Non-Equilibrium and Irreversible Electrodynamics

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Abstract

Classical electromagnetic theory of equilibrium systems can be put into correspondence with two topological constraints placed on a 4 dimensional variety of independent variables $\{x, y, z, t\}$. In terms of the exterior differential systems, the two topological constraints are $F - dA = 0$ and $J - dG = 0$. The topological formulation can be algebraically prolonged, leading to the independent concepts of topological torsion, $A\hat{}F$, and topological spin, $A\hat{}G$. Non-zero values for these 3-forms are indicators of non-equilibrium electrodynamic systems. These 3-forms are identically zero in equilibrium systems of Pfaff dimension 2. In the non-equilibrium systems, the direction fields of the 3-forms $A\hat{}F$ and $A\hat{}G$ can exhibit linking and separation of the systems into disconnected parts. The topological closure (exterior derivative) of these 3-forms define the topological Poincare Invariants. When the Poincare invariants vanish, the 3-forms have properties similar to the conservative charge current 3-form $J = dG$. Closed integrals of the closed 3-forms form rational topological period integrals (quantum numbers), even in the non-equilibrium situations. However when the Poincare invariants do not vanish they become the source of topological evolution, irreversible changes of phase, topological defects, and the creation of stationary states far from equilibrium. The possible continuous topological evolution of an equilibrium or non-equilibrium electromagnetic system is studied in terms of Cartan's magic formula acting on the set of exterior differential forms. This method extends the covariant formulation of tensor analysis to include non-adiabatic processes. Equivalence classes

of systems and processes can be constructed in terms of the Pfaff topological dimension of the set of exterior differential forms used to define an electromagnetic system.

1. INTRODUCTION

1.1. Topological Universality

It is a remarkable fact that the physical theories of Thermodynamics, Electrodynamics and Hydrodynamics all have similar topological foundations. These similarity features become evident, and useful, when the different disciplines are expressed in the universal language of Cartan's theory of exterior differential forms.

1. Each discipline utilizes the concept that a physical system can be encoded in terms of an exterior differential 1-form of Action, A.
2. Each discipline utilizes the concept that a process, or current, acting on the physical system, can be encoded to within a factor, ρ, by a contravariant direction field, V.
3. Each discipline has a dynamics that can be expressed in terms of continuous topological evolution based upon the Lie differential with respect to V. Warning: this topological dynamics is not always fully equivalent to that dynamics generated by the covariant differential of tensor analysis. The geometric dynamics of tensor analysis is a subset of the topological dynamics.

The arguments of the functions that define the physical system, the process, and the induced additional 1-forms, in this article are limited (with a few exceptions) to an ordered variety of $n = 4$ independent base variables, abstractly specified as $\{x, y, z, t\}$, and their differentials, $\{dx, dy, dz, dt\}$. It is presumed that other varieties of base variables $\{\xi^1, \xi^2, \xi^3, \xi^4\}$ can be represented in terms of diffeomorphic maps from $\{\xi^1, \xi^2, \xi^3, \xi^4\}$ to $\{x, y, z, t\}$. To a physicist, the base variables play the role of admissible coordinates if they are diffeomorphically related. However no specific geometric metric or connection is (necessarily) imposed on these varieties of pre-geometric dimension $n = 4$ base variables.

Although the main thrust of this article is to study topological features of non-equilibrium electrodynamic systems, it is necessary to preface that work with a few remarks about the topological properties of the continuum (section 1), and thermodynamics from a topological point of view (section 2).

1.2. Pfaff Topological Dimension

Perhaps one of the most important topological tools to be used within the theory of continuous topological evolution is the concept of Pfaff topological dimension. The maximum Pfaff dimension is equal to number of independent variables in the base variety, which in this article has been limited to $n = 4$. For a given 1-form of Action, $A = A_k(x, y, z, t)dx^k$ defined on the base variety of $\{x, y, z, t\}$, it is possible to ask what is the irreducible minimum number of independent functions $\theta(x, y, z, t)$ required to describe the topological features that can be generated by the specified 1-form, A. This irreducible number of functions is defined here in as the "Pfaff topological dimension" of the 1-form, A. For example, if

$$A = A_k dx^k \Rightarrow d\theta(x, y, z, t)_{irreducible}, \tag{1.1}$$

$$\text{such that } A_k = \partial\theta(x, y, z, t)/\partial x^k, \tag{1.2}$$

then only one function $\theta(x, y, z, t)$ is required to describe the Action, not four. In this example the irreducible Pfaff topological dimension of the 1-form, A, is 1, although the dimension of base variety is 4.

Relative to the Cartan topology [Baldwin 1991], the "Pfaff topological dimension" can be generated by each of the Pfaffian forms associated with each discipline. The irreducible Pfaff topological dimension for any given 1-form A is readily computed by constructing the Pfaff sequence of forms:

$$\text{Pfaff sequence } \{A, dA, A\hat{\ }dA, dA\hat{\ }dA\}. \tag{1.3}$$

The Pfaff topological dimension is is equal to the number of non-zero terms in the Pfaff sequence. For example, if the Pfaff sequence for a given 1-form A is $\{A, dA, 0, 0\}$ in a region $U \subset \{x, y, z, t\}$, then the Pfaff topological dimension of A is 2 in the region, U. The 1-form A, in the region U, then admits description in terms of only two, but not less than 2, independent variables, say $\{u^1, u^2\}$. For a differentiable map φ from $\{x, y, z, t\} \Rightarrow \{u^1, u^2\}$, the exterior differential 1-form defined on the target variety U of 2 pre-geometry dimensions as

$$A(u^1, u^2) = A_1(u^1, u^2)du^1 + A_2(u^1, u^2)du^2, \tag{1.4}$$

has a functionally well defined pre-image $A(x, y, z, t)$ on the base variety $\{x, y, z, t\}$ of 4 pre-geometric dimensions. This functionally well defined pre-image is obtained by functional substitution of u^1, u^2, du^1, du^2 in terms of $\{x, y, z, t\}$ as defined by the mapping ϕ. The process of functional substitution is called the pull-back.

$$A(x,y,z,t) = A_k dx^k = \varphi^*(A(u^1,u^2)) = \varphi^*(A_\sigma du^\sigma) \tag{1.5}$$

It may be true that the functional form of a 1-form, A, yields a Pfaff topological dimension equal to 2 globally over the domain $U \subset \{x,y,z,t\}$, except for sub regions where the Pfaff dimension of A is 3 or 4. These sub regions represent topological defects in the almost global domain of Pfaff dimension 2. Conversely, the Pfaff dimension of A could be 4 globally over the domain, except for sub regions where the Pfaff dimension of A is 3, or less. These sub regions represent topological defects in the almost global domain of Pfaff dimension 4. Applications of both viewpoints will be described below. The important concept of Pfaff topological dimension also can be used to define equivalence classes of physical systems and processes.

The concept of Pfaff "topological dimension" was developed more than 110 years ago (see page 290 of Forsyth [Forsyth (1890) 1959]), and has been called the "class" of a differential 1-form in the mathematical literature. More recent mathematical developments can be found in [Schouten 1949]. The method and its properties have been little utilized in the applied world of physics and engineering. Of key importance is the fact that the non-zero existence of the 3-form $A\hat{}dA = A\hat{}F$ of *Topological Torsion*, and the 4-form of *Topological Parity*, $dA\hat{}dA = F\hat{}F$ implies that the Pfaff topological dimension of the region is 3 or 4, respectively. Either value is an indicator that the physical system (in the sub region) is NOT in thermodynamic equilibrium. The concept of *topological parity*, $F\hat{}F$, has its foundations in the theory of Pfaff's problem, with a recognizable 4 dimensional formulation appearing in Forsyth [Forsyth (1890) 1959] page 100. The idea of *Topological Torsion*, $A\hat{}F$, is associated with the idea of magnetic helicity density, a concept that apparently had its electromagnetic genesis with the study of plasmas in WWII. However, the concept of helicity density is but one component of the four dimensional *Topological Torsion 4 vector*.

Recall that a space curve with non-zero Frenet - Serret torsion does not reside in a two dimensional plane. Non-zero Frenet - Serret torsion of a space curve is an indicator that the *geometrical* dimension of the space curve is at least 3. The fact that the Pfaff *topological* dimension of the 1-form, A, is at least 3, when $A\hat{}F$ is non-zero, is the basis of why the 3-form, $A\hat{}F$, was called "Topological Torsion". The idea of non-zero $A\hat{}F$ also appears in the theory of the Hopf Invariant [Bott 1994].

The concept of $A\hat{}F$ has also appeared in the differential geometry of connections, where the matrix valued 3-form is known as the Chern-Simons 3-form.

However, on varieties without connection or metric, the Chern-Simons concept is not well defined, but the Topological Torsion concept exists and is acceptable, for it does not depend upon the geometric features of metric and/or connection.

1.3. Deformation Invariants as Topological Properties

Topological properties are defined as invariants with respect to homeomorphisms. A more mundane definition is that a topological property is an invariant of a continuous deformation. Certain integral properties of an electromagnetic system are deformation invariants with respect to those continuous evolutionary processes that can be described by a singly parameterized vector field. The absolute deformation invariants lead to the fundamental topological conservation laws described in the physical literature of electromagnetism as the conservation of charge and the conservation of flux. Recall the definitions used to describe processes of continuous topological evolution.

> A continuous process is defined as a map from an initial state of topology $T_{initial}$ into a final state of perhaps different topology T_{final} such that the limit points of the initial state are permuted among the limit points of the final state (see p. 97 et.seq. [Lipschutz 1965]). If the ordering of the limit points is invariant, the process is uniformly continuous. If the ordering (as in a folding of a boundary) or the number of the limit sets is changed, the process is non-uniformly continuous.
>
> A topological deformation invariant is defined as an integral of an exterior differential p-form over a p dimensional manifold, or cycle, zpd, such that the Lie differential of the integral of the p-form ω with respect to a singly parameterized vector field, ρV^k, vanishes, for any choice of deformation parameter, ρ.

$$\text{Integral Deformation Invariant}: \quad L_{(\rho V^k)} \int_p \omega = 0 \quad any\ \rho \tag{1.6}$$

The requirements that a given p-form becomes a deformation invariant (and therefor a topological property, invariant with respect to homeomorphisms) is expressed in terms of certain topological constraints. Those objects or properties that remain the same under continuous deformation represent topological, not geometric, properties. For example, the number of holes in a thin rubber sheet stays the same as the rubber sheet perturbed by a deformation. The number of holes is a topological property. However, if the topological constraints required for continuous deformation are not satisfied, then topological change takes place. Topological change would require that the number of holes in the thin rubber sheet example were to change. Topological change can occur continuously or discontinuously. The focus in this article is on continuous topological change, and as will be demonstrated below, topological change is a necessary requirement for thermodynamic irreversibility [RMK 1976 b].

1.3.1. Absolute Integral Invariants

There are two types of invariant integrals, Absolute and Relative integral invariants. If the exterior p-form that forms the integrand is exact, the Absolute integral invariant places conditions only on the boundary of the domain of integration. It is these types of objects (Absolute integral invariants) that give a formality to those thermodynamic concepts whereby a physical system reaches equilibrium uniformly within its interior, and yet may couple with its exterior environment via fluxes across its boundary. Only effects related to the boundary are of consequence. For example, consider physical systems that can be defined by a 1-form of Action, A, such that the derived 2-form $F = dA$, is exact. It follows from Stokes theorem that the 2-dimension integral of F is an absolute integral deformation invariant with respect to *all* continuous processes that can be defined by a singly parameterized vector field, subject to a boundary condition that the net flux, $i(\rho V^k)F$, of F, across the 1-dimensional boundary of M is zero:

$$L_{(\rho V^k)} \int\int_M F = \int\int_M i(\rho V^k) dF + \int\int_M d(i(\rho V^k)F) \tag{1.7}$$

$$= 0 + \int_{boundary\ of\ M} i(\rho V^k)F \tag{1.8}$$

This concept is at the basis of the Helmholtz theorems of vorticity conservation (or angular momentum per unit mass) in hydrodynamics, and the conservation of flux in classical electromagnetism. Herein, this concept of deformation invariance of a topologically coherent structure will be written in the form of an exterior differential system [Bryant 1991], $F - dA = 0$. The exterior differential system is to be recognized as topological constraint. From Stokes theorem, the 2 dimensional domain of finite support for F can not, in general, be compact without boundary, unless the Euler characteristic vanishes. There are two exceptional cases for absolute invariance of the integral, and they occur when the integration domain is compact without boundary. Such two dimensional domains which have a zero Euler characteristic are the torus and the Klein-Bottle, but these situations require the additional topological constraint that $F\hat{}F \Rightarrow 0$. The fields in these exceptional cases must reside on these exceptional compact surfaces without boundary, which form topological coherent structures. Note that an evolutionary process could start with $F\hat{}F \neq 0$, and possibly evolve to a state with $F\hat{}F = 0$. If such residue states are compact without boundary, then they must be either tori or Klein bottles.

The same technique can be applied to non-exact but closed p-forms.

1.3.2. Relative Integral Invariants

If the integration of the exact 2-form, F, is over a closed two dimensional chain, designated as a 2 dimensional cycle, z2d (which may or may not be a 2 dimensional boundary), then the Integral is invariant for any deformation factor, ρ :

$$L_{(\rho V^k)} \int\int_{z2d} F = \int\int_{z2d} i(\rho V^k)dF + \int\int_{z2d} d(i(\rho V^k)F) = 0 + 0. \tag{1.9}$$

The two integrals on the right vanish, the first due to the fact that $dF = 0$, and the second due the fact that the closed integral over an exact form vanishes. Close integrals of exact p-forms are always relative deformation integral invariants. However, the same technique can be applied to non-exact but closed p-forms. For electromagnetism, there are several exact p-forms, each producing a relative deformation integral invariant. For example, the 3-form of charge-current density is exact, $J = dG$. The 4-forms that define the Poincare Invariants are exact: $F\hat{}F = d(A\hat{}G)$ and $F\hat{}G - A\hat{}J = d(A\hat{}G)$.

If the conditions of relative integral invariance are applied to an arbitrary 1-form of Action, then the relative integral invariance condition becomes

$$L_{(\rho V^k)} \int_{z1d} A = \int_{z1d} i(\rho V^k)dA + \int_{z1d} d(i(\rho V^k)A) \tag{1.10}$$

$$= \int_{z1d} i(\rho V^k)F + 0 \Rightarrow 0. \tag{1.11}$$

It follows the $i(\rho V^k)dA$ must be zero on the cycle $z1d$ for any deformation parameter ρ. Cartan has shown that this is the condition that implies the process ρV^k has a "Hamiltonian" representation [Cartan 1958 (1922)].

2. Topological Thermodynamics

As the emphasis in this article is on electromagnetic theory, this section on topological thermodynamics will be abbreviated, but is presented here in order to demonstrate how the universal topological viewpoint couples thermodynamic thinking to electromagnetism.

2.1. Cartan's Magic formula and the First Law of Thermodynamics

The topological view of thermodynamics described herein is based on three axioms.

1. Thermodynamic physical systems can be encoded in terms of a 1-form of covariant Action Potentials, $A_k(x, y, z, t)$, on a 4 dimensional abstract variety of ordered independent variables, $\{x, y, z, t\}$. The variety supports a volume element $\Omega_4 = dx\hat{}dy\hat{}dz\hat{}dt$.

2. Thermodynamic processes are assumed to be encoded, to within a factor, $\rho(x, y, z, t)$, in terms of contravariant vector direction fields, $\mathbf{V}_4(x, y, z, t)$.

3. Continuous topological evolution of the thermodynamic system can be encoded in terms of Cartan's magic formula (see p. 122 in [Marsden 1994]). The Lie differential, when applied to a exterior differential 1-form of Action, $A = A_k dx^k$, is equivalent *abstractly* to the first law of thermodynamics.

$$\begin{aligned}
&\text{Cartan's Magic Formula} & L_{(\rho\mathbf{V}_4)}A &= i(\rho\mathbf{V}_4)dA + d(i(\rho\mathbf{V}_4)A) && (2.1)\\
&\text{First Law of Thermodynamics} & &: W + dU = Q, && (2.2)\\
&\text{Inexact 1-form of Heat} & L_{(\rho\mathbf{V}_4)}A &= W + dU = Q && (2.3)\\
&\text{Inexact 1-form of Work} & W &= i(\rho\mathbf{V}_4)dA, && (2.4)\\
&\text{Internal Energy} & U &= i(\rho\mathbf{V}_4)A. && (2.5)
\end{aligned}$$

In effect, Cartan's methods can be used to formulate precise mathematical definitions for many thermodynamic concepts in terms of topological properties

-without the use of statistics or metric constraints. Moreover, the method applies to non-equilibrium thermodynamical systems and irreversible processes, again without the use of statistics or metric constraints.

In order to make the equations more suggestive to the reader, the symbolism for the variety of independent variables has been chosen to be of the format $\{x, y, z, t\}$, but be aware that no constraints of metric or connection are imposed upon this variety. For instance, it is NOT assumed that the base variety is euclidean.

2.2. Physical Systems: Equilibrium, Isolated, Closed and Open

Physical systems and processes are elements of topological categories determined by the Pfaff topological dimension (or class) of the 1-forms of Action, A, Work, W, and Heat, Q. For example, the Pfaff topological dimension of the exterior differential 1-form of Action, A, determines the various species of thermodynamic systems in terms of distinct topological categories. The are two topological thermodynamic categories that are determined by the closure (or differential ideal) of the 1-form of Action, $A \cup dA$, and the closure of the 3-form of topological torsion, $A\hat{}dA \cup dA\hat{}dA$. The first category is represented by a connected Cartan topology, while the second category is represented by a disconnected Cartan topology. The Cartan topology is discussed in detail in Chapter 3.

2.2.1. Connected Topology $A\hat{}F = 0$

- Equilibrium physical systems are elements such that the Pfaff topological dimension is 1.

  Isolated physical systems are elements such that the Pfaff topological dimension is 2, or less.

- Isolated systems of Pfaff dimension 2 need not be in equilibrium, but do not exchange radiation or mass with the environment.

2.2.2. Disconnected Topology $A\hat{}F \neq 0$

- Closed physical systems are elements such that the Pfaff topological dimension is 3. Closed systems can exchange radiation, but not mass, with the environment.

- Open physical systems are such that the Pfaff topological dimension is 4. Open physical systems can exchange both radiation and mass with the environment.

$$Systems \quad : \quad \text{defined by the Pfaff dimension of } A \tag{2.6}$$
$$dA = 0 \qquad \text{Equilibrium - Pfaff dimension 1} \tag{2.7}$$
$$A\hat{}dA = 0 \qquad \text{Isolated - Pfaff dimension 2} \tag{2.8}$$
$$d(A\hat{}dA) = 0 \qquad \text{Closed - Pfaff dimension 3} \tag{2.9}$$
$$dA\hat{}dA \neq 0. \qquad \text{Open - Pfaff dimension 4.} \tag{2.10}$$

Note that these topological specifications as given above are determined entirely from the functional properties of the physical system encoded as a 1-form of Action, A. The system topological categories do not involve a process, which is encoded (to within a factor) by some vector direction field, $\mathbf{V}_4$. However, the process $\mathbf{V}_4$ does influence the topological properties of the work 1-form W and the Heat 1-form Q.

2.3. Equilibrium vs. Non-Equilibrium Systems

The intuitive idea for an equilibrium system comes from the experimental recognition that the intensive variables of pressure and temperature become domain constants in an equilibrium state: $dP \Rightarrow 0$, $dT \Rightarrow 0$. A definition made herein is that the Pfaff topological dimension in the interior of a physical system which is in the equilibrium state is at most 1 [Bamberg 1992]. The Cartan topology generated by the elements of the Pfaff sequence for A is then a connected topology of one component, $\{A \neq 0,\ dA = 0,\ A\hat{}dA = 0,\ dA\hat{}dA = 0\}$. Although the Pfaff topological dimension of A is at most 2 in the isolated state, processes in the equilibrium state are such that the Work 1-form and the Heat 1-form must be of Pfaff dimension 1. For suppose $W = PdV$, then $dW = dP\hat{}dV \Rightarrow 0$ if the pressure is a domain constant. Similarly, suppose $Q = TdS$, then $dQ = dT\hat{}dS \Rightarrow 0$ if the temperature is a domain constant. Hence both W and Q are of Pfaff dimension 1 for this equilibrium example. If the Pfaff dimension of the 1-form of Action is 1, then $dA \Rightarrow 0$. It follows in this more stringent case that $W \Rightarrow 0$, hence the Pressure must vanish, and Heat 1-form is a perfect differential, $Q = d(U)$.

