Some Closed Form Solutions to the Navier Stokes Equations

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Abstract

An algorithm for generating a class of closed form solutions to the Navier-Stokes equations is suggested, with examples. Of particular interest are those exact solutions that exhibit intermittency, tertiary Hopf bifurcations, flow reversal, and hysteresis.

1. Introduction

The Navier-Stokes equations are notoriously difficult to solve. However, from the viewpoint of differential topology, the Navier-Stokes equations may be viewed as a statement of cohomology: the difference between two non-exact 1-forms is exact. Abstractly, the idea is similar to the cohomology statement of the first law of thermodynamics.

$$Q - W = dU \tag{1.1}$$

For the Navier-Stokes case, define the two inexact 1-forms in terms of the dissipative forces

$$W_D = \mathbf{f}_D \bullet d\mathbf{r} = \rho\{\nu\nabla^2\mathbf{V}\} \bullet d\mathbf{r} \tag{1.2}$$

and in terms of the advective forces of virtual work

$$W_V = \mathbf{f}_V \bullet d\mathbf{r} = \rho\{\partial\mathbf{V}/\partial t + grad(\mathbf{V} \bullet \mathbf{V}/2) - \mathbf{V} \times curl\mathbf{V}\} \bullet d\mathbf{r} \tag{1.3}$$

Then the abstract statement of cohomology, formulated as $W_V - W_D = -dP$, when divided by the common function, ρ, is precisely equivalent to an exterior differential system whose coefficients are the partial differential equations defined as the Navier-Stokes equations,

$$\{\partial\mathbf{V}/\partial t + grad(\mathbf{V} \bullet \mathbf{V}/2) - \mathbf{V} \times curl\mathbf{V}\} - \{\nu\nabla^2\mathbf{V}\} = -grad\, P/\rho \tag{1.4}$$

The cohomological constraint on the velocity field, $\mathbf{V}$, is such that the kinematically defined vector, $\mathbf{f}$,

$$\mathbf{f} = \mathbf{f}_V - \mathbf{f}_D \tag{1.5}$$

is a vector field that satisfies the Frobenius integrability theorem [1]. That is,

$$\mathbf{f}\bullet\, curl\, \mathbf{f} = 0 \text{ even though } \mathbf{v}\bullet\, curl\, \mathbf{v} \neq 0. \tag{1.6}$$

The meaning of the Frobenius criteria is that the vector $\mathbf{f}$ has a representation in terms of only two independent functions of $\{x, y, z, t\}$. The Navier-Stokes equations makes this statement obvious. One of these functions has a gradient, $gradP$, in the direction of the tangent vector to $\mathbf{f}$, and the other function, ρ, is a renormalization, or better, a reparametrization factor for the dynamical system represented by $\mathbf{f}$.

These observations suggest that there must exist certain constraint relationships on the functional forms that make up the components of any solution vector field, $\mathbf{V}$, (which usually does not satisfy the Frobenius condition in general) such that the covariant kinematic vector, $\mathbf{f}$, is decomposable in terms of at most two functions. If such a constraint equation can be found in terms of the component functions that represent $\mathbf{V}$, then its solutions may be easier to deduce than the direct solutions of the Navier-Stokes equations. For example, the constraint relation may involve only 1 partial differential equation rather than 3. In fact such a single constraint relation can be found by imposing a type of symmetry condition on the system, a symmetry condition that expresses the existence of a two dimensional (functional) representation for the vector field, $\mathbf{f}$. In this article attention will be focused on the two spatial variables, r and z, such that the solution examples will have a certain degree of cylindrical symmetry. As these solutions involve dissipative terms with a kinematic viscosity coefficient, ν, they are not necessarily equilibrium solutions of an isolated thermodynamic system.

Closed form solutions are few in number [3], but it appears that many of the known steady-state solutions to the Navier-Stokes equations fall into the following class of systems: Consider a variety $\{x, y, z, t\}$ with $r^2 = x^2 + y^2$. Consider three arbitrary functions, $\Theta(r, z)$ and $\Phi(r, z, t)$, and $\Lambda(r, z)$ which are defined in terms of two independent variables spatial variables, (r, z), and time. Define the flow field, $\mathbf{V}$ in cylindrical coordinates as,

$$\mathbf{V} = \Lambda(r, z)\mathbf{u}_z + \Theta(r, z)\mathbf{u}_r + \Phi(r, z, t)\mathbf{u}_\phi/r, \tag{1.7}$$

where $\mathbf{u}_\phi$ is a unit vector in the azimuthal direction. Note that this vector field does not necessarily satisfy the Frobenius theorem. Note that for simplicity, the only time dependence permitted is in the azimuthal direction.

