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1 Submersive equivalence classes for metric fields

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The $+++-$ signature of physical space-time focuses attention on the topologically distinct equivalence classes that permit a space and time partition of a four dimensional manifold [45]. By first utilizing the idea that all manifolds admit a global immersion into Euclidean space of possibly some higher dimension, a global metrical field $g_{\mu\nu}$ may be induced on space-time. Particular emphasis will be given to that unique global 1form $A = g_{4\mu} dq^{\mu}$ $(\mu = 1, ..., 4)$ associated with the minus (timelike) sign in the metric signature. The various equivalence classes of space-time may be determined by searching for all submersive maps to space-time (with the given metric field) that preserve the retrodictive (pull back) features of the exterior differential form A, its exterior derivative dA and the various intersections formed by exterior products of the two. (Recall that a submersion is a continuous map whose Jacobian coefficients are of maximal rank and onto.)

Every induced global metric field falls into one of four equivalence classes of submersive maps. These classes identify all equivalent geometries that preserve the Pfaffian reducibility [88] of A.

If A is closed, its exterior derivative vanishes: $dA = 0$. Such metrics are said to be of Class I.

If $dA \neq 0$, but $A\hat{}dA = 0$, the 1-form, A, integrable, and the metric field is said to be of Class II.

If $A\hat{}dA \neq 0$, but $dA\hat{}dA = 0$, the metric field is said to be of Class III.

If $dA\hat{}dA \neq 0$, the metric field is of Class IV.

The Pfaffian reducibility properties of A on a four-dimensional space-time are exhausted by the four classes presented above.

For a metric of Class I these reducibility statements mean that, if in some choice of coordinates q^{μ}; (which admits a global metric field) the 1-form A has the full representation as

$$
\begin{aligned}
A &= A_1(q)dq^1 + A_2(q)dq^2 + A_3(q)dq^3 + A_4(q)dq^4, &(1)\\
&\approx g_{01}(q)dt^1 + g_{02}(q)dq^2 + g_{03}(q)dq^3 + g_{00}(q)dq^0, &(2)
\end{aligned}
$$

then there exists a submersive map φ from a set of coordinates x^i to q^μ such that in the space of x^i coordinates, the 1-form appears as the differential of a single function $x^1 = \chi$; i.e., the pullback $\varphi^*(A)$ has the equivalent form, as a perfect differential,

$$\varphi^*(A) = d\chi. \tag{3}$$

All geometries (systems of coordinates) are equivalent to this topological class if they are submersively related.

For Class-II metrics the equivalence class of submersive maps yields the reduced form

$$\varphi^*(A) = \psi d\chi. \tag{4}$$

For class-III metrics the reduced form is $\varphi^*(A) = d\beta + \psi d\chi$, and for Class-IV metrics the reduced Pfaffian form is $\varphi^*(A) = \alpha d\beta + \psi d\chi$, a result which exhausts the possibilities for Pfaffian reduction over four dimensions.

Metrics of Class I support a global set of covariant vector lines continuously orthogonal to the spacelike partition of the metric field. The physical interpretation of the topological idea represented by global orthogonality is one of « infinite extension », or the existence of a long-range field. Moreover, this gradient field, $A = d\psi$, is globally normalizable. This 1-form A does not support a 2-form structure, in that $F = dA = 0$. (Although from the standpoint of differential topology there is no need to identify the 2-form F with the electromagnetic field, for purposes of rapid comprehension of the ideas to be presented the reader is invited to do so.) If A is identified as a representative of the electromagnetic-like potentials, the vanishing of F implies that no electromagnetic-like fields[1] exist in spaces that are of Class I. Metrics of Class I can support mass, as exemplified by the Robinson-Walker metrics and their attendant long-range fields. As $F = 0$, it is also true that $F\hat{}F = 0$, which via the electromagnetic-like identification of A implies that the pseudoscalar

$$F\hat{}F = 2(\mathbf{E}\cdot\mathbf{B})dx\hat{}dy\hat{}dz\hat{}dt \tag{5}$$

is zero, and remains zero for all elements of the class. The physical interpretation of this topological fact is that parity is an invariant for such systems.