Of particular interest herein are those regions of base variables for open, non-equilibrium, Turbulent physical systems, formed by the closure[1] of the 3-forms $A\hat{}dA$, $W\hat{}dW$, and $Q\hat{}dQ$. For such regions, the Pfaff topological dimension of the 1-forms, A, W, and Q, are all initially of Pfaff topological dimension 4, save for defect regions that are of Pfaff dimension 3. For example, evolutionary dissipative irreversible processes in such open systems can describe evolution to regions of base variables where the Pfaff topological dimension of the 1-form of Action, A, changes from 4 to 3. Such processes describe topological change in the physical system. For a given 1-form of Action, A, those regions of Pfaff topological dimension 3, once created, form topological "defect structures" in the closure of the 3-form, $A\hat{}F$. The defect structures of the 1-form of Action, A, (of Pfaff dimension 3) can behave as long lived (excited) states of the initial physical system, but they are far from equilibrium and are not isolated, for they are not of Pfaff topological dimension equal to 2 or less. Such excited states (of odd topological dimension) can admit extremal processes of kinematic perfection, and can have a Hamiltonian generator for the kinematics represented as first order ordinary differential equations. The Hamiltonian evolution remains contained in the defect structure, unless topological fluctuations destroy the kinematic perfection.

Such concepts can be applied to a model of cosmology (where the stars are the defect structures), to turbulent plasmas and fluids (where wakes are the defect structures), and to a better understanding of the arrow of time. Although the defects in the Turbulent non-equilibrium regime are not necessarily equilibrium structures, once formed and self organized as coherent topological structures of Pfaff dimension 3, they can evolve along extremal trajectories that are not dissipative, and may even have a Hamiltonian representation. These "stationary", if not long lived (excited) states of Pfaff dimension 3, indeed are states "far" from the equilibrium state, which requires a Pfaff dimension of 1. Note that the word "far" does not imply a "distance". The Pfaff dimension 3 and 4 sets are not even "connected" to the equilibrium states in a topological sense. The non-equilibrium states of a physical system that are "near-by" to the equilibrium state, are "connected" to the equilibrium state, and are of Pfaff dimension 2.

The descriptive words of self-organized states far from equilibrium are abstracted from the intuition and conjectures of I. Prigogine [Kondepudi 1998]. However, the topological theory presented herein presents for the first time a solid, formal, mathematical justification (with examples) for the Prigogine con-

[1] The closure of the p-form Σ is the union of Σ and $d\Sigma$, which Cartan has called a differential ideal.

jectures. Precise definitions of equilibrium and non-equilibrium systems, as well as reversible and irreversible processes can be made in terms of the topological features of Cartan's exterior calculus. Thermodynamic irreversibility and the arrow of time are well defined in a topological sense [RMK 2003], a technique that goes beyond (and without) statistical analysis. Thermodynamic irreversibility and the arrow of time requires that the evolutionary process produce topological change.

2.4. Multiple Components

One of the most remarkable properties of the Cartan topology [Baldwin 1991], generated by the elements of a Pfaff sequence, is due to the fact that when $A\hat{}dA = 0$, (Pfaff dimension 2 or less) the Cartan topology of the physical system is reducible to a single connected topological component. This single component need not be simply connected. On the other hand when $A\hat{}dA \neq 0$, (Pfaff dimension 3 or more) the the Cartan topology of the physical system is a disconnected topology of more than one topological component. The bottom line is that when the Pfaff dimension is 3 or greater (such that conditions of the Frobenius unique integrability theorem are not satisfied), solution uniqueness to the Pfaffian differential equation, $A = 0$, is lost. If there exist solutions, there is more than one. Such concepts lead to propagating discontinuities (signals), envelope solutions (Huygen wavelets), an edge of regression (the spinodal line of phase transitions) a lack of time reversal invariance, and the existence of irreducible affine torsion in the theory of connections. It is the opinion of this author that a dogmatic insistence that "useful" physical theories must produce a unique outcome from given set of initial conditions is a severe topological constraint which has hindered the understanding of irreversibility and non-equilibrium systems.

2.5. Processes

2.5.1. Continuous Processes

All continuous processes may be put into equivalence classes as determined by the vector fields, $\mathbf{V}$, that generate the evolution of the physical system, A. For example, for the 1-form, A, those vector fields that satisfy the transversal equation,

$$Associated : i(\rho\mathbf{V})A = 0 \tag{2.11}$$

are said to be elements of the "associated class" of vector fields relative to the form A. For such thermodynamic processes, the change of internal energy is locally zero.

Those vectors that satisfy the equations,

$$Extremal : i(\rho\mathbf{V})dA = 0 \tag{2.12}$$

are said to be elements of the extremal class of vector fields. For such processes, the virtual work vanishes, $W = 0$. It should be noted that the 2-form dA admits a unique extremal vector only on topological spaces of odd Pfaff topological dimension. Such topological spaces define a Contact manifold, which serves (in higher dimensional cases) as the 2n+1 dimensional state space of mechanics. If the Pfaff dimension of the 1-form of Action, A, is 4, then a unique extremal vector does not exist. Similarly, the non-zero 2-form, dA, of maximal rank, defines a symplectic manifold of even (2n+2) dimensions. However, on the symplectic manifold there does exists a unique vector direction field, the Topological Torsion vector (described below), but no extremal vector.

Vectors which are both extremal and associated are said to be elements of the characteristic class of vector fields [Klein 1962].

$$Characteristic : i(\rho\mathbf{V})A = 0 \quad and \quad i(\rho\mathbf{V})dA = 0 \tag{2.13}$$

Note that characteristic flow lines generated by $\mathbf{V}$ of the Characteristic class preserve the Cartan topology, for each form of the Cartan topological base is invariant with respect to the action of the Lie derivative relative to characteristic flows. Characteristics are often associated with wave phenomena, and propagating discontinuities. Note that extremal processes relative to A are characteristic processes relative to dA.

2.5.2. Reversible and Irreversible Processes

The Pfaff topological dimension of the exterior differential 1-form of Heat, Q, determines important topological categories of processes. From classical thermodynamics "The quantity of heat in a reversible process always has an integrating factor" [Goldenblatt 1962] [Morse 1964] . Hence, from the Frobenius unique integrability theorem, which requires $Q\hat{}dQ = 0$, all reversible processes are such that the Pfaff dimension of Q is less than or equal to 2. Irreversible processes are such that the Pfaff dimension of Q is greater than 2. A dissipative irreversible topologically *turbulent* process is defined when the Pfaff dimension of Q is 4.

$$\begin{aligned} Processes &= \text{defined by the Pfaff dimension } Q && (2.14) \\ Q\hat{}dQ &= 0 \qquad \text{Reversible - Pfaff dimension 2} && (2.15) \\ d(Q\hat{}dQ) &\neq 0. \qquad \text{Turbulent - Pfaff dimension 4.} && (2.16) \end{aligned}$$

Note that the Pfaff dimension of Q depends on both the choice of a process, $\mathbf{V}_4$, and the physical system, A, upon which it acts. As reversible thermodynamic processes are such that $Q\hat{}dQ = 0$, and irreversible thermodynamic processes are such that $Q\hat{}dQ \neq 0$, Cartan's formula of continuous topological evolution, $L_{(\rho\mathbf{V}_4)}A = Q$, can be used to determine if a given process, $\mathbf{V}_4$, acting on a physical system, A, is thermodynamically reversible or not:

$$\left[\begin{array}{l} \text{Reversible Processes } \rho\mathbf{V}_4: \; L_{(\rho\mathbf{V}_4)}A\hat{}L_{(\rho\mathbf{V}_4)}dA = 0, \\ \text{Irreversible Processes } \rho\mathbf{V}_4: L_{(\rho\mathbf{V}_4)}A\hat{}L_{(\rho\mathbf{V}_4)}dA \neq 0. \end{array}\right] \qquad (2.17)$$

Remarkably, Cartan's magic formula can be used to describe the continuous dynamic possibilities of both reversible and irreversible processes, in equilibrium or non-equilibrium systems, even when the evolution induces topological change, transitions between excited states, and changes of phase, such as condensations.

It is important to note that the direction field, $\mathbf{V}_4$, need not be topologically constrained such that it is singularly parameterized. That is, the evolutionary processes described by Cartan's magic formula are not necessarily restricted to vector fields that satisfy the topological constraints of kinematic perfection, such as $dx^k - v^k dt = 0$. A discussion of kinematic topological fluctuations, where $dx^k - v^k dt = \Delta x^k \neq 0$, and a system first order equations (as well as the kinematic fluctuations, $dv^k - a^k dt = \Delta v^k \neq 0$, and a system of second order equations) is described below.

2.5.3. Adiabatic Processes - Reversible and Irreversible

The topological formulation permits a precise definition to be made for both reversible and an irreversible adiabatic processes in terms of the topological properties of Q. On a geometrical space of N dimensions, a 1-form will admit N-1 vector fields such that $i(V_A)Q = 0$. Such processes V_A are defined as local adiabatic processes [Bamberg 1992]. The N-1 null vectors will form a distribution of adiabatic processes orthogonal to the 1-form Q. The distribution of adiabatic processes will not form a smooth hypersurface, unless the Pfaff dimension of Q is

2 or less. In other words the null curves (adiabats) form an smooth hypersurface in the equilibrium state. Note that all adiabatic processes are defined by vector direction fields, to within an arbitrary factor, $\rho(x, y, z, t)$. That is, if $i(V_A)Q = 0$, then it is also true that $i(\rho V_A)Q = 0$.

The differences between the inexact 1-forms of Work and Heat become obvious in terms of this topological format. Both 1-forms depend on the process and on the physical system. However, Work is always transversal to the process, as $i(\mathbf{V}_4)W = i(\mathbf{V}_4)i(\mathbf{V}_4)dA = 0$, but Heat is not always transversal, as $i(\mathbf{V}_4)Q = i(\mathbf{V}_4)dU \Rightarrow 0$, only for adiabatic processes. It is this fundamental difference between Heat, Q, and Work, W, that lead to the Carnot-like statements that it is possible to convert work into heat with 100% efficiency, but it is not possible to convert heat into work with 100% efficiency.

Local adiabatic direction fields, defined as null curves of Q, do not imply that the Pfaff dimension of Q must be 2. That is, it is not obvious that Q can be written in the form, $Q = TdS$, as is possible on the manifold of equilibrium states. From the Cartan formulation it is apparent that if Q is not zero, then

$$\begin{aligned} i(\mathbf{V}_A)L_{(\mathbf{V}_A)}A &= i(\mathbf{V}_A)i(\mathbf{V}_A)dA + i(\mathbf{V}_A)d(i(\mathbf{V}_A)A) \\ &= 0(by\ transversality) + i(\mathbf{V}_A)d(i(\mathbf{V}_A)A) = i(\mathbf{V}_A)Q \end{aligned} \tag{2.18}$$

If Q is not zero, an Adiabatic process requires that ,

$$\text{Adiabatic process } i(\mathbf{V}_A)Q = i(\mathbf{V}_A)d(i(\mathbf{V}_A)A)) \Rightarrow 0, \quad Q \neq 0 \tag{2.19}$$

$$\text{with a sufficient condition } = i(\mathbf{V}_A)A \Rightarrow 0. \tag{2.20}$$

The sufficient condition implies that the process $\mathbf{V}_A$ is an associated vector of the 1-form of Action. If the heat 1-form is zero, then the process is a reversible adiabatic process of a special type. A reversible process is defined such that the Pfaff dimension of Q is less than 3; or, $Q\hat{}dQ = 0$. Hence $i(\mathbf{V}_A)(Q\hat{}dQ) = 0$ for reversible processes. But

$$i(\mathbf{V}_A)(Q\hat{}dQ) = (i(\mathbf{V}_A)Q)\hat{}dQ - Q\hat{}i(\mathbf{V}_A)dQ, \tag{2.21}$$

which permits reversible and irreversible adiabatic processes to be well distinguished [2] when $Q \neq 0$:

[2] It is apparent that i(V)Q= 0 defines an adiabatic process, but not necessarily a reversible adiabatic process. This topological point clears up certain misconceptions that appear in the literature.

$$\text{Reversible Adiabatic Process} = -Q\hat{\ } i(\mathbf{V}_A)dQ \Rightarrow 0,\ i(\mathbf{V}_A)Q \Rightarrow 0, \quad (2.22)$$
$$\text{Irreversible Adiabatic Process} = -Q\hat{\ } i(\mathbf{V}_A)dQ \neq 0,\quad i(\mathbf{V}_A)Q \Rightarrow 0 \quad (2.23)$$

It is certainly true that if $L_{(\mathbf{V})}A = Q = 0$, *identically*, then all such processes are adiabatic, and reversible. (In sub-section, 2.6, it will be demonstrated how these thermodynamic ideas can be associated with the tensor processes of covariant differentiation and parallel transport.)

2.5.4. Processes classified by topological constraints

Cartan has shown that all Hamiltonian processes (systems with a generator of ordinary differential equations), $\rho\mathbf{V}_H$, satisfy the following equations of topological constraint on the work 1-form, W :

$$\text{A Hamiltonian } \mathbf{V}_H \text{ is either } \mathbf{V}_E \text{ or } \mathbf{V}_B \quad (2.24)$$
$$\text{Extremal Hamiltonian } \mathbf{V}_E \quad (2.25)$$
$$W_E = i(\rho\mathbf{V}_E)dA = 0 \qquad \text{Pfaff dimension} = 0 \quad (2.26)$$
$$\text{Bernoulli-Casimir Hamiltonian } \mathbf{V}_B \quad (2.27)$$
$$W_B = i(\rho\mathbf{V}_B)dA = d\Theta \quad \text{Pfaff dimension} = 1 \quad (2.28)$$

A special case occurs when the Bernoulli function is equal to the negative of the internal energy, for then the heat 1-form produced by this special Hamiltonian process vanishes.

For symplectic processes (which are not strictly Hamiltonian) the situation is a bit more intricate, but in all cases the Pfaff dimension of the Work 1-form is at most 1.

$$\text{Helmholtz Symplectic} \quad (2.29)$$
$$W_S = i(\rho\mathbf{V}_S)dA = d\Theta + \gamma \quad \text{Pfaff dimension} = 1 \quad (2.30)$$
$$dW_S = 0 \text{ as } \gamma \text{ is closed but not exact.} \quad (2.31)$$

The closed but not exact forms, γ, introduce non-uniqueness into the definition of the work 1-form. As $dQ = d(\Theta+\gamma+U) = 0$, for all three processes defined above, the are all reversible (see equation (2.15)), but each class of processes must satisfy an additional topological constraint if the process is to be locally adiabatic:

$$\text{Adiabatic process } i(\mathbf{V}_A)Q = i(\mathbf{V}_A)d(i(\mathbf{V}_A)A)) \Rightarrow 0, \quad Q \neq 0 \tag{2.32}$$

$$\text{with a sufficient condition } = i(\mathbf{V}_A)A \Rightarrow 0. \tag{2.33}$$

If $d\Theta = 0$, then $\rho\mathbf{V}_E$ is a characteristic process relative to the 2-form F. If the work 1-form is of Pfaff topological dimension 0, then the process is an extremal process relative to A (see equation 2.13).

Extremal processes cannot exist on a non-singular symplectic domain, because a non-degenerate anti-symmetric matrix (the coefficients of the 2-form dA) does not have null eigenvectors on space of even dimensions . Although unique extremal stationary states do not exist on the domain of Pfaff dimension 4, there can exist evolutionary invariant Bernoulli-Casimir functions, Θ, that generate non-extremal, "stationary"states. Such Bernoulli processes can correspond to energy dissipative symplectic processes, but they, as well as all symplectic processes, are reversible in the thermodynamic sense described below. The mechanical energy need not be constant, but the Bernoulli-Casimir function(s), Θ,are evolutionary invariant(s), and may be used to describe non-unique stationary state(s).

The equations, above, that define several familiar categories of processes, are in effect constraints on the topological evolution of any physical system represented by an Work 1-form, A. The Pfaff dimension of the 1-form of virtual work, $W = i(\mathbf{V})dA$ is 1 or less for all three categories. The Extremal constraint of equation (2.25) that the Pfaff dimension of W can be used to generate the Euler equations of hydrodynamics for a incompressible fluid. The Bernoulli-Casimir constraint of equation (2.27) can be used to generate the equations for a barotropic compressible fluid. The Helmholtz constraint of equation (2.31) can be used to generate the equations for a Stokes flow. All such processes are thermodynamically reversible as $dQ = 0$. None of these constraints on the Work 1-form, W, above will generate the Navier-Stokes equations, which require that the topological dimension of the 1-form of virtual work must be greater than 2.

A crucial idea is the recognition that non-equilibrium processes must be on domains of Pfaff dimension which support Topological Torsion, $A\hat{}dA \neq 0$, or Topological Spin, $A\hat{}G$, with attendant properties of non-uniqueness, envelopes, regressions, and projectivized tangent bundles. Such domains are of Pfaff dimension 3 or greater. Moreover, as described below, it would appear that thermodynamic irreversibility must support a non-zero Topological Parity 4-form, $dA\hat{}dA \neq 0$. Such domains are of Pfaff dimension 4 or greater.

Although there does not exist a unique gauge independent stationary state on the symplectic manifold of Pfaff dimension 4, remarkably there does exist a unique vector field on the symplectic domain, with components that are generated by the 3-form $A\hat{}dA$. This unique (to within a factor) vector field is defined as the Torsion Current, $\mathbf{T}_4$, and satisfies (on the 2n+2=4 dimensional manifold) the equation,

$$i(\mathbf{T}_4)dx\hat{}dy\hat{}dz\hat{}dt = A\hat{}dA \tag{2.34}$$

This (four component) vector field, $\mathbf{T}_4$, has a non-zero divergence almost everywhere, for if the divergence is zero, then the 4-form $dA\hat{}dA$ vanishes, and the domain is no longer a symplectic manifold! The Torsion vector, $\mathbf{T}_4$, can be used to generate a dynamical system that will decay to the stationary states $(div_4(\mathbf{T}_4) \Rightarrow 0)$ starting from arbitrary initial conditions. These processes are irreversible in the thermodynamic sense. It is remarkable that this unique evolutionary vector field, $\mathbf{T}_4$, is completely determined (to within a factor) by the physical system itself; e.g., the components of the 1-form, A, determine the components of the Torsion vector. Recall that if $L(\mathbf{v})A\hat{}L(\mathbf{v})dA \neq 0$, then the process V acting on the physical system, A, is irreversible. This topological definition implies that the three categories (above) of symplectic, Hamiltonian-Bernoulli or Hamiltonian-extremal processes, $\mathbf{V} \subset \mathbf{S}$, are reversible $(as\ L_{(\mathbf{S})}dA{=}dQ = 0)$. However, for evolution in the direction of the Torsion vector, $\mathbf{T}_4$, direct computation demonstrates that the fundamental equations lead to a conformal evolutionary process, a process which is thermodynamically irreversible:

$$L_{(\mathbf{T}_4)}A = \sigma A \quad\quad and \quad i(\mathbf{T}_4)A = 0, \tag{2.35}$$

such that

$$L_{(\mathbf{T}_4)}A\hat{}L_{(\mathbf{T}_4)}dA = Q\hat{}dQ = \sigma^2 A\hat{}dA \neq 0. \tag{2.36}$$

2.5.5. Topological Torsion and Irreversible Processes.

For any physical system encoded by a 1-form of Action, A, on a variety of 4 independent base variables, it is possible to construct the 3-form of Topological Torsion, $A\hat{}dA$. This 3-form is equivalent to the contraction of the Volume 4-form, Ω_4, with the 4 component vector, $\mathbf{T}_4$,

$$A\hat{}dA = A\hat{}F = i(\mathbf{T}_4)\Omega_4. \tag{2.37}$$

In other words, the direction field, $\mathbf{T}_4$, is defined entirely from the functional structure of the 1-form of Action, A, used to encode the physical system. Evolutionary processes can be defined to within a factor, ρ, by $\mathbf{T}_4$. This topological torsion process defined by, $\rho\mathbf{T}_4$, has remarkable properties.

