parameter as a function of the perturbation parameter, R. The order parameter is defined as the minimum dimension of the bounding ellipsoid. Although computed for a dynamical system, this figure is typical of phase changes that occur in material substances.

Figure 1 above displays the distorted toroidal surfaces upon which the streamlines reside, for a choice of parameters that leads to a divergence free flow bounded by a spherical ellipsoid, and $R < 2D$. The bounding surface has two singular points which define the polar axis of the torus of trajectories. The associated Poincare characteristic indicates non-chaotic behavior in the torus phase of Euler characteristic 0. The Poincare section appears to be of integer dimension 2, at most. When the Hopf ellipse is tangent to the bounding surface, then the surface has four singular points.

However, for the case where $R > 2D$,the transient solution trajectories produce a visual appearance of a two hole button surface of Euler characteristic -4.

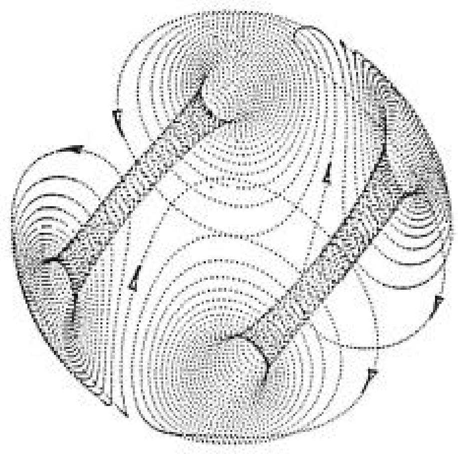

Figure 5: **Button phase R > 2D**

The chaotic orbits are shown in Figure 5 for a **portion** of this evolution history. The orbits are guided by the Hopf ellipse, but are not streamlines. Actually, for this case, the bounding surface has six singular points, two which formed the polar axis of the original torus, and four more which represent the intersection of the Hopf ellipse and the bounding surface, all of which can be determined analytically. The associated Poincare section indicates a "chaotic" behavior in that the point set of section is apparently of dimension greater than 2. Figure 5 displays an isometric projection for an orbit of Euler characteristic -2 and corresponds to the case ($2D < R < 4D$).

The associated Poincare section (given in Figure 6) indicates chaotic behavior for an appropriate choice of initial conditions. It is to be noted that the guiding center for the torsional solutions consists of those segments of the Hopf z-plane ellipse that are contained within the enveloping ellipsoid. The solution trajectories spiral about one segment of the z-plane ellipse with more or less a constant throat diameter until they approach the enveloping surface. At such a time, and intermittently, the orbit flips to the second segment of the Hopf z-plane ellipse, and settles down to a helical rotation with a more less constant throat diameter different from that of the previous leg of the orbit. It is this variable throat size that leads to the "chaotic" behavior of the orbit. Contrary to the observations of Bayers, et.al., the Poincare section exhibits island behavior in the sense that the throats of surface of Euler characteristic minus 2 appear to be forbidden regions. The Poincare section in Figure 6 consists of 16000 intercepts of a single orbit.

Figure 6: **Poincare Section R > 2D**

4 Remarks

It is to be noted that the solution set is bounded in a 3 dimensional sense even though the divergence is positive or negative. In fact, for the "non-isochoric" case, when $divV = -(3D + T)\ z$ the velocity field is harmonic in the sense that $div\ grad\ V - curl\ curl\ V = 0$. It follows that the steady flow is independent of the magnitude of the viscous dissipation coefficient, ν. However, such bounded steady harmonic "non-viscous" solutions are oblate, and never prolate. It is curious that "flexible" communication satellites that are spin stabilized about the prolate axis will tumble, while spin stabilization about the oblate axis is robust. A possible explanation of this result may be due to fact that the motions of such objects in the weakly viscous medium of outer space produce a "compact" domain similar to that described above, and the minimum dissipation configuration corresponds to a Stokes flow of the oblate saddle-node Hopf configuration.

For the divergence free case, and for certain values of the parameters, the flow solution resembles a long cigar shaped system that reminds this author of a primitive transverse wave packet called a "photon". Such a solution is a distortion of Figure 7.

For such an extreme prolate solution the ratio of kinetic energy to angular momentum magnitude is a constant times the rotational frequency, also typical of a "photon". The saddle node Hopf STF solution is also remarkable in that dissipative features can be "eliminated" by appropriate choices of the parameters. For example, when $\{2D + 2T + C]\} = 0$ and $R = 0$, the velocity field is harmonic. That is, $div\ grad\ V - curl\ curl\ V = 0$, again. It follows that the flow then is independent from the viscosity factor in the Navier -Stokes equations. Internal irreversible dissipation disappears!

Figure 7: **Photon phase swirl defect ?**

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