[1]For examples of the electromagnetic properties of nonclosed 1-form, A and a metric field, see [42].

Metrics of Class I support long-range parity-preserving phenomena, but do not support electromagneticlike fields.

Metrics of Class II support a global set of covariant vector lines continuously orthogonal to the spacelike partition of space-time, but now the field is not necessarily continuously normalizable, globally. The partition need not be orientable for metrics of Class II. The concept of global orthogonality again implies a physical feature of long range. For metrics of Class II, however, the induced 2-form structure $F = dA$ is not zero. Electromagnetic-like fields are admissible, but, remarkably enough, the 4-form Lagrange density $F\hat{}^*F = (\mathbf{B}\cdot\mathbf{H} - \mathbf{D}\cdot\mathbf{E})dx\hat{}dy\hat{}dz\hat{}dt$ is negative, and so, in a rest frame, the energy density is electric-like, and not magnetic-like. These facts are reminiscent of the properties of a Coulomb field.

Metrics of Class II support long-range parity-preserving phenomena, but in addition to the features of Class I, these metrics admit electromagnetic-like concepts of charge and electric fields. The Reissner-Nordstrom metric is an example of a metric of Class II.

Metrics of Class III do not admit a unique global (infinite extension) set of vector lines continuously orthogonal to the spacelike partition. In this sense the topology yields a physical interpretation in terms of a short-range field; i.e., a field which is not infinitely extendible. The 1-form A admits a nonzero 2-form F inducing an electromagnetic-like interpretation. Now, however, the Lagrange density $F\hat{}H = (\mathbf{B}\cdot\mathbf{H} - \mathbf{D}\cdot\mathbf{E})dx\hat{}dy\hat{}dz\hat{}dt$ can be either electric-like or magnetic-like[2]. The 4-form $F\hat{}F$ vanishes for metrics of Class III, so that the pseudoscalar $\mathbf{E}\cdot\mathbf{B}$ remains zero, an invariant, for all members of the class, and parity is conserved for such systems. The Godel metric is an example.

Metrics of Class IV again can be said to be short range for there does not exist a unique set of vector lines continuously orthogonal everywhere to the spacelike partition. Moreover, as the pseudo scalar $\mathbf{E}\cdot\mathbf{B}$ does not vanish for such metrics, parity need not be an invariant for such systems. The Kerr-Newman metric for a charged, spinning mass is an example of Class-IV metrics.

The topological analysis presented above yields a tempting correlation with the four basic «forces» of physics:

1. Long-range, parity-preserving, gravitational-like forces appear to be associated with metric fields of Class I.

2. Long-range parity-preserving, electromagnetic-like forces seem to be admissible by metric fields of Class II.

[2] Added 2004. The quantity H in this 1975 publication is now given the symbol G in works after 1980. H is reserved to represent topological torsion, $A\hat{}F$

3. Parity-preserving short-range forces, which may be associated with nonzero electric-like or magnetic-like energy densities, appear to belong to the topological equivalence class of metric fields of Class III.

4. Short-range parity-violating forces which may have nonzero electriclike and magnetic-like fields appear to belong to the topological equivalence class of metric fields of Class IV.

Although it smacks of «unified field syndrome », this author cannot resist presenting the interpretation that the topological Class I is the representative of gravitational forces; Class II is the representative of electromagnetic forces; Class III may be the representative of nuclear forces; and Class IV may be the representative of the weak forces. Although the interpretative work is in its infancy, the factual classification scheme of metric fields based on irreducible Pfaffian forms and their equivalence classes of submersive maps as presented above does suggest a way for possibly incorporating electric, nuclear and weak forces into the metrical theory of physics in terms of topological, not geometric properties. A study of the additional topological constraints implied by the zeros of the duals of A, F and the exterior derivatives of these duals in combinations such as $d*A$, $dH = d*F$, $A\hat{}H$ and $d(A\hat{}H)$ is in progress. For example, the topological constraint $d*A = 0$ is the analogue of the Lorentz- gauge condition. The topological constraint $dH = 0$ implies no charge currents and distinguishes the Schwarzschild metric (which is Class II, $A\hat{}F = 0,\ dH = 0$) from the Reissner-Nordstrom metric (which is also Class II, $A\hat{}F = 0,\ dH \neq 0$).