1. The topological torsion process, $\rho\mathbf{T}_4$, is a null vector for the 1-form of Action, and therefor defines a local adiabatic process:

$$\rho\mathbf{T}_4 \text{ is locally adiabatic: } i(\rho\mathbf{T}_4)A = i(\rho\mathbf{T}_4)Q = 0. \tag{2.38}$$

2. The topological torsion process, $\rho\mathbf{T}_4$, is a conformal vector for the 1-form of Action,

$$L_{(\rho\mathbf{T}_4)}A = i(\rho\mathbf{T}_4)dA = \Gamma A. \tag{2.39}$$

   with a conformal factor $\Gamma = divergence(\rho\mathbf{T}_4)$.

3. The topological torsion process, $\rho\mathbf{T}_4$, creates a thermodynamically irreversible process, when the 4-divergence of $\rho\mathbf{T}_4$ is not zero:

$$Q\hat{}dQ = \Gamma^2 A\hat{}dA \neq 0. \tag{2.40}$$

4. The topological torsion vector, $\rho\mathbf{T}_4$, is a reversible adiabatic characteristic vector relative to A when the 4-divergence, Γ, of $\rho\mathbf{T}_4$ is equal to zero.

$$\text{When } \Gamma = 0, \tag{2.41}$$
$$Q\hat{}dQ = \Gamma^2 A\hat{}dA = 0, \tag{2.42}$$
$$i(\rho\mathbf{T}_4)Q = 0 \tag{2.43}$$
$$i(\rho\mathbf{T}_4)A = 0, \; i(\rho\mathbf{T}_4)dA = 0. \tag{2.44}$$

As the divergence of $A\hat{}F$ is non-zero only if $dA\hat{}dA$ is not zero, it is apparent that such a domain is of Pfaff topological dimension 4, relative to A. Recall that a physical system of Pfaff dimension 4 corresponds to an open thermodynamic system. In this sense, thermodynamic irreversibility is an artifact of 4 dimensions. Recall that the 4-form $F\hat{}F$ is exact, and is equal to $d(A\hat{}F)$. Hence the 4 dimensional integral becomes a deformation invariant if the integral over the boundary of a domain satisfies the condition,

$$\int\int\int\int_M F\hat{}F = \text{is an Absolute Deformation invariant, if} \tag{2.45}$$

$$0 = \int\int\int_{boundary\ of\ M} (\rho V^k)(A\hat{}F) \tag{2.46}$$

Similarly for the 4-form $dQ\hat{}dQ$

$$\int\int\int\int_M dQ\hat{}dQ = \text{is an Absolute Deformation invariant, if} \tag{2.47}$$

$$0 = \int\int\int_{boundary\ of\ M} (\rho V^k)(Q\hat{}dQ) \tag{2.48}$$

The latter condition can be put into correspondence with a closed system where only radiation is passed through the boundary, but mass is not. If the absolute criteria is not satisfied, mass can be exchanged through the boundary with the environment.

Note that for a given 1-form of Action, A, it is possible to construct a matrix of N-1 null vectors, and then to compute the adjoint matrix of cofactors transposed to create the unique direction field (to within a factor), $\mathbf{V}_{NullAdjoint}$. Evolution in the direction of $\mathbf{V}_{NullAdjoint}$ does not represent an adiabatic process path, as $i(\mathbf{V}_{NullAdjoint})A \neq 0$. For a given A, the N-1 null vectors need not span a smooth hypersurface whose surface normal is proportional to a gradient field. The components of the 1-form may be viewed as the normal vector to an implicit hypersurface, but the implicit hypersurface is not necessarily defined as the zero set of some function.

2.6. The Lie differential $L_{(V)}$ and the Covariant differential $\nabla_{(V)}$

Koszul [Hermann 1968] has give a set of axioms that can be used to define a linear connection and a covariant derivative. The covariant derivative axioms require that

$$\nabla_{(fV)}\omega = f\ \nabla_{(V)}\omega, \tag{2.49}$$

$$\nabla_{(V)}f\omega = (\nabla_{(V)}f\ )\omega + f\ \nabla_{(V)}\omega. \tag{2.50}$$

This axiomatic representation of a covariant derivative and an affine connection should be compared to the Lie differential,

$$L_{(fV)}A = f\ L_{(V)}A + df\ (i(V)A), \tag{2.51}$$
$$L_{(V)}fA = (L_{(V)}f\ )A + f\ L_{(V)}A. \tag{2.52}$$

Only if the last term in the expansion of the Lie differential, $df\ (i(V)A)$, is zero does the formula for the Lie differential have an equivalent representation as a covariant derivative in terms of a connection. Suppose that $i(V)A = 0$, such that the Lie differential and the covariant differential are equivalent.

$$L_{(fV)}A = f\ L_{(V)}A = f\ \nabla_{(V)}A. \tag{2.53}$$

It follows that

$$\begin{aligned} L_{(fV)}A &= f\ L_{(V)}A + df\ (i(V)A) && (2.54)\\ &= f\ L_{(V)}A = f\ Q. && (2.55)\\ \text{But } i(V)Q &= f\ i(V)i(V)dA = 0 \Rightarrow i(V)Q, && (2.56)\\ \text{where } i(V)Q &= 0 \text{ defines an adiabatic process.} && (2.57) \end{aligned}$$

Hence, *all covariant derivatives with respect to a connection have an equivalent representation as an adiabatic process*!!!

Such covariant adiabatic processes need not be thermodynamically reversible, but suppose that the adiabatic process is such that

$$L_{(V)}A = Q = 0. \tag{2.58}$$

Then

$$dL_{(V)}A = L_{(V)}dA = dQ = 0, \tag{2.59}$$

and it follows that the adiabatic process is reversible, as $Q\hat{}dQ = 0$. However, the condition that Q be zero is the equivalent to the condition of parallel transport:

$$L_{(V)}\omega = \nabla_{(V)}\omega = 0. \tag{2.60}$$

The remarkable conclusion is that *the concept of parallel transport in tensor analysis is in effect an adiabatic, reversible process*!!!

These conclusions should have impact on the understanding (and limitations) of physical theories (gravity and gauge theories in particular) that rely on the covariant derivative, and the theory of connections. Such connection dependent theories can describe only adiabatic evolution.

3. Topological Electrodynamics

Following the preliminary topological discussions of thermodynamics presented above in section 1 and in section 2, it is now possible to apply a topological and thermodynamical perspective to the theory of non-equilibrium electromagnetic physical systems, and their irreversible and reversible evolutionary processes. The topological view of Electrodynamics described herein is based on three axioms.

1. Electrodynamic physical systems can be encoded in terms of an exterior differential 1-form of covariant Action Potentials per unit charge, $A(x,y,z,t) = A_k(x,y,z,t)dx^k$, on a 4 dimensional abstract variety of ordered independent variables, $\{x,y,z,t\}$. The variety supports a volume element $\Omega_4 = dx\hat{}dy\hat{}dz\hat{}dt$.

2. The 1-form of Potentials creates by exterior differentiation an exact exterior differential 2-form

$$F\{x,y,z,t\} - dA\{x,y,z,t\} = 0. \tag{3.1}$$

3. Electrodynamic currents $\mathbf{J}_4(x,y,z,t)$ are assumed to be encoded, to within a factor, ρ, in terms of contravariant vector direction fields, $\mathbf{V}_4(x,y,z,t)$. The electromagnetic charge current densities (or $N-1 = 3$-form densities, $J = i(\mathbf{J}_4)\Omega_4$), are globally closed, hence exact, and must be generated by the exterior derivative of some $N-2 = 2-$ form density $G\{x,y,z,t\}$:

$$J\{x,y,z,t\} - dG\{x,y,z,t\} = 0. \tag{3.2}$$

As all N-1 contravariant direction fields, $\mathbf{V}_4(x,y,z,t)$, admit an integrating factor ρ in the sense that

$$d(\rho i(\mathbf{V}_4)\Omega_4) = 0, \tag{3.3}$$

then, all closed or exact vector direction fields of the form $\rho i(\mathbf{V}_4)\Omega_4$ have zero divergence, and generate a conservation law usually described as an "equation of continuity". Arbitrary processes $\mathbf{V}_4$ in thermodynamics are not closed, and hence the most general thermodynamic processes are not exact.

Herein, it becomes evident that classical electromagnetism is equivalent to a set of topological constraints on a variety of independent variables. These topological constraints can be written in the language of two exterior differential systems [Bryant 1991]

$$\text{Conserved flux} \quad : \quad F - dA = 0 \Rightarrow L_{(\rho V^k)} \int_{z2d} F = 0 \quad (3.4)$$

$$\text{Conserved charge-current} \quad : \quad J - dG = 0 \Rightarrow L_{(\rho V^k)} \int_{z3d} J = 0. \quad (3.5)$$

It was established by Cartan that exterior differential systems are equivalent to a system of partial differential equations. The two exterior differential systems defined above will be used to generate two systems of partial differential equations, known as the Maxwell-Faraday equations and the Maxwell-Ampere equations, without the geometric constraints of a metric, or a connection.

The exact 2-form of thermodynamic field "intensities" $F(\mathbf{E},\mathbf{B})$ is defined in terms of inexact 1-form of potentials, A, in units of $\hbar/e$. The exact 3-form of charge current density, J, is defined in terms of the inexact 2-form density of thermodynamic field quantities, or "excitations", $G(\mathbf{D},\mathbf{H})$ in units of $\hbar$. The two form, F, historically is associated with forces, and the 2-form density, G, historically is associated with sources.

3.1. The 2-form of Field Intensities (E and B)

The first topological constraint $F - dA = 0$ leads to the 2-form of field intensities, $\mathbf{E}$ and $\mathbf{B}$, and the Maxwell-Faraday equations. Following the usual definitions of the exterior derivative, the components of the 2-form dA become

$$dA = \{\partial A_k/\partial x^j - \partial A_j/\partial x^k\}dx^j\hat{}dx^k = F_{jk}dx^j\hat{}dx^k = +\mathbf{B}_z dx\hat{}dy... + \mathbf{E}_x dx\hat{}dt... \quad (3.6)$$

where in usual engineering notation,

$$\mathbf{E} = -\partial\mathbf{A}/\partial t - grad\phi, \qquad \mathbf{B} = curl\ \mathbf{A}. \quad (3.7)$$

Be aware that the engineering notation, where the six components of the second rank covariant tensor, F_{jk}, are grouped into two 3 component vectors, is deceptive, for the diffeomorphic transformational properties of the field intensities ($\mathbf{E}$ and $\mathbf{B}$) are not that of Cartesian rank 1 three dimensional vectors, but are that of a second rank covariant tensor field.

Note that the Poincare lemma applied to the topological constraint ($ddA = dF = 0$) always leads to the first Maxwell pair of (Faraday Induction) equations: e.g.,

$$ddA = \{curl\ \mathbf{E} + \partial\mathbf{B}/\partial t\}_x dy\hat{}dz\hat{}dt - .. + .. - div\,\mathbf{B} dx\hat{}dy\hat{}dz\} \Rightarrow 0, \tag{3.8}$$

or

$$\{curl\ \mathbf{E} + \partial\mathbf{B}/\partial t = 0, \qquad div\,\mathbf{B} = 0\}. \tag{3.9}$$

This result is actually true for a variety of any dimension $\succeq 4$ and for any set of covariant symbols. Even if the dimension of the ordered domain exceeds 4, the concept of $ddA = dF = 0$, always yields the same set of partial differential equations relative to the first four base variables. The first Maxwell set of equations forms a nested set of PDE's on varieties of pre-geometric dimension > 4. The addition of new independent base variables does not change the format of the first four Maxwell PDE equations, but just adds to the set new PDE equations involving field components defined over the new ordered set of base variables. The concept of Faraday Induction is universal, and should not restricted to the science of electromagnetism. It is valid for any physical system which can be described by a 1-form of Action with a Pfaff dimension 2 or larger, such as a fluid with vorticity.

The very existence of the $\mathbf{E}$ and $\mathbf{B}$ fields implies that the 2-form dA does not vanish. Hence, the 2-form defines a symplectic manifold of at least Pfaff dimension 2. As the 2-form is exact, the symplectic 2-manifold cannot be compact without boundary. This result follows from Stokes theorem, with 2 exceptions: the Klein bottle and the torus. Except for these two exceptions, the exact symplectic domain of the electromagnetic field intensities, $\mathbf{E}$ and $\mathbf{B}$, must either be open, or if compact it has a boundary.

3.2. The 2-form density G of Field excitations and sources

The second topological constraint assumes the existence of a N-2 form density, G, as given by the expression, (note the signs)

$$G = G^{34} dx\hat{}dy... + G^{12} dx\hat{}dt... = -\mathbf{D}^z dx\hat{}dy - .. + \mathbf{H}^z dz\hat{}dt + ... \tag{3.10}$$

Exterior differentiation produces an N-1 form, $J = \mathbf{J}^z dx\hat{}dy\hat{}dt... - \rho dx\hat{}dy\hat{}dz$. Matching the coefficients of the exterior expression $dG = J$ leads to the Maxwell-Ampere equations,

$$curl\,\mathbf{H} - \partial\mathbf{D}/\partial t = \mathbf{J} \tag{3.11}$$

and

$$div\,\mathbf{D} = \rho. \tag{3.12}$$

The fact that J is exact leads to the charge conservation law, $dJ = ddG = 0$, or

$$\partial\mathbf{J}^x/\partial x + \partial\mathbf{J}^y/\partial y + \partial\mathbf{J}^z/\partial z + \partial\rho/\partial t = 0. \tag{3.13}$$

Recall that it should not be presumed that the set of symbols $\{x, y, z, t\}$ is on some Cartesian type of space. The arrangement and the exterior differential equations are dependent only on the ordering of the set $\{x, y, z, t\}$, and not the explicit form of the symbols.

3.3. Topological Features of Electrodynamics

3.3.1. Universal PDE's

Exterior differential systems of topological constraints are equivalent to systems of partial differential equations. The two sets of PDE,s that correspond to the exterior differential system, $F - dA = 0$, and the exterior differential system, $J - dG = 0$, have been shown above to be equivalent to the Maxwell-Faraday equations and to the Maxwell-Ampere equations, respectively. The PDE's are created without the geometric constraints, or choice, of metric, connection, or gauge. To summarize in engineering format:

$$\text{Maxwell Faraday PDE's} \quad curl\,\mathbf{E} + \partial\mathbf{B}/\partial t = 0, \qquad div\,\mathbf{B} = 0, \tag{3.14}$$

$$\text{Maxwell Ampere PDE's} \quad curl\,\mathbf{H} - \partial\mathbf{D}/\partial t = \mathbf{J}, \qquad div\,\mathbf{D} = \rho. \tag{3.15}$$

It is also true that the abstract form of these eight PDE's is universally the same, without additional features, or terms, for all pre-geometric dimensions of 4 or greater. The bottom line is that the system of PDE's that are called Maxwell's theory of Electromagnetism is a topological theory, independent from geometrical constraints.

3.3.2. Thermodynamic features

It is important to realize that the Maxwell-Faraday equations of "intensities" and the Maxwell-Ampere equations of "quantities" belong to two thermodynamic categories which are topologically distinct. In the first category, the 1-form of potentials can have non-unique but topologically closed components, A_c, which do not contribute to the 2-form of intensities, F, as $dA_c \Rightarrow 0$. Similarly, in the second category, the 2-form density of excitations can have non-unique but closed components, G_c, which do not contribute to 3-form density of charge-currents, J, as $dG_c \Rightarrow 0$. In the historic literature of the last 50 years, the non-uniqueness of the potentials has led to a large industry called gauge theories. The non-uniqueness of the excitations has not been so well developed.

3.3.3. Non-uniqueness and topological defects

These concepts of non-uniqueness can effect the topological properties of the solutions to the Maxwell electrodynamic system. The non-uniqueness can appear as discontinuities in solution amplitude and derivatives (electromagnetic signals), as a solution multi-valuedness (polarization), as envelope (Huygen wavelet or Cherenkov) solutions, and in many other physically recognizable topological properties related to the continuity, integrability, differentiability, compactness, and reality of the solutions. In short, the non-uniqueness of both the potentials, A, and the field excitations, G, lead to the concept of a species of topological defects as regions where topological refinements of uniqueness, integrability, differentiability, compactness, or reality fail. Of particular note is the fact that the solutions to the PDE's which involve topological defects need not be defined in a locally linear, and therefor unique, manner. The defects are sets upon which the solutions do not behave as tensors, for tensors require that different neighborhoods are related by diffeomorphisms and therefor impose uniqueness by linearity. The use of exterior differential forms circumvents these limitations of uniqueness demanded by the theory of tensor analysis, for differential forms are functionally well behaved in a retrodictive pullback sense [RMK 1976 b] relative to differential maps which are not one to one diffeomorphisms. Exterior differential forms are well behaved with respect to evolutionary processes that include topological change, and the production of topological defects.

3.3.4. Topological defects and "Quantization" in connected systems

Topological defects find representation in terms of closed, but not exact, homogeneous differential forms of topological dimension M immersed in spaces of pre-geometric dimension $N \succeq M$. The integrals over closed integration chains (closed cycles which may not be boundaries) of such closed, but not exact, exterior differential forms are the basis of deRham cohomology theory [deRham 1960]. The values of such closed integrals have rational ratios and provide a topological basis for "quantization". Integrals of exact k-forms over closed cycles or boundaries are always zero. Integrals of closed but not exact k-forms over cycles which are not boundaries are not zero. Integrals of closed but not exact k-forms over boundaries have zero values. In a topological theory of electromagnetism, the closed, but not exact, components of the 1-form of potentials, A, lead to 1-dimensional period integrals and the concept of the flux quantum as a 1-dimensional topological defect. Similarly, closed but not exact components of the 2-form of excitation densities, G, lead to 2-dimensional period integrals and the concept of the charge quantum as a 2-dimensional topological defect. These period integral concepts have been discussed elsewhere [RMK 1977].

3.3.5. Topological defects and "Quantization" in disconnected systems

Most of the emphasis in this article however is placed upon the non-classical, closed but not exact, exterior differential 3-forms and 4-forms, which can be constructed from A and G and their exterior derivatives and exterior products. The two most important of these constructions, in a domain of 4 pre-geometric dimensions, is defined in terms of the Topological Torsion 3-form, $A\hat{}F$, and the Topological Spin 3-form density, $A\hat{}G$. The 3-dimensional topological defects associated with the closed but not exact components of these 3-forms are defined as the topological torsion-helicity quantum and the topological spin-chirality quantum, respectively. It can be demonstrated that flux quantum period integrals are elements of topological defects in equilibrium thermodynamic systems, but the torsion-helicity quanta and the spin-chirality quanta are elements of topological defects in *non-equilibrium*, dissipative, thermodynamic systems, and can be created as artifacts of irreversible processes. It is the discovery of these non-equilibrium properties that demonstrates the utility of the topological perspective of electromagnetism. These ideas will be discussed in detail in section 4.

3.4. Signals and Propagating Topological Defects

The topological constraints, hence the Maxwell system of PDE's they generate, are valid in any frame of acceptable (diffeomorphically related) coordinates. So why is the Lorentz system of transformations so dominant in classical electromagnetic theory? In a remarkable piece of work [Fock 1964] V. Fock demonstrated that the point set upon which the solutions to the Maxwell PDE's are not uniquely defined forms a propagating amplitude discontinuity. He clearly formulated the idea that electromagnetic signals were such propagating field discontinuities. (This concept of a signal and its precise relationship to electromagnetic fields was not clearly defined by Einstein).

To quote Victor Fock:

> "The laws of propagation of light in empty space are thoroughly understood. They find their expression in the well-known equations of Maxwell. However, we are not interested in the general case of light propagation, but only in the propagation of a signal advancing with maximum speed; i.e., the propagation of a wave front. Ahead of the front of the wave all components of the field vanish. Behind it some of them are different from zero. Therefore, some of the field components must be discontinuous at the front. On the other hand, given the field (a solution to the equations) on some surface moving in space, the derivatives of the field are, in general, determined by Maxwell's equations. Hence the value of the field at an infinitely near surface is also (uniquely) determined (by analytical continuation) and discontinuities are impossible. The only case when this is not so is when the form and the motion of the surface satisfies certain special conditions subject to which the values of the derivatives is not determined by the values of the field components themselves. Such a surface is called a characteristic surface, or briefly, a characteristic. Thus discontinuities of the field can occur only on a characteristic, but since there must certainly be discontinuities at a wave front (the signal), such a front is clearly a characteristic."