Substitution of this format for $\mathbf{V}$ into the equation for $\mathbf{f}$ will yield a vector equation of the form

$$\mathbf{f} = \alpha(r, z)\mathbf{u}_z + \beta(r, z)\mathbf{u}_r + \gamma(r, z, t)\mathbf{u}_\phi/r. \tag{1.8}$$

The Pfaffian form $W = \mathbf{f} \circ d\mathbf{r}$ will become an expression in two variables if the azimuthal factor $\gamma(r, z, t)$ is constrained to the value zero. In other words, a single constraint on the functions, $\Theta(r, z)$ and $\Phi(r, z, t)$, and $\Lambda(r, z)$, defined by the equation $\gamma(r, z, t) = 0$, can be used to reduce the Pfaffian form to the expression

$$\alpha(r, z)dz + \beta(r, z)dr = -dP/\rho(r, z) \tag{1.9}$$

The left hand side represents a Pfaffian form in two variables, and therefore always admits an integrating factor. It is this idea that is used to find new solutions to the Navier-Stokes equations. First a solution to constraint equation is determined. Then the Cartan 1-form of total work is computed. The 1-form is either exact, or can be made exact by an appropriate integrating factor. If the 1-form is exact then the Pressure is obtained by integration. It the 1-form is not exact a suitable integrating factor is found, and that integrating factor represents a variable fluid density, ρ. For a given choice of integrating factor, the Pressure is again obtained by integration.

It is also useful to consider a rotating frame of reference defined by the equation

$$\boldsymbol{\omega} = \omega\mathbf{u}_z. \tag{1.10}$$

It is the choice of rotational axis that defines the cylindrical symmetry. For such rotating systems the same technique will insure that the flow field, $\mathbf{V}$, is a solution of the Navier-Stokes equations in a rotating frame of reference,

$$\begin{aligned} & \partial\mathbf{V}/\partial t + grad(\mathbf{V}\circ\mathbf{V}/2) - \mathbf{V}\times curl\mathbf{V} && (1.11) \\ = \; & -gradP/\rho + \nu\nabla^2\mathbf{V} - 2\boldsymbol{\omega}\times\mathbf{V} - \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}) && (1.12) \end{aligned}$$

By direct substitution, into the Navier-Stokes equation above, of the presumed format for the velocity field $\mathbf{V}$ yields an expression for $\gamma(r,z)$ in terms of the three functions $\Theta(r,z)$ and $\Phi(r,z)$, and $\Lambda(r,z)$:

$$\begin{aligned} \gamma(r,z) \; = \; & \{\partial\Phi/\partial t + \Lambda(r,z)\partial\Phi/\partial z + \Theta(r,z)(\partial\Phi/\partial r - 2\omega r) && (1.13) \\ & -\nu\{\partial^2\Phi/\partial z^2 + \partial^2\Phi/\partial r^2 - (\partial\Phi/\partial r)/r\}. && (1.14) \end{aligned}$$

Similar evaluations of the standard formulas of vector calculus in terms of the assumed functional forms for the velocity field lead to the useful expressions:

$$div\mathbf{V} = \partial\Theta/\partial r + \Theta/r + \partial\Lambda/\partial z \qquad (1.15)$$

$$curl\,\mathbf{V} = \{\partial\Phi/\partial r\,\mathbf{u}_z - \partial\Phi/\partial z\,\mathbf{u}_r\}/r + \{\partial\Theta/\partial z - \partial\Lambda/\partial r\}\,\mathbf{u}_\phi$$

$$\begin{aligned} curl\,curl\,\mathbf{V} \; = \; & \{-\partial^2\Lambda/\partial r^2 + \partial^2\Theta/\partial z\partial r\}\,\mathbf{u}_z + && (1.16) \\ & \{-\partial^2\Theta/\partial z^2 + \partial^2\Lambda/\partial z\partial r\}\,\mathbf{u}_r + \\ & \{-\partial^2\Phi/\partial z^2 - \partial^2\Phi/\partial r^2 + (\partial\Phi/\partial r/r)\}\,\mathbf{u}_\phi/r \end{aligned}$$