The object $A\hat{}H$, whose exterior derivative is related to the virial of Clausius, has the units of angular-momentum density (per unit source), but no simple physical identification has been made of this topological quantity.

1.1 Comments (as of 2004)

Following the original 1975 article, in 1992, it was realized that the methods of Pfaff topological dimension were topological ideas that did not require metric constraints. Hence the original article was modified, with some new additions that related to the topological differences based upon the signature of space time. I thought it remarkable that this article (below) was rejected outright by the editors, without referee comments. Yet the ideas in later years have become of interest to the physics community.

2 A Remark on the Symmetry Breaking of space time

This short note, as of 11/14/92, is an unpublished follow-on to the 1975 article presented above. The fundamental difference was that it is now recognized that "long range" is a metric idea, and Pfaff topological dimension is a topological idea. Hence topologically "long range" is better stated as "not reachable". The components of a system of Pfaff dimension greater than 2 are not topologically connected and therefor are not reachable in a continuously connected manner.

Abstract. The algebraic differences between the space-time signature (-,-,-,+) and the space-time signature (+,+,+,-) suggest that there may be a physical effect associated with such a symmetry breaking.

Introduction In this journal, almost twenty years ago [45] [46], an argument was presented to show how the properties of the four forces in physics could be deduced from the features of the four distinct Pfaffian equivalence classes of differential geometry that can be constructed on a space of four dimensions. The four equivalence classes were determined from the metric solutions, g_{mn} , to the Einstein field equations, by constructing a 1-form of action, A, in terms of the space time, g_{4m}, components of the metric field: $A = g_{4m}dx^m$. The methods of Pfaff reduction can be used to generate four equivalence classes in terms of the Pfaff dimension, or class, of this 1-form. Summarizing the previous results, the equivalence class of Pfaff dimension 1 class will support long range gravitation (mass) and is parity preserving. The second equivalence class of Pfaff dimension 2 will support both gravity (mass) and electromagnetism (charge) and is to be associated with long range parity preserving forces. The third equivalence class of Pfaff dimension 3 will support both mass and charge, but the forces - although parity preserving– are of short range. The last equivalence class of Pfaff dimension 4 involved short range interactions that can violate time reversal and symmetry breaking . Examples were given in terms of known solutions to the field equations.

Although the previous methods were motivated by ideas of differential geometry, it is now known that the concepts used to generate the four equivalence classes associated with the four forces are not of a geometrical nature, but instead the equivalence classes have their foundations in topology. Indeed, the older analysis concluded that two of the equivalence classes are to be associated with forces that are *long range*, in the sense of having distance limits going to infinity, while the other two equivalence classes are to be associated with forces that are of *short range*. However, the concept of distance is more of a geometrical idea, not a topological idea.

At the present time of writing this article it is perceived that the true nature of the equivalence classes is based on the topological issue of *connectedness*, and does not reflect the geometrical idea of *distance* necessarily. Following the work of Baldwin, two of the equivalence classes belong to a connected topology (Pfaff dimension 1 and 2), and the other two equivalence classes belong to a disconnected topology (Pfaff dimension 3 and 4). Hence the topological features of the strong and the weak forces do not involve short range, but instead reflect the concepts of accessibility. That is, the topology of the "long range" forces is connected, while the topology of the "short range" forces is disconnected. The topological idea of connectedness is to be exchanged for the geometrical idea of "long range". There is a difference between the concepts of whether or not the point b is not reachable by a continuous process and not reachable in a finite time.