Fock also demonstrated that the characteristic or singular point set for the system of Maxwell PDE's was given by the non-linear partial differential equation called the Eikonal equation. The Eikonal equation is a non-linear partial differential equation that consists of a (canonical) quadratic form with signature (+,+,+,-) or (-,-,-,+):

$$\text{Eikonal} \quad (\pm\partial\varphi/\partial x)^2 \pm (\partial\varphi/\partial y)^2 \pm (\partial\varphi/\partial z)^2 \mp 1/c^2(\partial\varphi/\partial t)^2 = 0. \tag{3.16}$$

These ideas were championed by Luneberg (and others) at the end of WWII.

The key idea in Fock's work is the result that the characteristic solutions remain characteristic solutions (discontinuities remain discontinuities - signals remain signals) only under a limited set of coordinate transformations. The laws of Maxwell as tensor equations are well behaved with respect to ALL diffeomorphisms, but the characteristic solutions to Maxwell's equations retain their topological properties only with respect to a restricted class of transformations. Fock proves that the linear class of transformations that preserve signals is the Lorentz group, for which a finite propagation speed is an invariant concept. However, the part of Fock's development that has been ignored is the idea that there also exists a non-linear transformation group (the Moebius fractional projective transformations) that also preserve discontinuities, signals, and the properties of characteristics. Such signals are not restricted to finite propagation speed. These results will be the topic of another presentation.

3.5. Characteristics

For the work presented herein, the source free system of forms $\Sigma_0 = \{F, G\}$, $\{J = 0\}$ will be treated as a closed ideal, $d\Sigma_0 = \{dF, dG\} = 0$. The Cartan technique will be used to search for two systems of dual vector fields that define point sets which are complimentary in a union and intersection sense. This problem is equivalent to finding those point sets upon which the field amplitudes admit discontinuities, but still satisfy Maxwell's equations. In other words, the solutions to Maxwell's equations are not unique upon the singular point set. There are two ways to go about this problem, the ray method (where a search is made for the characteristics of the differential ideal Σ_0) and the "dual" wave method [RMK 1991 d]. In this article only the wave method will be pursued in detail.

Consider the set of covariant fields k which annihilate the system of forms of the closed ideal in an intersection sense. These point sets are given by the covariant vectors

$$\mathbf{k}_4 = \{\mathbf{k}, \omega\} \tag{3.17}$$

such that the l-form, $k = \mathbf{k} \cdot d\mathbf{r} - \omega\, dt$, will have null intersections with the closed ideal of forms $\{\Sigma_0\}$; i.e., search for k such that

$$k\hat{}\{\Sigma_0\} = 0, \tag{3.18}$$

or

$$k\hat{}F = 0, \quad k\hat{}G = 0. \tag{3.19}$$

These equations of characteristics have an engineering format as the set

$$\begin{aligned} \mathbf{k} \times \mathbf{E} - \omega\mathbf{B} &= 0, \quad \mathbf{k} \cdot \mathbf{B} = 0, & (3.20) \\ \mathbf{k} \times \mathbf{H} + \omega\mathbf{D} &= 0, \quad \mathbf{k} \cdot \mathbf{D} = 0, & (3.21) \end{aligned}$$

from which it is apparent that $\mathbf{k}$ has the direction of the momentum flux $\mathbf{D} \times \mathbf{B}$. It is to be noted that these six independent equations for the covariant wave field $\mathbf{k}$ are the usual sets of equations which define the point sets upon which field discontinuities may exist [Kline 1965]. The 1-form, k, can be multiplied by any non-zero function without changing the format of the equations. In fact it is more convenient to use the covariant field in the format $k = \{\mathbf{k}, \omega\} = \omega\{\mathbf{n}, 1\}$. (The equivalent analysis in terms of rays, implies that the ray vector is in the direction of the energy flux, $\mathbf{E} \times \mathbf{H}$. The energy flux and the momentum flux are not necessarily in the same direction).

If the characteristic system is viewed as a set of six equations in 12 unknowns representing the components of the fields, then the problem does not admit a solution without further constraints. To alleviate this problem, it is convenient to impose geometrical constraints in the form of constitutive relations on the topological system. In tensor language, assume

$$G^{\mu\nu} = \chi^{\mu\nu\alpha\beta} F_{\alpha\beta} \tag{3.22}$$

Due to the antisymmetries of the forms, the 256 components of the tensor density, $\chi^{\mu\nu\alpha\beta}$ in four dimensions may be reduced to 36 distinct values, a reduction which leads to the 6 vector formalism of Sommerfeld [Sommerfeld 1952]; namely, in matrix format,

$$\begin{aligned} \mathbf{D} &= \theta([\epsilon] \circ \mathbf{E} + [\gamma] \circ \mathbf{B}) & (3.23) \\ \mathbf{H} &= \theta([\gamma^\dagger] \circ \mathbf{E} + [\mu^{-1}] \circ \mathbf{B}). & (3.24) \end{aligned}$$

When the constitutive matrix is complex it can be partitioned into real and imaginary parts. The various optical effects that have been measured in classical electromagnetism then can be into correspondence with symmetries induced by the diagonal and off diagonal matrix elements of the partitioned representation (See Post [Post 1983]).

$$\left| \begin{array}{c} D \\ H \end{array} \right\rangle = \left[\begin{array}{cc} [dielectric\_birefr.] & [Fresnel\_Fizeau] \\ [Fresnel\_Fizeau] & [magnetic\_birefr.] \end{array} \right] \circ \left| \begin{array}{c} E \\ B \end{array} \right\rangle + \tag{3.25}$$

$$\sqrt{-1} \left[\begin{array}{cc} [dielectric\_Faraday] & [Opt\_Act] \\ [Opt\_Act] & [magnetic\_Faraday] \end{array} \right] \circ \left| \begin{array}{c} E \\ B \end{array} \right\rangle \tag{3.26}$$

3.6. Fresnel - Kummer surfaces

The six constitutive equations given above may be used to eliminate half of the unknowns in the six equations for the characteristic system. For example, the elimination of **E** and **H** in the characteristic equations leaves a homogeneous system of six equations in six variables. The Cramers determinantal condition on this homogeneous set must be satisfied if a solution exists. The Cramers determinant condition, $\mathcal{H}(n_x, n_y, n_z) = 0$ leads to a quartic Kummer surface in the components of the three vector of reciprocal phase velocity, $\mathbf{n} = \mathbf{k}/\omega$. By using the matrix representation,

$$[\mathbf{n}^{\times}] = \left[\begin{array}{ccc} 0 & n_z & -n_y \\ -n_z & 0 & n_x \\ n_y & -n_x & 0 \end{array} \right] \tag{3.27}$$

the Kummer surface $(\theta = 1)$ can be defined by the equation:

$$\mathcal{H}(n_x, n_y, n_z) = det([\epsilon] + [\gamma][\mathbf{n}^{\times}] - [\mathbf{n}^{\times}][\gamma^{\dagger}] + [\mathbf{n}^{\times}][\mu^{-1}][\mathbf{n}^{\times}]) = 0. \tag{3.28}$$

In the three-dimensional space of variables (n_x, n_y, n_z) the implicit hypersurface $\mathcal{H} = 0$ is of fourth degree and creates an extension of the usual Fresnel wave surface to include not only anisotropic birefringence, but also electric and magnetic Faraday rotation, optical activity, and Fresnel-Fizeau phenomena in combination.

In the case of birefringence symmetries alone, the quartic function $\mathcal{H} = 0$ splits into two doubly degenerate quadratic factors which are the usual representations of the Fresnel ellipsoids, one ellipsoid for each state of polarization. In the case of the Lorentz symmetries, the Kummer surface is fourth order degenerate with the phase velocity of propagation equal for both states of polarization and for both directions of propagation.

The general Fresnel wave surfaces are distorted ellipsoids which do not have a center of symmetry. Hence an arbitrary line through the origin in reciprocal phase velocity space will intercept the Fresnel wave surface(s) in four distinct points representing four distinct phase velocities: two magnitudes in the outbound direction for each state of polarization, and two *different* velocities in the inbound direction, one for each state of polarization. A Maple based program (http://www22.pair.com/csdc/pdf/fresnel.pdf) is available for computing the Frenel wave surfaces.

The little appreciated result is that the propagation velocity of a signal can be sensitive to the direction of propagation, which of course implies that the constitutive equations do not have the symmetries of the Lorentz vacuum. These effects were experimentally verified in dual polarized ring laser experiments conducted by Sanders [RMK 1977 b]. When the solutions to the Kummer surface equation are not degenerate (four distinct roots to the fourth order polynomial), then the solutions to the wave equation are best described in terms of quaternions, and not real or complex vectors.

The constitutive technique demonstrates that there exist characteristic wave solutions to Maxwell's equations for which the propagation velocity is *not* fourfold degenerate. Each of two states of polarization can have two different speeds in the outbound direction, and two different phase speeds in the inbound direction. All four phase speeds can be distinct. It is a property of the Lorentz Vacuum constitutive map that these 4 speeds are degenerate; that is, the propagation speeds for each polarization and each direction are the same. The Lorentz constitutive map also insures that the fields satisfy both the Maxwell-Faraday and Maxwell-Ampere equations and the vacuum condition that charge-current 4 vector vanishes. Under these constraints, the fields satisfy the vector wave equation with the same phase speed, c. However, the Lorentz constitutive equations are not the only constitutive equations which have this property. In the next section, the concept of a Chiral vacuum will be studied. The only difference between the Lorentz Vacuum and the Chiral Vacuum is in the conformal factor that represents the radiation impedance.

3.7. The Chiral vacuum

In the Lorentz vacuum case, the classic assumption reduces the constitutive matrix to the format

$$\text{Lorentz vacuum} \tag{3.29}$$

$$\left| \begin{array}{c} \mathbf{D} \\ \mathbf{H} \end{array} \right\rangle = \left[\begin{array}{cc} \epsilon_0 [\mathbf{I}] & [\mathbf{0}] \\ [\mathbf{0}] & 1/\mu_0 [\mathbf{I}] \end{array} \right] \circ \left| \begin{array}{c} \mathbf{E} \\ \mathbf{B} \end{array} \right\rangle . \tag{3.30}$$

where ε_0 and $1/\mu_0$ are presumed to be constants. It is not difficult to demonstrate that with the Lorentz constitutive constraint, all fields, $\mathbf{D}, \mathbf{H}, \mathbf{E}, \mathbf{B}$ satisfy the vector wave equation of the form

$$\nabla^2 \mathbf{E} - \epsilon_0 \mu_0 \partial^2 \mathbf{E}/\partial t^2 = \nabla^2 \mathbf{E} - (1/c^2) \partial^2 \mathbf{E}/\partial t^2 = 0, \tag{3.31}$$

$$\epsilon_0 \mu_0 c^2 = 1. \tag{3.32}$$

However consider the constitutive constraint defined as the Chiral vacuum, where β/Z_0 is defined as the Chiral coefficient. The Chiral coefficient factor, β, can be either positive or negative. In a sense the chiral vacuum admits either a left handed or a right handed "rotation", associated with polarization. The factor θ can be interpreted as a conformal factor, and plays the role of an expansion or contraction. In both the Lorentz case and the Chiral case, the First Poincare invariant is the same.

$$\text{Chiral vacuum with } \mathrm{Z}_0 = \sqrt{\mu/\epsilon} \tag{3.33}$$

$$\left| \begin{array}{c} \mathbf{D} \\ \mathbf{H} \end{array} \right\rangle = \theta \left[\begin{array}{cc} \epsilon_0 [\mathbf{I}] & \beta/Z_0 [\mathbf{I}] \\ -\beta/Z_0 [\mathbf{I}] & 1/\mu_0 [\mathbf{I}] \end{array} \right] \circ \left| \begin{array}{c} \mathbf{E} \\ \mathbf{B} \end{array} \right\rangle . \tag{3.34}$$

It again is straight forward to show the fields are solutions of the *same* vector wave equations. The difference between the two vacuum constitutive assumptions is that the determinants (related to the cube of the radiation impedance) are not the same.

$$\text{Determinant } \theta \left[\begin{array}{cc} \epsilon_0 [\mathbf{I}] & \beta/Z_0 [\mathbf{I}] \\ -\beta/Z_0 [\mathbf{I}] & 1/\mu_0 [\mathbf{I}] \end{array} \right] = \theta^6 (1/Z_0 + \beta^2/Z_0)^3. \tag{3.35}$$

Note that β can be either positive or negative, implying that is a left handed and a right hand chirality difference in the constitutive matrix. The factor β

implies a rotation of some sort in 6 dimensions, and the factor θ corresponds to an expansion factor.

The question arises as to the value of β^2/Z_0? A possible suggestion motivated the Bateman solutions (which are discussed below) is that $\beta^2/Z_0 = 1/Z_{Hall}$. If this conjecture is valid then the reciprocal of the radiation impedance for the Chiral vacuum is $(1+2\alpha)$ times larger than the reciprocal radiation impedance for the Lorentz vacuum. The number $\alpha \sim 1/137$ is the fine structure constant.

The Fresnel-Kummer wave surface equation for the characteristic of the Maxwell equations may be written as the polynomial,

$$(R^2 - 1)^2 = 0, \tag{3.36}$$

where $R^2 = {n_x}^2 + {n_y}^2 + {n_z}^2 = \mathbf{n} \circ \mathbf{n}$ represents the norm of the projectivized wave vector (index of refraction vector), $\mathbf{n} = \mathbf{k}/\omega$. Solutions of the characteristic polynomial yield the phase velocities of propagation in terms of the magnitude of the reciprocal index of refraction vector, $\mathbf{n}$. The Kummer surfaces represent two superposed spheres in $\mathbf{n}$ space. Both solutions imply that the 4-vector of charge current density is zero. Hence the interaction energy density, $A\hat{}J$, is zero.

> These are a rather startling results for they demonstrate that the Lorentz vacuum and the Chiral vacuum can be formally indistinguishable, except for the impedance of free space (which is related to the determinant of the constitutive tensor and therefor to the chiral and expansion coefficients).

4. Non-Equilibrium Electromagnetic Systems

4.1. Topological Spin and Topological Torsion

The exterior differential forms that make up the electromagnetic system on a geometric domain of 4 dimensions consist of the primitive 1-form, A, and the primitive N-2 form density, G, their exterior derivatives, and their algebraic intersections defined by all possible exterior products. The complete Maxwell system of exterior differential forms (the Pfaff sequence for the Maxwell system on 4 geometric dimensions) is given by the set:

$$\{A; F = dA, G; J = dG, A\hat{}F, A\hat{}G, A\hat{}J; F\hat{}F, G\hat{}G\}. \tag{4.1}$$

These differential forms and their unions may be used to form a topological base on the domain of independent variables. The Cartan topology constructed on this system of forms has the useful feature that the exterior derivative may be interpreted as a limit point, or closure, operator in the sense of Kuratowski (see p. 72 in [Lipschutz 1965]). The exterior differential systems that define the Maxwell-Ampere and the Maxwell-Faraday equations above are essentially topological constraints of closure.

The complete Maxwell system of differential forms (which assumes the existence of A and G and C2 differentiability) also generates two other exterior differential systems,

$$(F\hat{}G - A\hat{}J) - d(A\hat{}G) = 0, \tag{4.2}$$

$$F\hat{}F - d(A\hat{}F) = 0, \tag{4.3}$$

which prolong the primary (exact) exterior differential systems,

$$F - dA = 0, \tag{4.4}$$

$$J - dG = 0. \tag{4.5}$$

Each of the forms, A, G, $A\hat{}G$, $A\hat{}F$, can have closed but not exact components. The two 4-forms $(F\hat{}G - A\hat{}J)$ and $(F\hat{}F)$ are exact and have closed integrals which are evolutionary (relative) invariants of continuous deformations. The closed integrals therefor describe topological properties.

The first 3-form density, $A\hat{}G$, with physical units of $\hbar$, is called the "topological spin" (or chirality) [RMK 1977] and the second 3-form, $A\hat{}F$, with physical units of $(\hbar/e)^2$, is called the "topological torsion" (or helicity) [RMK 1990]. These two exterior 3-forms, $A\hat{}G$ and $A\hat{}F$ are not usually found in discussions of classical electromagnetism. The 3-forms are abstractly defined (on a space of 4 geometric dimensions with a volume element, $\Omega_4 = dx\hat{}dy\hat{}dz\hat{}dt$) in terms of exterior multiplication, but can be given realization in terms of 4 component engineering variables, $\mathbf{S}_4$ and $\mathbf{T}_4$.

$$\text{Topological Spin density} \quad : \quad A\hat{}G = i(\mathbf{S}_4)\Omega_4 \tag{4.6}$$

$$\mathbf{S}_4 = [\mathbf{S}, \sigma] = [\mathbf{A} \times \mathbf{H} + \mathbf{D}\phi, \mathbf{A} \circ \mathbf{D}], \tag{4.7}$$

$$\text{Topological Torsion vector} \quad : \quad A\hat{}F = i(\mathbf{T}_4)\Omega_4 \tag{4.8}$$

$$\mathbf{T}_4 = [\mathbf{T}, h] = -[\mathbf{E} \times \mathbf{A} + \mathbf{B}\phi, \mathbf{A} \circ \mathbf{B}]. \tag{4.9}$$

These constructions should be compared with the exact charge current 4-vector density, J, with a 4 component engineering representation, $\mathbf{J}_4 = [\mathbf{J}, \rho]$. The concepts of the topological Spin density (current) and the topological Torsion vector have had almost no utilization in applications of classical electromagnetic theory. Each construction depends explicitly on the existence of the 1-form of Action-potentials.

In previous sections it was noted that the closed components of the 1-form of Action do not effect the components of the 1-form of intensities, $F = dA = d(A_c + A_0) = 0 + d(A_0)$. However, these "gauge" additions do influence the topological dimension of the 1-form of Action. For example, let A_0 be of Pfaff Topological dimension 2, representing an equilibrium system where $A_0\hat{}dA_0 = 0$. Then by addition of a closed component to the original action, $A = A_c + A_0$ could have a topological dimension of 3, as

$$A\hat{}dA = (A_c + A_0)\hat{}dA_0 = A_c\hat{}dA_0 \neq 0. \tag{4.10}$$

So the addition of a closed component to the 1-form of Action could change the system from an equilibrium system to a non-equilibrium system. The 4-form $dA\hat{}dA$ is not influenced by the (gauge) addition to the original 1-form of Action.

$$dA\hat{}dA = dA_0\hat{}dA_0. \tag{4.11}$$

In the example below, a 1-form representing a Bohm-Aharanov-Abrikosov singular "vortex" string, $\gamma = b(ydx - xdy)/(x^2 + y^2)$, is added to a 1/r potential for a point source. The bare m/r "Coulomb" potential, $A_0 = m/\sqrt{(x^2 + y^2 + z^2)}dt$ exhibits no Topological Torsion, but does exhibit Topological Spin. The 1/r potential term implies that $dA_0 \neq 0$. Hence the 1-form of Action representing a bare "coulomb" potential, is not in equilibrium, but does represent a connected "isolated" topology of Pfaff dimension 2. The combined 1-form of Action,

$$A = b(ydx - xdy)/(x^2 + y^2) + m/\sqrt{(x^2 + y^2 + z^2)}dt \tag{4.12}$$

even though $d\gamma = 0$, is of Pfaff dimension 3, not 2. The addition of the BAA term changes the topology of the 1-form from a connected topology (Pfaff dimension 2) to a disconnected topology (Pfaff dimension 3). The Topological Torsion 3-form $A\hat{}F$ depends on both b and m, and is zero if $b = 0$, or if $m = 0$, reducing the Pfaff dimension of the modified 1-form back to 2. If $b = 0$ and $m \neq 0$, the 3-form $A\hat{}G$ is not zero.