$$\begin{aligned} \mathbf{V}\times curl\,\mathbf{V} \; = \; & \Theta\{\partial\Theta/\partial z - \partial\Lambda/\partial r\}\,\mathbf{u}_z + && (1.17) \\ & \Lambda\{\partial\Lambda/\partial r - \partial\Theta/\partial z\}\,\mathbf{u}_r + \\ & \{(1/r^2)\,grad\,(\Phi^2/2) - \{\Lambda\partial\Phi/\partial z + \Theta\partial\Phi/\partial r\}\,\mathbf{u}_\phi/r \end{aligned}$$

$$\begin{aligned} grad(\mathbf{V}\bullet\mathbf{V})/2 \; = \; & \{\Theta\partial\Theta/\partial z + \Lambda\partial\Lambda/\partial z + \Phi(\partial\Phi/\partial z)/r^2\}\mathbf{u}_z + && (1.18) \\ & \{\Theta\partial\Theta/\partial r + \Lambda\partial\Lambda/\partial r + \Phi(\partial\Phi/\partial r)/r^2 \; - \Phi^2/r^3\}\mathbf{u}_r \end{aligned}$$

$$grad(div\mathbf{V}) = \{\partial^2\Theta/\partial z\partial r + (\partial\Theta/\partial z)/r + \partial^2\Lambda/\partial z^2\}\mathbf{u}_z + \{\partial\Theta^2/\partial r^2 + \partial^2\Lambda/\partial r\partial z + (\partial\Theta/\partial r)/r - \Theta/r^2\}\mathbf{u}_r \tag{1.19}$$

It is remarkable that many solutions to the Navier-Stokes equations then can be found by using the following algorithm: Choose a functional form $\Phi(r,z)$ of interest and then deduce functions $\Lambda(r,z)$ and $\Theta(r,z)$ to satisfy the azimuthal constraint,

$$\gamma(r,z,t) = 0. \tag{1.20}$$

The flow field $\mathbf{V}$ so obtained is therefore a candidate solution to the compressible, viscous, three dimensional Navier-Stokes equations for a system with a density distribution, ρ and a pressure, P. The components of flow field so determined then permit the evaluation of the coefficients of the Pfaffian form

$$W = \alpha(r,z)dz + \beta(r,z)dr \tag{1.21}$$

If the expression is not a perfect differential, then use the standard methods of ordinary differential equations to find an integrating factor, $\rho(r,z)$. The integrating factor represents the density distribution of the resulting Navier-Stokes solution. The Pressure follows by integration.

This method is demonstrated in the next section for the known viscous vortex examples reported in Lugt. In addition, several new closed form exact solutions are generated by the technique. Among these closed form solutions are exact solutions to the Navier Stokes equations (in a rotating frame of reference) that exhibit the bifurcation classifications for N = 3 as given by Langford [2]. In particular, exact, non-truncated solutions are given that represent the trans-critical Hopf bifurcation, the saddle-node Hopf bifurcation, and the hysteresis Hopf bifurcation. It has been long suspect that many phenomena in hydrodynamics exhibit Hopf bifurcation; now these exact solutions to the Navier-Stokes equation formally justify this position, and are especially interesting for the understanding of slightly perturbed Poiseuille flow and the onset of turbulence in a pipe.

2. Examples

In the following examples, the vector field specified has been used to compute the various terms in the Navier-Stokes equations. The algebra has been simplified by use of a symbolic computation program written in the Maple syntax. For each example, the two vector components that make up the work one form have been evaluated and are displayed with the solution. For the divergence free cases, the pressure function also has been computed. First, known solutions are exhibited, and are shown to be derived from the above technique. Then a few new solutions are exhibited.

2.1. Old solutions

2.1.1. Example 1. The Rankine Vortex

$$\Phi(r,z) = a + (b+\omega)r^2, \quad \Theta = 0, \quad \Lambda = 1 \tag{2.1}$$

$$\mathbf{f}_V = \{-(a+br^2)^2/r^3\}\mathbf{u}_r + \{0\}\mathbf{u}_\phi/r + \{0\}\mathbf{u}_z \tag{2.2}$$

$$\mathbf{f}_D = \{0\}\mathbf{u}_r + \{0\}\mathbf{u}_\phi/r + \{0\}\mathbf{u}_z \tag{2.3}$$

This flow is a solution independent of the kinematic viscosity coefficient (the velocity field is harmonic, as $\mathbf{f}_D = 0$) and therefore could be construed as an equilibrium solution. This solution, for a and b equal to piecewise constants, will generate the Rankine vortex.