These ideas are most readily understood in terms of the Cartan topology built on a Pfaffian system, and its differential closure. Such an exercise is presented in Appendix A, and is a result of P. Baldwin for a single Pfaffian A [1]. Another method emphasizing the topological features is to realize that the existence of a global 1-form of Action, A, on a space of N+1 dimensions induces a line bundle on the variety N. Intrinsic geometric concepts and certain topological properties can be evaluated in two ways. The first method uses techniques of fiber bundle theory [13] [12], but the second method generates all of the interesting features more simply from the Jacobian matrix of the vector field adjoint to the global 1-form, A. The two even dimensional equivalence classes mentioned in the older article, and discussed in the appendix below, are elements of the Chern characteristic classes for the line bundle. These sets have global properties, and therefore carry topological significance. These concepts of Pfaff equivalence classes have application not only to the microcosm of atoms and elementary particles, as well as the cosmological arena of galaxies, but also to the mundane physics of hydrodynamics. Such methods have been used recently to obtain a better understanding of the production of wake patterns, and the creation and decay of turbulence in fluids.

Signature Symmetry Breaking

However, over the years a new feature of the analysis has appeared, and it is to this new feature that this letter is directed. Note that in the 1975 reference, the signature of the quadratic form was taken to be { +,+,+,- }. The question now arises: Is there a symmetry to be broken if one considers the often used but opposite signature { -,-,-,+ }. The idea is that the wave equation

$$+\partial^2\psi/\partial x^2 + \partial^2\psi/\partial y^2 + \partial^2\psi/\partial z^2 = +(1/c^2)\partial^2\psi/\partial t^2, \tag{6}$$

has a set of characteristics which satisfy the partial differential system:

$$+(\partial\psi/\partial x)^2 + (\partial\psi/\partial y)^2 + (\partial\psi/\partial z)^2 = +(1/c^2)(\partial\psi/\partial t)^2. \tag{7}$$

Hence, there are two ways to write this constraint as an algebraic variety (a null set) :

$$+(\partial\psi/\partial x)^2 + (\partial\psi/\partial y)^2 + (\partial\psi/\partial z)^2 - (1/c^2)(\partial\psi/\partial t)^2 = 0, \quad (8)$$

$$-(\partial\psi/\partial x)^2 - (\partial\psi/\partial y)^2 - (\partial\psi/\partial z)^2 + (1/c^2)(\partial\psi/\partial t)^2 = 0 \quad (9)$$

Each quadratic form is the complete mirror symmetry (the negative) of the other, but it turns out that the signatures are intrinsically different from a topological point of view in the neighborhood of the null variety.

The analytic question that remains is: Does this symmetry of space time signatures have distinguishable consequences? The physical question is: Are there experiments that can be done to distinguish the symmetry breaking between {-,-,-,+} and {+,+,+,-}?

The analytic answer, based on the idea that the Clifford Algebras of such systems are not isomorphic to one another [5], is yes! The mathematical argument is similar to that used to distinguish the two species of angular momentum algebras in quantum mechanics, an argument which is based on the different signatures of the raising or lowering operators (commutator or anti-commutator brackets) for Bosons vs. Fermions. The fact that the differences in angular momentum signature are physically observable implies that the differences in space-time signatures may also be measurable.

Consider the Clifford Algebra with signature {+,+,+,-}. As discussed in reference [2], this algebra is isomorphic to the algebra of 4x4 matrices with real numbers as matrix elements. This matrix algebra is the usual representation used for waves in 4 dimensions. Next consider the Clifford Algebra with signature {-,-,-,+}. This algebra is isomorphic to the algebra of 2x2 matrices with quaternions as matrix elements. The non-abelian quality of the quaternions makes this algebra have extraordinary differences from the algebra of 4x4 matrices over the real numbers.

This positive analytic result which breaks the symmetry between the two space-time signatures implies there must be a physical difference between the two types of space-time, one with signature {+,+,+,-}, and the other with signature {-,-,-,+}. These differences imply that there exist two species of waves. What are they? A possible answer was first given by Schultz [90] who found exact quaternionic solutions to Maxwell's equations that indicated that the speed of propagation in the inbound and outbound directions would be different for such waves. This result was in agreement with the ring laser experiments of Sanders [52]. These sets of experiments indicated that the electromagnetic four fold degeneracy of the Lorentz equivalence class could be broken such that all four waves of left - right polarizations and of to - fro propagation directions would propagate at four distinct speeds. A further more general analysis on the macroscopic parity and time reversal symmetry breaking effects in electromagnetic systems was presented in reference [67]. The question of whether or not these waves, or the effects of {+,+,+,-} vs. {-,-,-,+} signatures, produce any quantum or hydromechanical effects is open.