4.1.1. Coulomb plus vortex singularity.

Consider the Potentials for the combined vortex singularity and Coulomb 1/r potential:

$$A = [by/(x^2+y^2), -bx/(x^2+y^2), 0, -(1/4\pi\epsilon)m/r], \quad (4.13)$$
$$r = \sqrt{x^2+y^2+z^2}. \quad (4.14)$$

The induced fields (assuming $\mathbf{D} = \varepsilon\mathbf{E}$, $\mathbf{B} = \mu\mathbf{H}$) are:

$$\mathbf{E} = (1/4\pi\epsilon)m[x,y,z]/r^3 \quad (4.15)$$

$$\mathbf{B} = [0,0,0] \quad (4.16)$$

$$\text{Topological Torsion } \mathbf{T}_4 = (1/4\pi\epsilon)mb[-zx, -zy, (x^2+y^2), 0]/(r^3(x^2+y^2)). \quad (4.17)$$

$$Poincare2 = 0 \quad (4.18)$$

$$\mathbf{J}_4 = [0,0,0,0] \quad (4.19)$$

$$\rho = 0 \quad (4.20)$$

$$\text{Topological spin } \mathbf{S}_4 = (1/4\pi\epsilon)^2\epsilon m^2/r^4[x,y,z,0]. \quad (4.21)$$

$$Poincare1 = -(1/4\pi\epsilon)^2\epsilon m^2/r^4 \quad (4.22)$$

$$\mathbf{E}\times\mathbf{H} = [0,0,0]. \quad (4.23)$$

$$(\mathbf{J}\circ\mathbf{E}) = 0 \quad (4.24)$$
$$(\mathbf{E}\circ\mathbf{B}) = 0, \quad (4.25)$$
$$(\mathbf{A}\circ\mathbf{B}) = 0, \quad (4.26)$$
$$(\mathbf{A}\circ\mathbf{D}) = 0 \quad (4.27)$$

4.2. The Poincare Topological Invariants

The exterior derivatives of the 3-forms of topological Spin and topological Torsion produce two exact 4-forms, $F\hat{}G - A\hat{}J$ and $F\hat{}F$, whose closed integrals are topological objects which generalize the conformal invariants [Whittaker 1944] of a Lorentz system, as discovered by Poincare and Bateman. Note that these topological properties of invariance with respect to continuous deformations are valid even in the non-equilbrium domain of dissipative charge-currents and radiation.

In the format of independent variables $\{x, y, z, t\}$, with a volume element Ω_4, the exterior derivative, acting on the 3-forms as a topological limit point generator, can be related to the classic 4-divergence of the 4-component Topological Spin and Topological Torsion vectors, $\mathbf{S}_4$ and $\mathbf{T}_4$.

$$\begin{aligned} Poincare\ 1 &= d(A\hat{}G) = F\hat{}G - A\hat{}J \\ &= \{div_3(\mathbf{A}\times\mathbf{H} + \mathbf{D}\phi) + \partial(\mathbf{A}\circ\mathbf{D})/\partial t\}\ \Omega_4 \\ &= \{(\mathbf{B}\circ\mathbf{H} - \mathbf{D}\circ\mathbf{E}) - (\mathbf{A}\circ\mathbf{J} - \rho\phi)\}\ \Omega_4, \quad (4.28) \\ &= 0 \ \text{ for the example in the previous section.} \quad (4.29) \end{aligned}$$

$$\begin{aligned} Poincare\ 2 &= d(A\hat{}F) = F\hat{}F \\ &= \{div_3(\mathbf{E}\times\mathbf{A} + \mathbf{B}\phi) + \partial(\mathbf{A}\circ\mathbf{B})/\partial t\}\ \Omega_4 \\ &= \{2\mathbf{E}\circ\mathbf{B}\}\ \Omega_4 \quad (4.30) \\ &\neq 0 \text{ for the example in the previous section.} \quad (4.31) \end{aligned}$$

The Poincare invariants are, in effect, the evolutionary source terms for the 3-forms of topological spin, $A\hat{}G$, and topological torsion, $A\hat{}F$. When the Poincare invariants are zero, the closed integrals of the electromagnetic 3-forms of $A\hat{}G$ and $A\hat{}F$ become additional topologically coherent configurations invariant with respect to all evolutionary processes of continuous deformation.

The first term in the first Poincare invariant has a coefficient function which represents twice the difference between the magnetic energy density and the electric energy density of the electromagnetic field in a Lagrangian sense:

$$\text{Topological Field Lagrangian:}\ \ F\hat{}G = (\mathbf{B}\circ\mathbf{H} - \mathbf{D}\circ\mathbf{E})\ \Omega_4. \quad (4.32)$$

The second term in the first Poincare invariant has a coefficient function which is defined as the interaction energy density:

$$\text{Topological Interaction: } A\hat{}J = (\mathbf{A} \circ \mathbf{J} - \rho\phi)\ \Omega_4. \tag{4.33}$$

In Lagrangian variational methods, the 4-form $F\hat{}F$, which defines the second Poincare invariant, has been related to the concept of Topological Parity:

$$\text{Topological Parity: } F\hat{}F = +\{2\mathbf{E} \circ \mathbf{B}\}\ \Omega_4. \tag{4.34}$$

Using the example system, (**??**) permits the construction of the Topological Spin current and its divergence relative to the Lorentz-Lorenz constitutive constraint.

4.3. Topological Torsion and Spin quanta

When either Poincare deformation invariant vanishes, the corresponding closed 3-dimensional integrals of $A\hat{}G$ and $A\hat{}F$ become deRham period integrals. The closed, but not exact, components of each 3-form can be put into correspondence with "quantized" topological defects.

The topological Spin quantum is defined as the closed integral of those closed but not exact components of the 3-form $A\hat{}G$ (which represent the kernel of the first Poincare 4-form),

$$\text{Spin quantum} = \iiint_{z3d} A\hat{}G \text{ with units n } \hbar. \tag{4.35}$$

The period integrals $\iiint_{z3d} A\hat{}G$ are deformation invariants (hence define a topological property) with rational ratios. The notation $z3d$ designates a closed integration chain defined in regions where $d(A\hat{}G) = 0$.

Similarly, when the second Poincare invariant vanishes, the closed integral of the 3-form of Torsion-Helicity becomes a deformation invariant with quantized values:

$$\text{Torsion quantum} = \iiint_{z3d} A\hat{}F \text{ with units m } (\hbar/e)^2. \tag{4.36}$$

The period integrals $\iiint_{z3d} A\hat{}F$ are deformation invariants (hence define a topological property) with rational ratios. In this case, The notation $z3d$ designates a closed integration chain defined in regions where $d(A\hat{}F) = 0$.

It is important to realize that the topological conservation laws (deformation invariants with respect to homeomorphisms) are valid in a plasma as well as in the vacuum, subject to the conditions of zero values for the Poincare invariants. On the other hand, topological evolution and transitions between "quantized" states of Spin-chirality or Torsion-helicity require that the respective Poincare invariants are not zero.

The 3-forms, $A\hat{}G$, and $A\hat{}F$, are not necessarily closed, nor exact. Their exterior differentials (divergences) are not necessarily zero. The values of the 4-forms created by exterior differentiation of these 3-forms define the integrands of the topological Poincare invariants. As these 4-forms are exact by construction, their closed integrals are always relative integral deformation invariants and thereby define topological properties. The 3-forms are not necessarily, in themselves, deformation invariants.

However, when the Poincare invariants vanish (zero divergence) the closed integral of the corresponding 3-form generates a topological quantity (Topological Spin or Topological Torsion respectively) which is also a deformation invariant. In such situations, the 3-forms are closed, but not necessarily exact. Hence their closed integrals generate deRham period integrals [Flanders 1963] [deRham 1960], and have rational ratios. Such is the stuff of topological quantization, which is independent from scales.

The theory of 3-forms and their period integrals was investigated with respect to electromagnetism and other field theories by the present author, first with respect to the three form defined below as topological spin, $A\hat{}G$, and then later with respect to the 3-form of topological torsion, defined as $A\hat{}F$. The first application of $A\hat{}F$ was in the field of turbulence [RMK1976], where it was conjectured that the transition from streamline flow (uniquely integrable in the sense of Frobenius, such that $A\hat{}F = 0$) to a turbulent flow (not uniquely integrable in the sense of Frobenius, $A\hat{}F \neq 0$) must involve a topological change. Although the interest was focused on hydrodynamics, the electromagnetic format was always used to establish a credence level in the computations that were done by hand. In the modern world of symbolic calculators on your desktop, this algebraic tedium has been alleviated. See http://www22.pair.com/csdc/pdf/maxwell.pdf

On domains where the Pfaff topological dimension is 3 (and not 4) there exists a 3 dimensional period integral of the topological torsion, which is related to the Hopf invariant (see p. 228 in [Bott 1994]), $\iiint_{3\_cycle} A\hat{}F$. It will be demonstrated below that if the domain is of Pfaff dimension 3, then evolutionary processes in the direction of the electromagnetic charge current 4 vector, J, leave the integral of the Topological Torsion current over a 3 dimensional boundary as an evolutionary invariant. Even more remarkable is the fact that such a statement is valid in domains where the Pfaff dimension is 4, not 3, if the current flow is on the surface defined by $(\mathbf{E} \bullet \mathbf{B}) = 0$. See the section below on the Hopf Invariant.

4.4. Chirality and Orientation

The idea of a chiral constitutive map indicates that there exists a right handed or a left handed "rotation" that does not affect the First Poincare invariant. The First Poincare invariant defines the orientation (or inbound versus outbound directions) which is insensitive to the sense of the chiral factor. If the orientation is fixed by the First Poincare invariant, there are two admissible "rotations" or polarizations permitted by the chirality factor. On the other hand, if the second Poincare invariant is fixed, then there can be two "translation" directions permitted for First Poincare invariant, which will be related to either an expansion or a contraction.

4.5. Topologically Transverse waves

The vanishing of the topological Spin 3-form is a topological constraint on the domain that defines topologically transverse electric (TTE) waves: the vector potential, $\mathbf{A}$, is orthogonal to $\mathbf{D}$, in the sense that $\mathbf{A} \circ \mathbf{D} = 0$. The vanishing of the topological Torsion 3-form is a different topological constraint on the domain that defines topologically transverse magnetic (TTM) waves: the vector potential, $\mathbf{A}$, is orthogonal to $\mathbf{B}$, in the sense that $\mathbf{A} \circ \mathbf{B} = 0$. (In fluid dynamics, $\mathbf{A} \circ \mathbf{B} = 0$ is called the helicity). When both 3-forms vanish, the topological constraint on the domain defines topologically transverse (TTEM) waves. The geometric definitions of transverse waves may or may not be equivalent to the topological definitions. For classic real fields this double constraint would require that vector potential, $\mathbf{A}$, is collinear with the field momentum, $\mathbf{D} \times \mathbf{B}$, and in the direction of the wave vector, $\mathbf{k}$. Theoretical examples (and dielectric waveguide experiments) indicate that a TTEM solution does not radiate. The result gives a possible explanation for the paradox that an electron in the Bohr model is accelerated, yet does not seem to radiate. This conjecture will be discussed elsewhere.

Note that if the 2-form F was not exact, such topological concepts of transversality would be without meaning, for the 3-forms of Topological Spin and Topological Torsion depend upon the existence of the 1-form of Action-Potentials, A. The torsion vector $\mathbf{T}_4$ and the Spin vector $\mathbf{S}_4$ are "associated" vectors to the 1-form of Action in the sense that

$$i(\mathbf{T}_4)A = 0 \quad and \quad i(\mathbf{S}_4)A = 0, \tag{4.37}$$

a result will prove to be of importance in the description of a topological basis for superconductivity. Evolution in the direction of either $\mathbf{T}_4$ or $\mathbf{S}_4$ is adiabatic.

4.6. Whittaker solutions.

It is possible to deduce the Maxwell-Faraday equations starting from the assumption that the 1-form of Action can be written in terms of two complex functions, α, and β, such that

$$A = \alpha d\beta - \beta d\alpha \Rightarrow \mathbf{A} = \alpha\nabla\beta - \beta\nabla\alpha, \quad \phi = -(\alpha\partial\beta/\partial t - \beta\partial\alpha/\partial t) \tag{4.38}$$

then the derived 2-form $F = 2d\alpha \hat{} d\beta$ generates the complex field intensities,

$$\mathbf{E} = (\partial\alpha/\partial t)\nabla\beta - (\partial\beta/\partial t)\nabla\alpha \quad and \quad \mathbf{B} = \nabla\alpha \times \nabla\beta,$$

Further assume a "Self Dual Chiral" constitutive map of the form

$$\text{Self dual Chiral vacuum with } \mathrm{Z}_0 = \sqrt{\mu/\epsilon} \tag{4.39}$$

$$\left| \begin{matrix} \mathbf{D} \\ \mathbf{H} \end{matrix} \right\rangle = \begin{bmatrix} 0 & \sqrt{-1}/Z_0\,[\mathbf{I}] \\ -\sqrt{-1}/Z_0\,[\mathbf{I}] & 0 \end{bmatrix} \circ \left| \begin{matrix} \mathbf{E} \\ \mathbf{B} \end{matrix} \right\rangle. \tag{4.40}$$

such that the resulting "self-dual" excitations constructed from the two arbitrary functions indeed satisfy the Maxwell-Ampere equations and the vacuum requirement that the charge current densities vanish. The construction yields:

$$\mathbf{H} = (\sqrt{-1}/Z_0)(\partial\alpha/\partial t)\nabla\beta - (\partial\beta/\partial t)\nabla\alpha \quad and \quad \mathbf{D} = -(\sqrt{-1}/Z_0)\nabla\alpha \times \nabla\beta. \tag{4.41}$$

The functions α and β may be used to construct an arbitrary function, $F(\alpha, \beta)$, and remarkably enough, the arbitrary function $F(\alpha, \beta)$ satisfies the Eikonal equation,

$$(\nabla F)^2 - \varepsilon\mu(\partial F/\partial t)^2 = 0. \tag{4.42}$$

From experience with Eikonal solutions and wave equations, it might be thought that Eikonal solutions are sufficient. However, the Bateman conditions (see below) are necessary, for the candidate solution

$$\alpha = (x \pm iy)/(z - ct), \quad \beta = (r - ct), \quad r = \sqrt{x^2 + y^2 + z^2)}. \tag{4.43}$$

satisfies the Eikonal equation, but not the Bateman conditions. When the Bateman conditions are satisfied, the solutions represent TTEM modes in the vacuum, and do not radiate. For arbitrary functions the algebra of electromagnetism can become quite complex. A Maple symbolic mathematics program for computing the various terms is available (see http//www22.pair.com/csdc/pdf/maxwell.pdf).

4.7. Conjugate and Minimal Surface Wave Functions

4.7.1. Bateman's Conjugate String Solutions

In 1914, in a small monograph entitled Electrical and Optical Wave Motion [Bateman 1914], H. Bateman introduced a number of interesting solutions to Maxwell's equations that emulate propagating singular strings (not plane waves). Bateman is perhaps more famous for his work on the equations that describe the decay chains of radioactive species. However, as pointed out by Whittaker [Whittaker 1944], it was Bateman who determined in 1910 that the Maxwell equations were invariant with respect to the conformal group, a much wider group than the Lorentz transformations. Bateman in 1910 also recognized the relationship of his work to the tensor calculus of Ricci and Levi-Civita, several years before the Einstein development of general relativity.

Transcribing the Bateman ideas to more modern form, consider the two Maxwell equations in the form $dF = 0$ and $dG = 0$. Use the chiral vacuum constitutive equations (3.34) to create the pair,

$$curl\,\mathbf{E} + \partial\mathbf{B}/\partial t = 0, \tag{4.44}$$

$$curl\,\mathbf{H} - \partial\mathbf{D}/\partial t = (1/\mu)curl\,\mathbf{B} + \varepsilon\partial\mathbf{E}/\partial t - \beta/Z_0\,\{curl\mathbf{E} + \partial\mathbf{B}/\partial t\} = 0. \tag{4.45}$$

The charge current 4 vector vanishes when both E and B satsify the vector wave equation, and the result is independent from the chiral factor, β/Z_0.

Form the inner 3D product of the first equation with $\mathbf{B}/\mu$ and the second with $\mathbf{E}$, to create the equation

$$div(\mathbf{E}\times\mathbf{B}) + \partial(1/2\mathbf{B}^2 + 1/2\mu\varepsilon\mathbf{E}^2)/\partial t = 0, \tag{4.46}$$

which is equivalent to an equation of continuity of the form $div(\rho\mathbf{v}) + \partial\rho/\partial t = 0$. By comparison, this "equation of continuity" yields

$$\rho\mathbf{v} = \mathbf{E}\times\mathbf{B} \quad \text{where} \quad \rho = (1/2\mathbf{B}^2 + 1/2\mu\varepsilon\mathbf{E}^2). \tag{4.47}$$

with the energy density given by the expression for ρ. Bateman finds the extraordinary result that

$$\rho^2(1/\mu\varepsilon - \mathbf{v} \circ \mathbf{v}) = \rho^2(c^2 - \mathbf{v} \circ \mathbf{v}) \quad (4.48)$$

$$= 1/4(\mathbf{E}^2 - c^2\mathbf{B}^2)^2 + c^2(\mathbf{E} \circ \mathbf{B})^2. \quad (4.49)$$

He remarks that "the rate at which energy flows through the field is less that the velocity of light", unless the two Poincare invariants on the RHS vanish. All of this follows with or with out the chiral factor, β/Z_0.

The key idea according to Bateman was to find complex 3-dimensional vector solutions, $\mathbf{M} = \mathbf{E}+ic\mathbf{B}$, to Maxwell's equations that satisfy the complex constraint equation,

$$\mathbf{M} \circ \mathbf{M} = (\mathbf{E}^2 - c^2\mathbf{B}^2) + 2\sqrt{-1}c(\mathbf{E} \circ \mathbf{B}) = 0. \quad (4.50)$$

Such solutions were defined by Bateman as self conjugate solutions. Note that the two factors are related to the first and second Poincare invariants. The importance of the null Poincare invariants becomes obvious, as they furnish the requirement that the field energy propagates with the speed of light.

In modern language of differential forms, Bateman determined that if two functions $\alpha(x, y, z, t)$ and $\beta(x, y, z, t)$ used to define the 2-form

$$F = d\alpha\hat{}d\beta = 1/2(\alpha d\beta - \beta d\alpha), \quad (4.51)$$

such that the field components

$$\mathbf{E} = (\partial\alpha/\partial t)\nabla\beta - (\partial\beta/\partial t)\nabla\alpha \quad and \quad \mathbf{B} = \nabla\alpha \times \nabla\beta, \quad (4.52)$$

satisfied the Maxwell Faraday equations, $curl\, \mathbf{E} + \partial\mathbf{B}/\partial t = 0$, then the N-2 form defined by construction as

$$G = i(d\alpha)\hat{}i(d\beta)\Omega = i(d\alpha)\hat{}i(d\beta)dx\hat{}dy\hat{}dz\hat{}dt \quad (4.53)$$

would satisfy the Maxwell Ampere equations $curl\, \mathbf{H} - \partial\mathbf{D}/\partial t = 0$ with

$$\mathbf{H} = (\partial\alpha/\partial t)\nabla\beta - (\partial\beta/\partial t)\nabla\alpha \quad and \quad \mathbf{D} = \nabla\alpha \times \nabla\beta. \quad (4.54)$$

This construction implies an unusual constitutive map of the form $\mathbf{D} = -\sqrt{-1}\gamma\mathbf{B}$ and $\mathbf{H} = \sqrt{-1}\gamma\mathbf{E}$. From the discussion above, it is apparent that the result utilizes the chiral factor γ in the "vacuum" constitutive equations given above. Any pair of functions can be added to can be added to a solution of the wave equation, and Maxwell's equations for a chiral vacuum are satisfied.

4.8. Minimal Surfaces

4.8.1. Bateman's Minimal Surface Solutions

Now of even more extraordinary interest is the fact that Bateman's self conjugate solutions, based on a complex vector with zero square, define a minimal surface. Before going on note that the norm of the self conjugate vector solution is given by the expression

$$\mathbf{M}^{*}\circ\mathbf{M} = [c^{2}\mathbf{B}^{*}\circ\mathbf{B} + \mathbf{E}^{*}\circ\mathbf{E}] \tag{4.55}$$

which is proportional to the energy density of the field. If this term is non-zero, it will turn out that the minimal surface will be regular.

From Osserman [Osserman 1986], a generalized minimal surface in E^{N} is a non-constant map from a 2-manifold, M, with a conformal structure (over regular regions) such that the coordinates of E^{N} are harmonic on M. Let the map be defined by

$$\Psi : \{\alpha, \beta\} \Rightarrow \mathbf{X}^{k}(\alpha + \sqrt{-1}\beta) \tag{4.56}$$

Define the complex vector

$$\mathbf{M} = \partial(\mathbf{X}^{k}(\alpha+\sqrt{-1}\beta))/\partial\alpha - \sqrt{-1}\partial(\mathbf{X}^{k}(\alpha+\sqrt{-1}\beta))/\partial\beta \Rightarrow \mathbf{E}+\sqrt{-1}c\mathbf{B} \tag{4.57}$$

Then if $\mathbf{M}\circ\mathbf{M} = 0$ the two functions α and β form a set of isothermal coordinates on a minimal surface. The induced metric $g_{\alpha\beta} = \sum_{k} \partial\mathbf{X}^{k}/\partial\alpha\circ\partial\mathbf{X}^{k}/\partial\beta$ generates a conformal structure, and if $\mathbf{M}^{*}\circ\mathbf{M} \neq 0$, the minimal surface is regular (without self intersections or pinch points). For real $\mathbf{E}$ and $\mathbf{B}$ the minimal surfaces are always regular except when both components vanish identically (No field intensities). For complex $\mathbf{E}$ and $\mathbf{B}$ the minimal surfaces can have singularities, which are of particular interest from a topological point of view.

The self conjugate (minimal surface) condition requires that the $\mathbf{E}$ field is orthogonal to the $\mathbf{B}$ field, and the electric and magnetic energy densities are equal. The conclusion is reached that propagating electromagnetic waves can be associated with minimal surfaces. The associated minimal surface is always regular and without singularities for real, non-zero $\mathbf{E}$ and $\mathbf{B}$. When the $\mathbf{E}$ and $\mathbf{B}$ fields are complex, which in a physical sense implies the existence of elliptical polarization, another interpretation is possible.

Note that the product of $\mathbf{M} \circ \mathbf{M}$, if not zero, forms another complex vector in the complex domain. This two component vector defines a holomorphic curve in terms of the Poincare invariants, $P1$ and $P2$.

$$\Theta = \mathbf{M} \circ \mathbf{M} = (\mathbf{E}^2 - c^2\mathbf{B}^2) + 2\sqrt{-1}c(\mathbf{E} \circ \mathbf{B}) = P1 + 2\sqrt{-1}cP2. \tag{4.58}$$

The magnitude of the complex number, Θ, is

$$\Theta^*\Theta = (\mathbf{E}^2 - c^2\mathbf{B}^2)^2 + 4c^2(\mathbf{E} \circ \mathbf{B})^2 \tag{4.59}$$

$$= (P1)^2 + 4c^2(P2)^2 \tag{4.60}$$

which is 4 times the RHS of Bateman's equation of continuity, (4.49).

This association of electromagnetic wave propagation with minimal surface theory was apparently unknown to Bateman, and not appreciated by the present author until only very recently, following a nth re-reading of Osserman's book on "A Survey of Minimal Surfaces" [Osserman 1986]. According to Osserman, the complex 3-vector representations of minimal surfaces were known to Enneper and Weierstrass. A study of the minimal surfaces generated in E4 is given by Kommerell. The minimal surfaces so generated in E4 by this class of vector fields will have 3-dimensional images that are not always regular. In general, two dimensional non-regular surfaces may have "singularities" consisting of "curves of double points" created by intersections of two local surface patches, or of "triple" points consisting of intersections of three local surface patches, or of curves of double points which terminate on "Pinch" points within the interior of the surface. These three types of self intersection singularities are the only three "stable" singularities in the sense of Whitney.

Recall that Whitney proved that any N manifold can be embedded in 2N+1 euclidean space, and immersed in a 2N euclidean space. The induced surfaces may be orientable or non-orientable. The non-orientable examples are characterized by the Klein-Bottle, or the Projective Plane, and the orientable surfaces by the Sphere. Each surface may have tubular handles, holes and distortions. Of interest to this work are not just any surface, but those surfaces which in particular are minimal surfaces. If the surface has no singularities, then the surface is said to be regular or embedded. The constraint of regularity implies that the surface normal vector never goes to zero over the surface, or the induced metric on the surface is always invertible. This implies that are always two linearly independent directions on a regular domain of the surface. If the lines of self intersection are divergence free on the domain (meaning that they stop or start only on boundary points, or are closed upon themselves, then the surface is said to be immersed in 3-Dimensions. The points where the divergence of the lines of intersection is not zero are defined as Pinch points. Such surfaces cannot be immersed in 3-D. The Pinch points are signatures of the fact the surface resides in 4-Dimensions (as an immersion), and cannot be immersed in 3-Dimensions. A flow vector field may have domains where it is irrotational or solenoidal, and these domains may be separated by a surface. If the surface of separation is a minimal surface, then the flow on this surface is harmonic. The minimal surface need not be regular, and may have lines of self-intersection. These lines of surface self-intersections (lines of singular double points) are not necessarily solenoidal. In fact, the Pinch points are points where the lines of self-intersection terminate not on themselves and not on a boundary, but in the surface interior. The Pinch points may be viewed as the "sources" of the divergence of the lines of self-intersection. The classic example is given by Whitney's Umbrella or Cross-cap, the last of the only three possible stable, but singular, mappings.

4.8.2. Spinor solutions.

It is perhaps of more interest to realize that the components of the Eikonal quadratic form can be interpreted as an isotropic null vector, for which E. Cartan gave the name Spinor [Cartan 1966]. This topological result of modern electromagnetism emphasizes the fact that spinors play a role in wave phenomena at all scales, from the micro world of quantum mechanics to the macro world of cosmology. The Eikonal equation is a quadratic form with the same signature as the Minkowski metric line element. Einstein's specification that light travels along a null geodesic follows from the constraint of a metric geometry. However, the characteristic solutions representing propagating discontinuities follow from the system of PDE's with out regard to a metric. Spinor solutions for the characteristics are not tensors, for spinors have an ambiguity under reflections with respect to a sign. Superposition of (conjugate) spinors can be arranged to produce vectors, which are tensors. These facts are realized in spinor states of circular polarization which can be combined into vector states of linear polarization. These concepts will be detailed in the section 5.2.5 below.

Applications of constitutive equations and their topological features to the generation of the singular solution sets of the PDE's (describing domains of propagating non-uniqueness) and the interpretation in terms of electromagnetic signals are given in [RMK 1991 d]. A notable result is the demonstration of exact (quaternion - spinor) solutions for which the 4-fold degeneracy of propagating signals is broken, not only for polarization, but also for propagation direction. The propagation speed of an electromagnetic signal outbound is not necessarily the same as the propagation speed inbound! The signal speed need not be the same in both directions, in direct confrontation with an axiom of special relativity. The results were verified in dual polarized ring laser experiments [RMK 1977 b].

4.8.3. An Example from the Hopf Map

An important topological feature of the two 3-forms, $A\hat{}G$ and $A\hat{}F$, that distinguishes them from the 1-form A and the 2-forms F and G, is that both 3-forms vanish on domains which are of Pfaff topological dimension 2 or less. That is, the 3-forms of Topological spin and Topological Torsion (and their exterior differentials) vanish for equilibrium electrodynamic systems. The two 3-forms supply important topological information in non-equilibrium plasmas.

It can be demonstrated for non-equilbrium electrodynamic systems of Pfaff topological dimension 3, there can exist electromagnetic domains for which the magnetic field $\mathbf{B}$ is irreducibly 3 dimensional. Moreover, there can exist propagating modes for such fields such that magnetic field must have a longitudinal component. This result settles the somewhat controversial arguments that have appeared in the literature in the last few years concerning the possible existence of longitudinal magnetic fields. Longitudinal (non-transverse) magnetic fields with three irreducible non-zero components do not exist if the 3-form of Topological Torsion current is identically zero. Existence of a non-equilibrium solution that admits a topological 3-form $A\hat{}F$ and demonstrates a propagating solution with a longitudinal component of $\mathbf{B}$ is given in Section 6.2.4

5. Evolutionary Processes and Electromagnetism

5.1. Reversible equilibrium and non-equilibrium processes

Consider an arbitrary process defined by the contravariant 4 vector field $\mathbf{J}_4 = \rho\{\mathbf{V};\mathbf{1}\}$ which will cause the evolution of a physical system defined by a 1-form of Action, A. For an electromagnetic 1-form of Action, the 1-form of virtual

Work, $W = (i(\mathbf{J}_4)dA)$, generates an expression that defines the components of the Lorentz force law:

$$W = i(\mathbf{J}_4)dA = (\rho\{\mathbf{E} + \mathbf{V} \times \mathbf{B}\}_k dx^k - \{\rho\mathbf{V} \bullet \mathbf{E}\}dt). \tag{5.1}$$

The question now arises as to the Pfaff topological dimension of the 1-form of virtual work, W, and its physical implications. The simplest case is when the Pfaff dimension of the virtual work 1-form is zero (globally). Then each of the four components of the virtual work 1-form must vanish. Such a constraint is impossible if the Pfaff dimension of the 1-form of Action is 4, for then there do not exist and eigenvectors of the matrix of coefficients, F_{mn}, with zero eigen values.

5.1.1. The Hamiltonian Extremal Class

The Hamiltonian extremal class is defined by those processes (or currents) such

$$W = i(\mathbf{J}_4)dA = 0 \tag{5.2}$$

The Pfaff dimension of the Work 1-form, W, is zero. When the Pfaff dimension of the Action 1-form is 3, the induced 2-form $F = dA$ is of rank 3 on a four dimensional domain. That is, the coefficients of the 2-form form a 4x4 anti-symmetric matrix, with two null eigenvalues. There exists, then, two distinct eigenvectors with a null eigen value. These two eigenvectors define two evolutionary fields which are extremal vector fields in the sense of the calculus of variations. That is, both vector fields produce a zero value for the Virtual Work 1-form. Such extremal evolutionary processes leave the integral of the 1-form of Action stationary, and are called "extremal" fields. (If the integration path is not closed, then the stationary variational principle imposes the additional requirement that $\rho\{\mathbf{A} \cdot \mathbf{V} - \phi\} \Rightarrow 0$ on the boundary. The integration path must start and terminate on the zero set of this (Lagrange) function.)

One of the two vector fields with null eigen values is not only extremal, but also it is characteristic, in the sense that not only does the virtual work vanish locally, but so does the internal energy of interaction. Such vectors are both extremal (virtual work is zero) and associated (interaction internal energy is zero). The added boundary condition in the calculus of variations is not required for the characteristic field. This *unique characteristic* evolutionary vector field, is proportional to the Torsion vector for situations where the Pfaff topological dimension of A is 3. Evolution along the direction of the characteristic vector field is always adiabatic.

For evolution in the direction of any extremal vector field (which may or may not be adiabatic), the components of the virtual work 1-form will vanish; e.g., the Pfaff dimension of the Virtual work is zero. Such an evolutionary process defines what is call the "perfect" plasma process in classical electromagnetic theory, for then there is no "ohmic" dissipation, ($\{\rho\mathbf{V}\bullet\mathbf{E}\} = 0$) and the Lorentz force vanishes, $\rho\{\mathbf{E}+\mathbf{V}\times\mathbf{B}\}\Rightarrow 0$. This condition is subsumed as the classic constraint for the sophomore problem of a charge particle moving in crossed magnetic and electric fields. If the extremal process is bit adiabatic and extremal, it is a characteristic process.

Extremal and Characteristic vector fields always admit a Hamiltonian representation for the Action 1-form.

$$A \Rightarrow pdq - h(p,q,t)dt. \tag{5.3}$$

5.1.2. The Hamiltonian Bernoulli Class

The Hamiltonian Bernoulli process satisfies the equation

$$W = i(\mathbf{J}_4)dA = d\Theta. \tag{5.4}$$

The Pfaff dimension of the Work 1-form, W, is unity. The Bernoulli process is necessarily adiabatic. The kinematic generating function is $\mathbf{\Theta}$.

5.1.3. The Helmholtz Symplectic Class (the Master Equation)

A somewhat more general class of evolutionary possibilities exists on those domains where the Pfaff dimension of the 1-form of Action is 3, but the Pfaff dimension of the virtual work 1-form is 1, not zero. In this case the virtual work 1-form must be closed, $d(i(\mathbf{W})d\mathrm{A})\Rightarrow 0$. Such a constraint defines the Helmholtz class of evolutionary processes, and leads to the "conservation of vorticity" in the hydrodynamic case, and to the concept of "frozen in magnetic flux" in the electromagnetic case. The closure condition implies that the Lorentz force need not be zero, but it should have zero curl: $d(\rho\{\mathbf{E}+\mathbf{V}\times\mathbf{B}\}_k dx^k - \{\rho\mathbf{V}\bullet\mathbf{E}\}dt) = 0$. First consider that case where ρ is a constant. Then, the necessary condition to satisfy the closure condition, for arbitrary displacements of the independent variables, is that

$$curl\,\{\mathbf{E}+\mathbf{V}\times\mathbf{B}\} = 0 \tag{5.5}$$

and similarly

$$\partial\{\mathbf{E}+\mathbf{V}\times\mathbf{B}\}/\partial t+grad\{\mathbf{V}\cdot\mathbf{E}\}=0. \tag{5.6}$$

Substituting the Maxwell result, $curl\,\mathbf{E}=-\partial\mathbf{B}/\partial t$, leads to the Master equation of the Imperfect Plasma:

$$-\partial\mathbf{B}/\partial t+curl\{\mathbf{V}\times\mathbf{B}\}=0. \tag{5.7}$$

Such evolutionary processes are defined as symplectomorphisms in the modern literature.

For non-constant "charge density distributions", ρ, the Helmholtz closure constraint requires that

$$\rho(\mathbf{E}+\mathbf{V}\times\mathbf{B})=-grad(\Theta). \tag{5.8}$$

In this case, the Master equation is modified slightly to account for a non-constant distributions, ρ :

$$-\partial\mathbf{B}/\partial t+curl\{\mathbf{V}\times\mathbf{B}\}=grad\,\ln\rho\times(\mathbf{E}+\mathbf{V}\times\mathbf{B}). \tag{5.9}$$

In elementary physics the scaling function ρ is to recognized as the charge density distribution. In elementary mathematics, the scaling function ρ is to be recognized as the integrating factor. In topological situations ρ can play the role of a deformation parameter.

There are two important cases that must be examined for the Helmholtz class of evolutionary processes. Either the Pfaff dimension is even and of dimension 4, or it is odd and of Pfaff dimension 3. If the Pfaff dimension of the Action is 4, then the symplectic *manifold* condition of maximal rank over the 4 dimensional domain requires that the second Poincare invariant is not zero:

$$dA\hat{}dA=2(\mathbf{E}\bullet\mathbf{B})dx\hat{}dy\hat{}dz\hat{}dt\neq 0. \tag{5.10}$$

The Helmholtz evolutionary process condition requires that

$$\rho(\mathbf{E}\bullet\mathbf{B})=\mathbf{B}\bullet grad\,\Theta\neq 0 \tag{5.11}$$

Therefore, there must exist a gradient (of pressure or temperature, Θ) in the direction of the $\mathbf{B}$ field lines. Similarly, there is ohmic dissipation if the motion is in the direction of the gradient, for then

$$\rho(\mathbf{E}\bullet\mathbf{V})=\mathbf{V}\bullet grad\,\Theta. \tag{5.12}$$

There is no "ohmic" dissipation in the direction orthogonal to $grad\,\Theta$.

Physically, then, in a symplectic system exhibiting symplectic evolution, there must exist a component of electric force **E** that accelerates charged particles along the magnetic **B** field lines, and that component of force, as an artifact of the symplectic constraints, must be the ultimate source of the magnetic dynamo. A similar situation holds in hydrodynamics where fluid mass can be accelerated along the lines of vorticity. For the extremal, non-symplectic case, the Lorentz force must vanish, and there is no magnetic dynamo action.

From the argument developed above for symplectic systems, the Bernoulli-Casimir energy function Θ is either of the type TS and/or of the type PV. For a solid, assume the former representation dominates. Then the "Lorentz force" must have the form of a spatial gradient of the temperature, $\rho(\mathbf{E} + \mathbf{V} \times \mathbf{B}) = grad(kT)$. For motion that is along the magnetic field lines, the term $\mathbf{V} \times \mathbf{B} \Rightarrow 0$. Then, incorporating the empirical Ohmic relation, $\mathbf{j} = \sigma\mathbf{E}$, it is apparent that the symplectic case leads to a derivation of flux equations in the Thompson format for thermal power:

$$\mathbf{j} = (1/\rho\sigma)grad(kT) \tag{5.13}$$

The suggestion is that the source of magnetic dynamo forces in plasmas is to be associated with a temperature gradient and the existence of differential velocity fluctuations in a symplectic system.

5.2. The Topological Hall effect

In the Bernoulli case of a Plasma process, the integrand must be proportional to an exact 2-form, $d\omega$. There is one obvious candidate, the 2-form, F :

$$i(\beta V)dG = \beta\{(\mathbf{J} - \rho\mathbf{V})^x dy\hat{}dz - ... + (\mathbf{J} \times \mathbf{V})^x dx\hat{}dt... = \sigma_{Hall}F. \tag{5.14}$$

The conductivity coefficient σ_{Hall} in the expression must be a domain constant. Comparing the components of the equation of constraint yields the properties of a Bernoulli Plasma process:

$$\text{Bernoulli Plasma process,} \quad \mathbf{J}_B = \rho\mathbf{V} + (\sigma_{Hall}/\beta)\mathbf{B}, \tag{5.15}$$

$$\text{and,} \quad (\mathbf{J}_B \times \mathbf{V}) = (\sigma_{Hall}/\beta)\mathbf{E}. \tag{5.16}$$

$$(\sigma_{Hall}/\beta)(\mathbf{J}_B \circ \mathbf{E}) \Rightarrow 0 \quad (5.17)$$
$$(\sigma_{Hall}/\beta)(\mathbf{V} \circ \mathbf{E}) \Rightarrow 0 \quad (5.18)$$
$$(\sigma_{Hall}/\beta)(\mathbf{E} \circ \mathbf{B}) \Rightarrow 0. \quad (5.19)$$

Thus the Bernoulli plasma process leads to a current $\mathbf{J}_B$ which is orthogonal to the $\mathbf{E}$ field and whose magnitude is proportional to the $\mathbf{B}$ field. To quote Landau and Lifshitz [Landau 1960] "As we see, it (the Hall effect) gives rise to a current perpendicular to the electric field, whose magnitude is proportional to the magnetic field." The conclusion is that the Bernoulli Plasma process generates a Hall effect, and requires that the second Poincare coefficient must vanish. It follows that the Topological Hall effect exists in non-equilibrium systems where the 1-form A cannot be of Pfaff dimension 4, but can be of Pfaff dimension 3. Bernoulli plasma processes are not dissipative in the sense that such that $(\mathbf{J}_B \circ \mathbf{E}) = 0$.

The appearance of a magnetic conductance, σ_{Hall}, is novel to the topological format of electromagnetism as presented herein, and is deduced from the sole assumption that the Plasma current defines a process direction field that preserves the closed integrals of the 2-form, G. Plasma processes do not change the net charge within the closed integration domain. That is, charges can be produced only in equal and opposite pairs by a "Plasma process". The total net charge is preserved.

The conclusion is that the Hall effect is a topological property of electromagnetism, and can appear at all scales, from the microworld to the macroworld to the cosmological world. In the next section, and in domains where the non-equilibrium 3-forms of Topological Torsion and Topological Spin are closed, it is demonstrated how the Topological Hall effect can have an impedance multiplied by a rational fraction. That is, the rational fraction Hall impedance is a topological result, independent from quantum theory.