Appendix A : The Cartan Topology

Starting in 1899, Cartan [7] [10] [8] [9] developed his theory of exterior differential systems built on the Grassmann algebraic concept of exterior multiplication, and the novel calculus concept of exterior differentiation. These operations are applied to sets called exterior p-forms, which are often described as the objects that form an integrand under the integral sign. The Cartan concepts may still seem unconventional to the engineer, and only during the past few years have they slowly crept into the mainstream of physics. There are several texts at an introductory level that the uninitiated will find useful [2] [19] [3] [10] [34] [91] [30]. A reading of Cartan's many works in the original French will yield a wealth of ideas that have yet to be exploited in the physical sciences. It is not the purpose of this article to provide such a tutorial of Cartan's methods, but suffice it to say the "raison d'être" for these, perhaps unfamiliar but simple and useful, methods is that they permit topological properties of physical systems and processes to be sifted out from the chaff of geometric ideas that, at present, seem to dominate the engineering and physical sciences. Many of Cartan's works have been translated by David Delphnich.

Cartan built his theory around an exterior differential system, Σ, which consists of a collection of 0-forms, 1-forms, 2-forms, etc.. He defined the closure of this collection as the union of the original collection with those forms which are obtained by forming the exterior derivatives of every p-form in the initial collection. In general, the collection of exterior derivatives will be denoted by dΣ, and the closure of Σ by the symbol, Σ^c, where

$$\Sigma^c = \Sigma \cup d\Sigma. \tag{10}$$

Cartan's interest in this closure was that founded on the idea that he was able to prove that the system of 1-forms adjoint to the closure were completely integrable. The result allowed him to devise schemes for prolonging a non-integrable system until it became integrable.

For notational simplicity in this article the systems of p-forms will be assumed to consist of the single 1-form, A. Then the exterior derivative of A is the 2-form $F = dA$, and the closure of A is the union of A and F: $A^c = A$∪F. The other logical operation is the concept of intersection, so that from the exterior product it is possible to construct the set $A\hat{}F$ defined collectively as H: $H = A\hat{}F$. The exterior derivative of H produces the set defined as $K = dH$, and the closure of H is the union of H and K : $H^c = H \cup K$.

This ladder process of constructing exterior derivatives, and exterior products, may be continued until a null set is produced, or the largest p-form so constructed is equal to the dimension of the space under consideration. The collection of sets so generated is defined as a Pfaff sequence. The largest rank of the sequence determines the Pfaff dimension of the domain (or class of the form, [88]], which is a topological invariant.

The idea for evolutionary systems is that the 1-form A (in general the exterior differential system, Σ) generates on space-time, {x,y,z,t}, four equivalence classes of points that act as domains of support for the elements of the Pfaff sequence, $\{ A, F, H, K\}$. The union of all such points will be denoted by $X = A \cup F \cup H \cup K$. The fundamental equivalence classes are given specific names:

$$\begin{aligned}
\text{Topological ACTION} &: A &(11)\\
A &= A_\mu dx^\mu &(12)\\
\text{Topological VORTICITY} &: F = dA &(13)\\
dA &= F_{\mu\nu} dx^\mu \hat{} dx^\nu &(14)\\
\text{Topological TORSION} &: H = A\hat{} dA &(15)\\
A\hat{} dA &= H_{\mu\nu\sigma} dx^\mu \hat{} dx^\nu \hat{} dx^\sigma &(16)\\
\text{Topological PARITY} &: K = dA\hat{} dA &(17)\\
dA\hat{} dA &= K_{\mu\nu\sigma\tau} dx^\mu \hat{} dx^\nu \hat{} dx^\sigma \hat{} dx^\tau. &(18)
\end{aligned}$$

The Cartan topology is constructed from a basis of open sets, which are defined as follows: First consider the domain of support of A. Define this "point set" by the symbol A. A is the first open set of the Cartan topology. Next construct the exterior derivative, $F = dA$, and determine its domain of support. Next, form the closure of A by constructing the union of these two domains of support, $A \cup F = A^c$. $A \cup F$ forms the second open set of the Cartan topology.