5.3. Topological Superconductors.

Post (see the appendix in [RMK 1991 c]) recognized that one of the most remarkable features of electrodynamics of the micro world compared to the macro world is that when written in terms of the MKS system of units, the fine structure constant becomes,

$$\alpha = 2\pi e^2/4\pi\varepsilon\, hc = 1/2(\mu/\varepsilon)^{1/2}/(h/e^2). \tag{5.20}$$

This formula demonstrates that α is a ratio of two fundamental impedances, the free-space impedance,

$$Z_0 = (\mu/\varepsilon)^{1/2} = 376.730313\Omega, \tag{5.21}$$

and the Hall impedance,

$$Z_{Hall} = h/e^2 = 25812.81491\Omega. \tag{5.22}$$

The relation between the quantum mechanical entities and the free space impedance as given by the equation,

$$\alpha = 1/2(Z_0/Z_{Hall}), \tag{5.23}$$

elevates the importance of the free space impedance, Z_0

The objective is to define superconductivity in a topological manner which incorporates the "quantization" features of deRham cohomology theory. The idea follows from the recognition that the Hall impedance exhibits rational fraction behavior. This rational fraction behavior was predicted on topological grounds by E. J. Post [Post 1983]. The implication is that superconductivity is related to topological defect structures. There are three ways to construct an impedance Z (with physical dimensions h/e^2) from period integrals [RMK 1991 c].

$$\begin{array}{rl} \text{Ordinary Superconductors :} & \text{Impedance } \mathsf{Z}_1 = \oint A / \iint_{z2} G \\ \text{Anyon High Tc ?} & \mathsf{Z}_2 = \iiint_{z3} A\hat{}G / (\iint_{z2} G)^2 \\ \text{Fractional Hall :} & \mathsf{Z}_3 = \iiint_{z3} A\hat{}F / \iiint_{z3} A\hat{}G \end{array} \tag{5.24}$$

In order to produce rational fractions, the closed integrals must be period integrals, where the integrands are closed in an exterior differential sense over the closed domains (cycles of 1, 2 or 3 dimensions) of integration. The closure condition on the first impedance Z_1 requires that $dA \Rightarrow 0$, which implies that the domain excludes the field intensities (Meissner repulsion). The closure condition on the third impedance, Z_3 requires that both Poincare invariants must vanish, but **E** and **B** fields are permitted in the domain of integration (as is observed in the Hall effect).

The conjecture to be explored herein is that a supercurrent corresponds to the case where the electromagnetic interaction energy density, $A\hat{}J$, vanishes in a topological sense. The motivation for such an assumption is founded upon the observation that if the 3-form of charge current density, J, was proportional to either the 3-form of topological torsion, $J = A\hat{}F$, or the 3-form of topological spin, $J = A\hat{}G$, then it follows that the interaction energy density of classical field theory will vanish, $A\hat{}J \Rightarrow 0$. Assume that a supercurrent contains components proportional to topological torsion 3-form and the topological spin 3-form. In order for the components of such a charge current 4-vector to be closed (and exact) the respective Poincare invariants of its components must be zero. Under such constraints, the closed integrals of the closed 3-forms of topological torsion and topological spin have rational (quantized) values, and become "deformable" topological invariants. Although the geometrical dimension of space time is 4, the constrained system has topological Pfaff dimension 3 and is not an equilibrium system. The evolutionary processes represented in terms of the divergence free forms of $\mathbf{T}_4$ and $\mathbf{S}_4$ are not irreversible. Such formulations, therefor, are possible candidates for non-dissipative supercurrents.

A third case would consider those situations where the 3-form charge current density has components proportional to those components of the 1-form of potentials which are elements of a spinor, $J = \lambda J_{spinor}$. The 3-form can always be multiplied by an integrating factor such that the rescaled spinor current has zero divergence. Similarly, suppose the 1-form A_{spinor} (to within a factor) also has the same spinor component functions. Then the interaction density vanishes, as

$$\text{Spinor London current:} \quad J_{spinor} = i(\lambda A_{spinor})\Omega \tag{5.25}$$
$$\text{Interaction Energy density:} \quad A\hat{}J = \lambda \langle A_{spinor} \circ A_{spinor} \rangle \ \Omega_4 \Rightarrow 0. \tag{5.26}$$

Hence, a charge current 3-form composed of three parts, such that

$$\text{Total Supercurrent } J_{\text{supercurrent}} = J_{spinor} + A\hat{}F/\lambda + A\hat{}G/\eta, \tag{5.27}$$
$$\text{With Interaction Energy density } A\hat{}J = A\hat{}J_{\text{supercurrent}} \Rightarrow 0, \tag{5.28}$$

is a candidate for a superconducting current, which intuitively has no interaction energy density.

If the Action 1-form is divided by a suitable by a quadratic Holder norm, then the Jacobian matrix of the Action 1 can be computed. The matrix has a determinant zero, if the homogeneity index is 1. The matrix defines the equivalent to the Shape matrix in differential geometry, and its Cayley–Hamilton similarity invariants define the curvatures generated by the zero set of the Cayley-Hamilton characteristic polynomial. If the current $J_{adjoint}$ is defined as the product of the adjoint of the shape matrix times the homogeneous coefficients of the 1-form of Action, then the homogeneous interaction energy

$$\text{Homogeneous Interaction energy}: A_{\text{homogeneous}}\hat{}J_{\text{adjoint}} \tag{5.29}$$

is exactly equal to the cubic curvature similarity invariant.

5.4. The Torsion Vector and Irreversible Evolutionary Processes (Pfaff dimension 4)

Assume that the Pfaff dimension of the domain of interest is 4, hence the space is symplectic. However, consider evolutionary fields that are not constrained to be symplectic such that $dW \neq 0$. Direct evaluation of the virtual work 1-form, $W = i(\mathbf{V}_4)dA$ yields (the Lorentz force)

$$W = i(\mathbf{V}_4)dA = (\{\rho\mathbf{E} + \mathbf{J}x\mathbf{B}\}_k dx^k - \{\mathbf{J} \bullet \mathbf{E}\}dt) \tag{5.30}$$

The obvious first choice for the evolutionary vector field has been based on the classic assumption that $\mathbf{V}_4$=[$\mathbf{J}$;ρ] $\Rightarrow \rho[\mathbf{V}; 1]$ The expression for virtual work becomes

$$W = \rho(\{\mathbf{E} + \mathbf{V}x\mathbf{B}\}_k dx^k - \{\mathbf{V} \bullet \mathbf{E}\}dt). \tag{5.31}$$

However, another perhaps not so obvious a candidate for a solution vector field is the expression for the Torsion current. That is, examine the evolution along the unique four dimensional vector field,

$$\mathbf{T}_4 = -\{(\mathbf{E} \times \mathbf{A} + \mathbf{B}\phi); \mathbf{A} \bullet \mathbf{B}\}. \tag{5.32}$$

The expression for virtual work becomes

$$W = i(\sigma\mathbf{T}_4)dA = \sigma(\{(\mathbf{A} \bullet \mathbf{B})\mathbf{E} + (\mathbf{E} \times \mathbf{A}) \times \mathbf{B}\}_k dx^k - \{\mathbf{E} \bullet \mathbf{B}\phi\}dt) = \sigma(\mathbf{E} \bullet \mathbf{B})A. \tag{5.33}$$

The torsion current is an associated field relative to the 1-form of Action, in the sense that

$$i(\sigma\mathbf{T}_4)A \Rightarrow 0. \tag{5.34}$$

Evolution in the direction of the Torsion vector does not produce any internal energy of interaction, even though the process is not extremal. The $\mathbf{T}_4$ process is adiabatic. In Pfaff dimension 4, the Torsion vector is not extremal, but adiabatic. It is amazing that $\mathbf{T}_4$ decays to a process which is characteristic; i.e., a process which is adiabatic and homogeneous of degree zero.

It follows that the Lie derivative of the Action along the direction of the Torsion current is an isovector process in the sense that

$$L_{(\sigma\mathbf{T}_4)}A = \Gamma A = \sigma(\mathbf{E} \bullet \mathbf{B})A = Q. \tag{5.35}$$

By direct computation,

$$L_{(\sigma\mathbf{T}_4)}dA = d\Gamma\hat{}A + \Gamma dA = dQ \tag{5.36}$$

from which it follows that

$$Q\hat{}dQ = \Gamma^2 A\hat{}dA. \tag{5.37}$$

If the topological parity $\Gamma = \sigma(\mathbf{E} \bullet \mathbf{B})$ does not vanish, then the Torsion current $\sigma\mathbf{T}$ represents an irreversible non-conservative process. For such processes the heat 1-form, Q, does not admit an integrating factor.

The formula $L_{(\sigma\mathbf{T}_4)}A = \Gamma A$ was the fundamental principle used by the present author in1974 to describe "An Extension of Hamilton's Principle to Include Dissipative Systems" [RMK 1974] [RMK 1975 d, RMK 1975]. It was not known at that time the such processes implied the existence of a symplectic structure, nor the fact that these processes were not symplectomorphisms.

5.5. More on Characteristic Vectors

On the 4D manifold, those point sets upon which $(\mathbf{E} \bullet \mathbf{B}) = 0$ are special, for then the symplectic manifold becomes a contact manifold, and admits a unique direction field on the 3D submanifold, a Hamiltonian extremal vector field. From the 4D point of view, the rank of the matrix of 2-form coefficients, dA, is 2 (not 4), which implies that there are two null eigen vectors (on 4 D). One of these eigen vectors is the Hamiltonian field such that $i(\mathbf{V})dA = 0$, but $i(\mathbf{V})A = U$ is unspecified. In other words, one of the null eigenvectors on the 4D space is extremal, but not associated.

The other null eigenvector is both and extremal and associated; that is, it is a characteristic vector field. This special vector field (to within a factor) is given by the Torsion Vector field, $\mathbf{T}_4 = -\{(\mathbf{E} \times \mathbf{A} + \mathbf{B}\phi); \mathbf{A} \bullet \mathbf{B}\}$ subject to the closure condition that $Div\mathbf{T}_4 = 2(\mathbf{E} \bullet \mathbf{B}) = 0$. In other words, the Torsion Vector defines the adiabatic direction field in space-time, which preserves every element of the Pfaff sequence locally.

$$
\begin{aligned}
\text{If } (\mathbf{E} \bullet \mathbf{B}) &= 0, \text{ then } \sigma \mathbf{T}_4 = \mathbf{C}_4, \text{ a characteristic vector} && (5.38)\\
i(\sigma \mathbf{T}_4)A &= 0, \; i(\sigma \mathbf{T}_4)dA = 0, \; L_{(i\mathbf{T}_4)}A = 0, && (5.39)
\end{aligned}
$$

Characteristic vectors relative to A are adiabatic and satisfy the equations

$$
\begin{aligned}
i(\mathbf{C}_4)A &= 0, \; i(\mathbf{C}_4)dA = 0, \; L_{(\mathbf{C}_4)}A = 0 && (5.40)\\
i(\mathbf{C}_4)F &= 0, \; i(\mathbf{C}_4)dF = 0, \; L_{(\mathbf{C}_4)}F = 0 \\
i(\mathbf{C}_4)G &= 0, \; i(\mathbf{C}_4)dG = i(\mathbf{C}_4)J = 0.
\end{aligned}
$$

The characteristic field is usually associated with propagating discontinuities, or signals. (Wave phenomena)

6. Hopf Maps and Minimal Surfaces

Consider the map from the 4 dimensional variety $\{x, y, z, ct\}$ to the 3 dimensional variety, $\{u, v, w\}$, given by the formulas (the Hopf map):

$$
\{u, v, w\} = \{2(xz + cty), 2(ctx - yz), (x^2 + y^2) - (z^2 + (ct)^2)\}. \tag{6.1}
$$

The three differentials $\{du, dv, dw\}$ admit three exterior differential 1-forms (of 4 components) to be defined on the variety $\{x, y, z, ct\}$ in terms of the gradient of the 3 functions, $\{u, v, w\}$ with respect to the 4 independent variables. A 4th exterior differential 1-form can be defined as the Hopf adjoint 1-form to these 3 gradient fields. That is search for the 1-form A such that $A\hat{}du\hat{}dv\hat{}dw$ is a non-zero volume element. The Hopf Action 1-form is defined as

$$
A_{Hopf} = ydx - xdy + ctdz - \; czdt \tag{6.2}
$$

and has a Pfaff topological dimension of 4. The form

$$
A_{Hopf} = ydx - xdy - ctdz + \; czdt \tag{6.3}
$$

is equally acceptable, and will change the sign of the second Poincare invariant.

A Holder norm can be defined in terms of the coefficients of the 1-form as

$$\lambda = (ax^p + by^p + kz^p + m(ct)^p)^{n/p}. \tag{6.4}$$

where the constant coefficients a, b, k, m, p, n are, in general, arbitrary. The Holder norm is a function which is homogeneous of degree n. Division of an Action 1-form by the Holder norm produces a homogeneous 1-form of homogeneity degree (2/n) for any a, b, k, m, or p. In that which follows, it will be assumed that $a = b = 1,\ k = \pm 1,\ m = \pm 1$ and $p = 2$. Focus will be placed on two quartic $(n = 4)$ examples constructed from the squares of certain quadratic forms, λ and β, of different index (the index is defined as the number of minus signs in the quadratic form).

$$\begin{aligned}
\lambda_{index\ 0} &= (+(ct)^2 + x^2 + y^2 + z^2) && (6.5)\\
\lambda_{index\ 1} &= (-(ct)^2 + x^2 + y^2 + z^2) && (6.6)\\
\beta_{index\ 1} &= ((ct)^2 + x^2 + y^2 - z^2) && (6.7)\\
\beta_{index\ 2} &= (-(ct)^2 + x^2 + y^2 - z^2). && (6.8)
\end{aligned}$$

6.1. Homogeneous Hopf Action 1-forms (degree -2, index 0)

Divide the Hopf Action 1-form, A_{Hopf} by $\lambda^2_{index\ 0}$ to create a Homogeneous (degree -2) Hopf 1-form of Action of index 0:

$$A_{Homogeneous\ Hopf} = (ydx - xdy + ctdz - \ czdt)/\lambda^2_{index\ 0}. \tag{6.9}$$

Note that the standard Hopf map is

$$\begin{aligned}
\{u, v, w\} &= \{2(xz + cty), 2(ctx - yz), (x^2 + y^2) - (z^2 + (ct)^2)\} && (6.10)\\
&= \{2(xz + cty), 2(ctx - yz), (\beta_{index\ 2})\}. && (6.11)
\end{aligned}$$

Following the definitions for index 0, compute[3] the components of the **E** and **B** fields to yield:

$$\mathbf{E} = [+c(xz + cty), c(yz - ctx), -(\beta_{index\ 2})/2]4/\lambda_0^3 \tag{6.12}$$

$$\mathbf{B} = -[xz + cty, yz - ctx, -(\beta_{index\ 2})/2]4/\lambda_0^3 \tag{6.13}$$

[3] See http://www22.pair.com/csdc/pdf/maxwell.pdf

It is apparent that the **B** field is irreducibly 3 dimensional almost everywhere, and that

$$d(A\hat{}F) = \mathbf{E} \circ \mathbf{B} = -4c/\lambda_0^4 \neq 0. \tag{6.14}$$

The **E** fields and the **B** fields are anti-collinear, and components proportional to the coefficients of the Hopf map. It follows that the 1-form of Action, A, is of Pfaff dimension 4, and the electromagnetic system is an open non-equilibrium thermodynamic system.

It is remarkable that the derived 2-form has coefficients (**E** and **B**) that are proportional to the same functions that define the Hopf Map. Using the minus ambiguity (parity) sign, leads to the classic result that $\mathbf{E}^2 = C^2\mathbf{B}^2$, but with the not-usual result that the **E** field is anti-parallel to the **B** field (If the positive ambiguity (parity) sign is used, the **E** and **B** fields are parallel):

The components of the Topological Torsion 4 vector for the example system become equal to :

$$\begin{aligned} \mathbf{T}_4 &\equiv [\mathbf{T},h] = +2c[x, y, z, ct]/\lambda_0^4, & (6.15) \\ di(\mathbf{T}_4)\Omega_4 &\neq 0 \text{ Divergence of } \mathbf{T}_4 \text{ is not zero.} & (6.16) \end{aligned}$$

The Torsion vector does not have zero divergence, hence the second Poincare invariant, $-4c/\lambda_0^4\Omega_4$, is not zero. Hence, although the process generated by $\mathbf{T}_4$ is locally adiabatic, as $i(\mathbf{T}_4)A = 0$ (see equation (2.20)), it must be thermodynamically irreversible (see equation (2.40)).

Subject to the Lorentz constitutive constraints, it is straight forward to show that the charge current density generated by the Maxwell Ampere relations is zero, only when the abnormal phase velocity constraint, $\varepsilon\mu c^2 + 1 = 0$, is satisfied.

$$\begin{aligned} & \text{Charge-Current density (Hopf Action, index 0, degree -2)} & (6.17) \\ J^x &= 4(6xzct + 5c^2t^2y - yx^2 - y^3 - yz^2)(\varepsilon\mu c^2 + 1)/\mu\lambda_0^4 & (6.18) \\ J^y &= 4(-6yzct - 5c^2t^2x + xy^2 + x^3 + xz^2)(\varepsilon\mu c^2 + 1)/\mu\lambda_0^4 & (6.19) \\ J^z &= -8ct\beta(\varepsilon\mu c^2 + 1)/\mu\lambda_0^4 & (6.20) \\ \rho &= 0 & (6.21) \\ d(J) &= d(i(\mathbf{J}, \rho)\Omega_4 = 0 \text{ Divergence of } [\mathbf{J},\rho] \text{ is always zero.} & (6.22) \end{aligned}$$

The proposed 1-form does NOT furnish a valid realization for the Lorentz vacuum constraint, in that it produces a non-zero zero charge current density, unless the "phase velocity" constraint, $(\varepsilon\mu c^2 + 1) = 0$. This constraint is abnormal unless the coefficients ε or μ are negative. Hence the scaled Hopf Action, where the scaling is of signature zero, does **not** describe a charge current free vacuum, for real positive values of ε, μ, and C. On the other hand, if it is presumed that the domain is such that say μ, or ε, is negative, then the Hopf Map, scaled as above, does generate charge-current free wave solutions. Negative ε appears to hold in metals and the Earth's ionosphere. Recent announcements indicate constructions that yield negative μ. [Physics Today May 2000]. However, for situations where ε or μ are negative, the Hopf wave solutions imply that the Spin angular momentum $A\hat{}G$ is not a deformation invariant (hence Spin angular momentum of the field is not conserved.)

The components of the Topological Spin current can be evaluated as

$$\text{Topological Spin example (Hopf Action, index 0, degree -2)} \tag{6.23}$$
$$S^x = -2x(3c^2t^2 + x^2 + y^2 - 3z^2)/\mu\lambda^5 \tag{6.24}$$
$$S^y = -2y(3c^2t^2 + x^2 + y^2 - 3z^2)/\mu\lambda^5 \tag{6.25}$$
$$S^z = -2z(3y^2 + 3x^2 - z^2 + c^2t^2)/\mu\lambda^5 \tag{6.26}$$
$$\sigma = -2t(3y^2 + 3x^2 - z^2 + c^2t^2)/\mu\lambda^5 \tag{6.27}$$
$$d(A\hat{}G) = d(i(\mathbf{S}, \sigma)\Omega_4 = 0 \tag{6.28}$$
$$\text{Divergence of } S_4 \text{ is zero} \tag{6.29}$$

As demonstrated by the example solution above, the existence of irreducibly three component, propagating, magnetic field solutions to Maxwell's equations can be accommodated without invoking a special gauge condition, or a constraint of a particular group, or claiming that Maxwell electromagnetism must be modified, or replaced by a Yang-Mills theory. All that is required is that the classical Maxwell theory must not be constrained to an equilibrium state.

6.2. Homogeneous Hopf Action 1-forms (degree -2, index 1)

Next, the Hopf Action 1-form, A_{Hopf} is divided by λ^2 to create a Homogeneous (degree -2) Hopf 1-form of Action.

$$A_{Homogeneous\ Hopf} = (ydx - xdy + ctdz - czdt)/\lambda^2_{index\ 1} \tag{6.30}$$

Note that the Hopf map is

$$\{u, v, w\} = \{2(xz + cty), 2(ctx - yz), (x^2 + y^2) - (z^2 + (ct)^2)\} \quad (6.31)$$
$$= \{2(xz + cty), 2(ctx - yz), (\beta_{index\ 2})\}. \quad (6.32)$$

Following the definitions for index 1, compute the components of the **E** and **B** fields to yield:

$$\mathbf{E} = [+c(xz - cty), c(yz + ctx), -(\beta_{index\ 1})/2]4/\lambda^3 \quad (6.33)$$

$$\mathbf{B} = -[xz + cty, yz - ctx, -(\beta_{index\ 1})/2]4/\lambda^3 \quad (6.34)$$

For this choice, it is remarkable that the derived 2-form has coefficients (**E** and **B**) that are proportional to different Hopf Maps. The Action 1-form is the same as above, but with a different denominator. This fact leads to the classic result that $\mathbf{E}^2 = C^2\mathbf{B}^2$, but now the **E** field is not collinear with the **B** field. The solution has magnetic helicity as $\mathbf{A} \circ \mathbf{B} \neq 0$ and is radiative in the sense that the Poynting vector, $\mathbf{E} \times \mathbf{H} \neq 0$.

It is apparent that the **B** field is irreducibly 3 dimensional almost everywhere, and that

$$d(A\hat{}F) = \mathbf{E} \circ \mathbf{B} = -4c/\lambda^4 \neq 0. \quad (6.35)$$

It follows that the 1-form of Action, A, is of Pfaff dimension 4, and the electromagnetic system is an open non-equilibrium thermodynamic system. It may come as a surprise, but both the **E** and the **B** fields satisfy the vector wave equation.

$$(\nabla^2 - 1/c^2\partial^2/\partial t^2) \left| \begin{matrix} \mathbf{E} \\ \mathbf{B} \end{matrix} \right\rangle = 0. \quad (6.36)$$

The components of the Topological Torsion 4 vector for the example system become equal to :

$$\mathbf{T}_4 \equiv [\mathbf{T}, h] = 2c[x, y, z, ct]/\lambda^4, \quad (6.37)$$
$$di(\mathbf{T}_4)\Omega_4 \neq 0 \text{ Divergence of } \mathbf{T}_4 \text{ is not zero.} \quad (6.38)$$

The Torsion vector does not have zero divergence, hence the second Poincare invariant, $-4c/\lambda^4\Omega_4$, is not zero. Hence, although the process generated by $\mathbf{T}_4$ is locally adiabatic, as $i(\mathbf{T}_4)A = 0$ (see equation (2.20)), it must be thermodynamically irreversible (see equation (2.40)).

Similar to the investigation described above for the index zero Hopf vectors, it is natural to ask if these **E** and **B** fields admit a Lorentz symmetry constitutive constraint such that vacuum state is charge current free. Again, such a condition implies that the fields are solutions of the vector wave equation. The charge current density has the format:

$$\text{Charge-Current density (Hopf Action, index 1, degree -2)} \quad (6.39)$$
$$J^x = 4(-6xzct + 5c^2t^2y + yx^2 + y^3 + yz^2)(\varepsilon\mu c^2 - 1)/\mu\lambda^4 \quad (6.40)$$
$$J^y = -4(6yzct + 5c^2t^2x + xy^2 + x^3 + xz^2)(\varepsilon\mu c^2 - 1)/\mu\lambda^4 \quad (6.41)$$
$$J^z = 8ct\beta(\varepsilon\mu c^2 - 1)/\mu\lambda^4 \quad (6.42)$$
$$\rho = 0 \quad (6.43)$$
$$d(J) = d(i(\mathbf{J}, \rho)\Omega_4 = 0 \text{ Divergence of } [\mathbf{J},\rho] \text{ is always zero.} \quad (6.44)$$

The direct computation of the Maxwell Ampere equations indicates that, indeed, $dG = J = 0$, if the phase velocity constraint vanishes, $\varepsilon\mu C^2 - 1 = 0$. Hence the scaled Hopf Action, where the scaling is of index one, **does** describe a charge current free vacuum, for real positive values of ε, μ, and C. Hence the proposed 1-form furnishes a valid realization for the Lorentz vacuum constraint, in that it produces a zero charge current density.

The local interaction energy density, $A\hat{}J = (\mathbf{A} \circ \mathbf{J} - \rho\phi)\ \Omega_4$, is zero for the example non-equilibrium system. It should be noted that fluctuations in the phase velocity function, $(\varepsilon\mu c^2 - 1) = 0$ can induce currents. The components of the Topological Spin current can be evaluated (for as

$$\text{Topological Spin example (Hopf Action, index 1, degree -2)} \quad (6.45)$$
$$S^x = -2x(3c^2t^2 + x^2 + y^2 - 3z^2)/\mu\lambda^5 \quad (6.46)$$
$$S^y = -2y(3c^2t^2 + x^2 + y^2 - 3z^2)/\mu\lambda^5 \quad (6.47)$$
$$S^z = -2z(3y^2 + 3x^2 - z^2 + c^2t^2)/\mu\lambda^5 \quad (6.48)$$
$$\sigma = -2t(3y^2 + 3x^2 - z^2 + c^2t^2)/\mu\lambda^5 \quad (6.49)$$
$$d(A\hat{}G) = d(i(\mathbf{S}, \sigma)\Omega_4 = 0 \quad (6.50)$$
$$\text{Divergence of } S_4 \text{ is zero} \quad (6.51)$$

It is importance to note that the exterior derivative of $A\hat{}G$ vanishes. Hence the integral of the 2-form of topological spin over a closed 3 chain yields 3 dimension period integrals, and is topologically quantized in this example. It is conjectured that fluctuations of the "perfect" vacuum phase relations, where $\varepsilon\mu C^2 - 1 \neq 0$, are associated with ZPF. Note that there are possible charge-current free (singular wave solutions) that are governed by curves in space time. These curves are generated by the intersection of the three surfaces created by setting each of the coefficients of the current density equal to zero. These solutions are valid for any phase velocity and could be a source of "needle" radiation.

The solution given above is not free of Topological Torsion, $A\hat{}F$, and there is a non-zero value of the second Poincare invariant, $\mathbf{E}\cdot\mathbf{B} \neq 0$. However, the Spin 3-form $A\hat{}G$ is also non-zero, but it has, subject to the phase constraint, a zero 4-divergence. (The first Poincare invariant is zero.) The divergence of the Spin 3-form, has 2 parts. The first part is twice the conventional Lagrange density of the fields, $(\mathbf{B}\cdot\mathbf{H} - \mathbf{D}\cdot\mathbf{E})$. The second part is the interaction between the potentials and the charge currents, $(\mathbf{A}\cdot\mathbf{J} - \rho\phi)$. When the divergence of the 3-form is zero, then the closed integrals of Topological Spin are deformation invariants, and have closed integrals with rational (quantized) ratios. That is, with regard to any singly parametrized vector field, V, describing an evolutionary process,

$$\begin{aligned} L_{(\beta V)} \int_{z3} (A\hat{}G) &= \int_{z3} i(\beta V)d(A\hat{}G) + \int_{z3} d(i(\beta V)A\hat{}G) \qquad (6.52) \\ &= 0 + 0 \supset \textit{evolutionary invariance.} \end{aligned}$$

The function β is an arbitrary deformation parameter.

As demonstrated by the example solution above, the existence of irreducibly three component, propagating, magnetic field solutions to Maxwell's equations can be accommodated without invoking a special gauge condition, or a constraint of a particular group, or claiming that Maxwell electromagnetism must be modified, or replaced by a Yang-Mills theory. All that is required is that the classical Maxwell theory must not be constrained to an equilibrium state.

6.3. Hopf Vectors and Spinors

Consider the map from R(X,Y,Z,S) to R3(u,v,w) given by the formulas:

$$\begin{aligned} \mathbf{H}1 &: R4 \Rightarrow R3 \qquad (6.53) \\ [u1, v1, w1] &= [2(XZ + YS), 2(XS - YZ), (X^2 + Y^2) - (Z^2 + S^2)]. \qquad (6.54) \end{aligned}$$

These formulas define the format of a Hopf map. The 3 component Hopf vector **H**1 is real and has the property that

$$\mathbf{H}1 \cdot \mathbf{H}1 = (u1)^2 + (v1)^2 + (w1)^2 = (X^2 + Y^2 + Z^2 + S^2)^2. \tag{6.55}$$

Hence a real (and imaginary) 4 dimensional sphere maps to a real 3 dimensional sphere. That is, the constrained Hopf map is equivalent to the map of a 3 dimension sphere to a 2 dimensional sphere. If the functions $[u1, v1, w1]$ are defined as $[x/ct, y/ct, z/ct]$, then the 4D sphere of unit radius, $(X^2 + Y^2 + Z^2 + S^2)^2 = 1$, maps to the 4D light cone.

The Hopf map can also be represented in terms of complex functions by a map from C2 to R3, as given by the formulas:

$$\mathbf{H}1 = [u1, v1, w1] = [\alpha \cdot \beta^* + \beta \cdot \alpha^*, i(\alpha \cdot \beta^* - \beta \cdot \alpha^*), \alpha \cdot \alpha^* - \beta \cdot \beta^*. \tag{6.56}$$

The variables α and β also can be viewed as two distinct complex variables defining ordered pairs of the four variables $[X, Y, Z, S]$. For example, the classic format given above for $\mathbf{H}1_\pm$ can be obtained from the expansion, $\alpha = X + iY$, $\beta = Z + iS$. Other selections for the ordered pairs of (X, Y, Z, S) (along with permutations of the 3 vector components) give distinctly different Hopf vectors. For example, the ordered pairs, $\alpha = X + iZ$, $\beta = Y + iS$, give

$$\mathbf{H}2 = [2(YX - SZ), X^2 + Z^2 - Y^2 - S^2, -2(ZY + SX)], \tag{6.57}$$

which is another Hopf vector, a map from R4 to R3, but with the property that **H**2 is orthogonal to **H**1 :

$$\mathbf{H}2 \cdot \mathbf{H}1 = 0. \tag{6.58}$$

Similarly, a third linearly independent orthogonal Hopf vector **H**3 can be found

$$\mathbf{H}3 = [X^2 + Y^2 - Z^2 - S^2, -2(YX + SZ), 2(-ZX + SY)], \tag{6.59}$$

such that

$$\mathbf{H}2 \cdot \mathbf{H}1 = \mathbf{H}3 \cdot \mathbf{H}2 = \mathbf{H}2 \cdot \mathbf{H}3 = 0, \tag{6.60}$$

$$\mathbf{H}1 \cdot \mathbf{H}1 = \mathbf{H}2 \cdot \mathbf{H}2 = \mathbf{H}3 \cdot \mathbf{H}3 = (X^2 + Y^2 + Z^2 + S^2)^2. \tag{6.61}$$

The three linearly independent Hopf vectors can be used as a basis of R3 excluding those points where the quartic form vanishes. The mapping functions (u, v, w) of the Hopf vector can be differentiated with respect to (X, Y, Z, S) to produce a set of three exact 1- form whose coefficients form 3 independent 4 component vectors on R4. A 4th linearly independent vector can be created algebraically and forms the "adjoint" field for the given Hopf vector. This direction field can be used to construct a non-integrable 1-form, A, of Pfaff dimension 4. These three exact 1-forms and the non-integrable 1-form can be used as a basis frame for the space. The exterior derivatives of the basis frame produce the usual Cartan connection which is not affine-torsion free in its subspaces. By this mechanism the differential structure of R4 as induced by the Hopf map is determined.

For **H**1, the 4 independent 1 forms are given by the expressions:

$$d(u1) = 2Zd(X) + 2Sd(Y) + 2Xd(Z) + 2Yd(S) \tag{6.62}$$

$$d(u2) = 2Sd(X) - 2Zd(Y) - 2Yd(Z) + 2Xd(S) \tag{6.63}$$

$$d(u3) = 2Xd(X) + 2Yd(Y) - 2Zd(Z) - Sd(S) \tag{6.64}$$

$$A_{Hopf} = \{-Yd(X) + Xd(Y) - Sd(Z) + Zd(S)\}. \tag{6.65}$$

A Frame Matrix can be generated by the coefficients of the 4 independent 1-forms. It is some interest to examine the properties of the adjoint 1-form, A, defined hereafter as the Hopf 1-form. The Hopf 1-form is of Pfaff dimension 4. It is also of interest to consider factors Λ that are of the format of the Holder norm, where n and p are integers, and (a,b,k,m) are arbitrary constants:

$$\Lambda = (aX^p + bY^p + kZ^p + mS^p)^{n/p}. \tag{6.66}$$

The exponent n determine the homogeneity of the resulting 1-form, which is given below an ambiguous format (the plus or minus sign) representing different orientations,

$$A_\pm = A(\pm)/\Lambda = \{\pm(Yd(X) - Xd(Y)) - Sd(Z) + Zd(S)\}/\Lambda. \tag{6.67}$$

When $n = p = 2$, the scaling factor becomes related to the classic quadratic form. The scaled Hopf 1-form, A, is then homogeneous of degree zero. For arbitrary n and p, the 3-form of topological (Hopf) torsion becomes:

$$Topological\ Torsion = (A_{\pm})\hat{}d(A_{\pm}) = i(\mathbf{T}_4)d(X)\hat{}d(Y)\hat{}d(Z)\hat{}d(S), \tag{6.68}$$

where the topological torsion 4 vector is equal to:.

$$\mathbf{T}_4 = \pm[X, Y, Z, S]/\Lambda. \tag{6.69}$$

The Torsion vector, $\mathbf{T}_4$, for the Hopf map is proportional to the position vector from the four dimensional origin and represents an expansion or a contraction process. The factor Λ depends upon the integers n and p as well as the constants (a,b,k,m).

The Topological Parity 4-form, whose coefficient is the 4 divergence of the Torsion vector, $\mathbf{T}_4$, becomes

$$Topological\ Parity = d(A_{\pm})\hat{}(d(A_{\pm}) \tag{6.70}$$

$$= -4(\pm)(n-2)d(X)\hat{}d(Y)\hat{}d(Z)\hat{}d(S)/\Lambda^2. \tag{6.71}$$

It is most remarkable that for n=2, any p and any (a,b,k,m), the topological parity vanishes; the scaled Hopf 1-form is of Pfaff dimension 3, not 4. In such cases the ratios of the integrals of the topological torsion 3 form over various closed manifolds are rational, and the closed integrals of the 3-form are topological deformation invariants. (coherent structures).

Also note that if the scaling factor is restricted to values such that $n = 4,\ p = 2,\ a = b = k = m = 1,$ then the Frame matrix is unimodular, and the scaled Hopf 1-form is homogeneous of degree -2, relative to the substitution $X \Rightarrow \gamma X$, *etc.* (A somewhat different definition of homogeneity relative to the volume element will be given below.) Emphasis is to be placed on those examples for which $n = 4,\ p = 2,\ a = b = k = 1,\ \ m = \pm 1.$

6.3.1. Spinors as linear combinations of Hopf Maps

The 3D isotropic (null) complex position vector, $[z1, z2, z3]$ can be decomposed into a real and an imaginary part, such that both parts have the same magnitude and are orthogonal. In short, the Cartan Spinor, [Cartan 1966] can be represented as the complex sum of two Hopf vectors. The spinors come in two triples of the form:

$$|\boldsymbol{\sigma}_{12}\rangle = |\mathbf{H}1\rangle + i\ |\mathbf{H}2\rangle \qquad with \quad \langle\boldsymbol{\sigma}_{12}| \circ |\boldsymbol{\sigma}_{12}\rangle = 0 \tag{6.72}$$

$$|\boldsymbol{\sigma}_{23}\rangle = |\mathbf{H}2\rangle + i\ |\mathbf{H}3\rangle \qquad with \quad \langle\boldsymbol{\sigma}_{23}| \circ |\boldsymbol{\sigma}_{23}\rangle = 0 \tag{6.73}$$

$$|\boldsymbol{\sigma}_{31}\rangle = |\mathbf{H}3\rangle + i\ |\mathbf{H}1\rangle \qquad with \quad \langle\boldsymbol{\sigma}_{31}| \circ |\boldsymbol{\sigma}_{31}\rangle = 0. \tag{6.74}$$

These complex combinations of Hopf vectors can be used to generate solutions for which the topological torsion vanishes, and yet the topological spin is finite and quantized.

The 3D complex Spinors constructed as above are isotropic null vectors, and as such represent a minimal surface (see page 63 in [Osserman 1986]).

6.3.2. Lack of time reversal symmetry

It should be noted that if the Action 1-form in the above example for a Hopf index 1 map is subjected to the time reversal operation in its coefficients ($T \Rightarrow -T$), the new Action 1-form does **NOT** describe a charge current free vacuum, for real positive values of ε, μ, and C and the Lorentz constitutive constraint. The positive t generates zero charge-current density, while the -t format generates a non-zero charge current density.

6.3.3. Twistors composed by superposing two index 1 Hopf 1-forms

By superposing (adding or subtracting) two different, index 1, Hopf 1-forms (which will be shown below to be equivalent to a Penrose twistor solution) it is possible to construct a vacuum (charge current free wave) solution to the Maxwell system, subject to the constraint that the phase speed satisfies the phase velocity equation, $(\varepsilon\mu C^2 - 1) = 0$.

As an example consider another Hopf 1-form of index 1 formulated as

$$A = \{CTd(X) + Zd(Y) - Yd(Z) - XCd(T)\}/\lambda_1^4. \tag{6.75}$$

Similar formulas for the field intensities can be determined as above. Note that the parity of the Hopf forms to be superposed can be the same or different. If the parity of the two superposed Hopf 1-forms are opposite, then without consideration of the phase constraint, the Topological Torsion of the "twistor" 1-form vanishes, $A\hat{}F = 0$. Yet the quantized topological spin3-form $A\hat{}G$ does not vanish, and moreover, subject to the phase constraint, the closed integrals of the Spin 3 form are conserved. This result implies that such a construction yields "quantized" values for the Spin integrals. These formulations can be compared with the Penrose twistor definitions in terms of differential forms [Penrose 1999].

6.4. The Bohm-Aharanov Index and the Hopf Invariant

There are two topological concepts that can be constructed from the 1-form of Action, or Maxwell potentials, A. The first involves domains where the 2-form $F = dA \Rightarrow 0$ (the electromagnetic intensities vanish). The second involves domains where the 4-form $F\char94F = dA\char94dA \Rightarrow 0$. These constraints produce the conditions of closure on the 1-form A and the 3-form $A\char94dA$ but neither differential form need be exact. For example if the domain $dA \Rightarrow 0$ defines holes in a piece of a connected surface, then for closed cycles that bound the hole and are in the domain $dA = 0$, the value of the line integral, $\oint_{1-cycle} A \circ dl \simeq \text{integer}\, 2\pi$. This integer is the Bohm-Aharanov index and is to be associated with the flux-quantum h/e in quantum mechanics. It is an index that counts the number of "holes" in the compliment to the domain of the field intensities.

Consider the evolution of the integral of the Action 1-form over a cycle relative to the 4-vector V

$$L_{(V)} \oint_{1-cycle} A = \oint_{1-cycle} i(V)(dA) + \oint_{1-cycle} d[i(V)(A)] = \oint_{1-cycle} W + 0 \quad (6.76)$$

If the Action in a relative integral invariant relative to the evolution generated by the field V, then the RHS must be zero. The statement is equivalent to the concept that the virtual work 1-form, W, be exact. Such is the Cartan constraint that the evolutionary field has a Hamiltonian representation.

The three dimensional integral $\iiint_{closed\,3-cycle} A\char94F$ also has values that "count" the number of obstructions in the domain where $F\char94F = 0$. This integer is defined as the Hopf Invariant (see p. 228 in [Bott 1994, Bott]). Both of these topological objects are constructed from harmonic forms, and are properties of the potentials that generate the intensive variables.

Consider the evolution of the 3-form $A\char94F$ in the direction of the charge current 4 vector, $J = dG$.

$$L_{(J)} \iiint_{z3d} (A\char94F) = \iiint_{z3d} i(J)(dA\char94dA) + \iiint_{z3d} d[i(J)(A\char94F)] \quad (6.77)$$

$$= \iiint_{z3d} 2(\mathbf{E} \circ \mathbf{B}) i(J)\, \Omega_4 + 0 \quad (6.78)$$

On domains of Pfaff dimension 3 relative to A, the RHS vanishes, as $(\mathbf{E} \circ \mathbf{B}) = 0$. Hence the integral of the 3-form $A\hat{}F$ over a closed cycle is an evolutionary invariant with respect to evolution in the direction of the charge-current 4 vector. (The Hopf invariant is a relative integral invariant). Note that if the integration domain is a boundary, and if $d\{(\mathbf{E} \circ \mathbf{B})i(J)\,\Omega_4\} = 0$, then by Stokes theorem, the integral over a boundary of $A\hat{}F$ is an evolutionary invariant. The requirement $2(\mathbf{E} \circ \mathbf{B})i(J)\,\Omega_4 = 0$ is equivalent to the idea that the current 4-vector is confined to the hyper-surface, $\mathbf{E} \circ \mathbf{B} = 0$.

6.5. Reprise

Note that although all of the symbols used above are familiar in the realm of electromagnetism, the topological results and formulas obtained apply to any set of symbols, representing an arbitrary physical system, for example a fluid. The Faraday Maxwell equations are universal ideas on continuous physical systems of C2 functions.

For a computational examples in Maple, see

http://www22.pair.com/csdc/pdf/maxwell.pdf

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