Next construct the intersection $H = A\hat{} F$, and determine its domain of support. Define this "point set" by the symbol H. H forms the third open set of the Cartan topology. Now follow the procedure established in the preceding paragraph. Construct the closure of H as the union of the domains of support of H and $K = dH$. The construction forms the fourth open set of the Cartan topology. In 4 dimensions, the process stops, but for N > 4, the process may be continued.

Now consider the basis collection of open sets that consists of the subsets,

$$B = \{A, A^c, H, H^c\} = \{A, A \cup F, H, H \cup K\} \tag{19}$$

The collection of all possible unions of these base elements, and the null set, $\{0\}$ generate the Cartan topology of open sets:

$$Topen = \{X, 0, A, H, A^c, H^c, A \cup H, A \cup H^c, A^c \cup H\}. \tag{20}$$

These nine subsets form the open point sets of the Cartan topology constructed from the domains of support of the Pfaff sequence $\{A, F, H, K\}$.

Table 2. The Cartan T4 Topology

A 1-form in 4D: $A = A_k(x)dx^k$

$X = \{A, \quad F = dA, \quad H = A\hat{}F, \quad K = F\hat{}F\}$

Basis subsets $\{A,\ K_{Cl}(A),\ H,\ K_{Cl}(H)\} = \{A, A \cup F, H, H \cup K\}$

$T(open) = \{X,\ \emptyset,\ A,\ H, A \cup F,\ H \cup K,\ A \cup H,\ A \cup H \cup K,\ A \cup F \cup H\}$

$T(closed) = \{\ \emptyset, X, F \cup H \cup K, A \cup F \cup K,\ H \cup K,\ A \cup F,\ F \cup K,\ F, K\}$

Subset	**Limit Pts**	**Interior**	**Boundary**	**Closure**
σ	$d\sigma$	.	$\partial\sigma$	$\sigma \cup d\sigma$
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
A	F	A	F	$A \cup F$
F	$\emptyset$	$\emptyset$	F	F
H	K	H	K	$H \cup K$
K	$\emptyset$	$\emptyset$	K	K
$A \cup F$	F	$A \cup F$	$\emptyset$	$A \cup F$
$A \cup H$	F, K	$A \cup H$	$F \cup K$	X
$A \cup K$	F	A	$F \cup K$	$A \cup F \cup K$
$F \cup H$	K	H	$F \cup K$	$F \cup H \cup K$
$F \cup K$	$\emptyset$	$\emptyset$	$F \cup K$	$F \cup K$
$H \cup K$	K	$H \cup K$	$\emptyset$	$H \cup K$
$A \cup F \cup H$	F, K	$A \cup F \cup K$	K	X
$F \cup H \cup K$	K	$H \cup K$	F	$F \cup H \cup K$
$A \cup H \cup K$	F, K	$A \cup H \cup K$	F	X
$A \cup F \cup K$	F	$A \cup F$	K	$A \cup F \cup K$
X	F, K	X	$\emptyset$	X

(21)

The closed sets of the Cartan topology are the compliments of the open sets :

$$Tclosed = \{0, X, F \cup H^c, A^c \cup K, H^c, A^c, F, K\}. \tag{22}$$

It is apparent that the Cartan topology as given in Table 1 is composed of the union of two subsets which are both open and closed ($X = A^c \cup H^c$), a result that implies that the

Cartan topology is not connected, unless the Topological Torsion, H, and hence its closure, vanishes. This extraordinary result has a number of physical consequences.

It is possible to compute the limit points for every subset relative to the Cartan topology. The classical definition of a topological limit point is that a point p is a limit point of the subset Y relative to the topology T if and only if for every open set which contains p there exists another point of Y other than p [27]. The results of this definition are presented in Table I which is due to P. Baldwin [1] (Also see Chapter 5 in vol 1.). Note that the Cartan exterior derivative is a *limit point operator* relative to the Cartan topology. In this sense, the Field Intensities of electromagnetism, **E** and **B**, generated as elements of $F = dA$, are the limit sets of the potentials, A.

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