As the flow is isochoric ($divV = 0$), the steady pressure can be determined by quadrature, and is given by the expression,

$$P = 1/2(b^2r^4 + 4abr^2 ln(r) - a^2)/r^2 \tag{2.4}$$

2.1.2. Example 2. Diffusion Balancing Advection.

$$\Phi(r,z) = a + br^{2+m/\nu}, \quad \Theta(r,z) = m/r, \quad \Lambda = 1, \ \ \omega = 0 \tag{2.5}$$

$$\begin{aligned}\mathbf{f}_V \ &= \ \{-m^2 - (a + br^{(2\nu+m)/\nu})^2/r^3\}\mathbf{u}_r + \\ &\quad\ \{br^{(2\nu+m)/\nu}m(2\nu+m)/\nu r^2\}\mathbf{u}_\phi/r + \{0\}\mathbf{u}_z\end{aligned} \tag{2.6}$$

$$\mathbf{f}_D = \{0\}\mathbf{u}_r + \{br^{(2\nu+m)/\nu}m(2\nu+m)/\nu r^2\}\mathbf{u}_\phi/r + \{0\}\mathbf{u}_z \tag{2.7}$$

In this case the Laplacian of the vector field is not zero, but the dissipative parts exactly cancel the advective parts in the coefficient of the azimuthal field, thereby satisfying the constraint condition. As the functions depend only on r, the integrability (gradient) condition is satisfied, and these solutions obey the Navier-Stokes equations for a system of constant density. The Pressure function may be computed as

$$\begin{aligned} P &= (-\nu b(4a(m+\nu) + bmr^{(2\nu+m)/\nu)})r^{(2\nu+m)/\nu)} - \\ &\quad (m(\nu+m)(a^2+m^2))/(2m(\nu+m)r^2) \end{aligned} \tag{2.8}$$

The solutions are cataloged in Lugt. As these solutions explicitly involve the kinematic viscosity, ν, they cannot be equilibrium solutions to isolated systems. Instead they represent steady state solutions, far from equilibrium. A special case exists for $m/\nu = -2$.

2.1.3. Example 3. Burger's Solution, but with Helicity and Zero Divergence.

$$\Phi(r,z) = k(1 - e^{-ar^2/2\nu}), \quad \Theta(r,z) = -ar, \quad \Lambda = U + 2az, \;\; \omega = 0 \tag{2.9}$$

$$\begin{aligned} \mathbf{f}_V &= \{-(ke^{(-ar^2/2\nu)} + r^2 a - k)(ke^{(-ar^2/2\nu)} - r^2 a - k)/r^3\}\mathbf{u}_r + \\ &\quad \{kra^2/\nu \; e^{(-1/2ar^2/\nu)}\}\mathbf{u}_\phi/r + \\ &\quad \{2(U+2az)a\}\mathbf{u}_z \end{aligned}$$

$$\mathbf{f}_D = \{0\}\mathbf{u}_r + \{-kra^2/\nu \; e^{(-1/2ar^2/\nu)}\}\mathbf{u}_\phi/r + \{0\}\mathbf{u}_z \tag{2.10}$$

This solution corresponds to a modification of Burger's solution and exhibits a 3-dimensional flow (in 2-variables) in which the diffusion is balanced by convection to give azimuthal cancellation. The Burgers solutions has been modified to exhibit zero divergence. This flow in a non-rotating frame of reference exhibits a helicity.

$$Helicity = (U + 2az)(ka/\nu)e^{(-1/2ar^2/\nu)} \tag{2.11}$$

2.2. New Solutions

2.2.1. Example 4. A Beltrami Type Solution

$$\Phi(r,z) = r^2\cos(z/a), \quad \Theta(r,z) = r\sin(z/a), \quad \Lambda(r,z) = 2a\cos(z/a), \;\; \omega = 0 \tag{2.12}$$

$$\mathbf{f}_V = \{-r\}\mathbf{u}_r + \{0\}\mathbf{u}_\phi/r + \{-4a\cos(z/a)sin(z/a)\}\mathbf{u}_z \tag{2.13}$$

$$\mathbf{f}_D = \nu/a^2[\{-r\,sin(z/a)\}\mathbf{u}_r + \{-rcos(z/a)\}\mathbf{u}_\phi/r + \{-2a\,cos(z/a)\}\mathbf{u}_z] \tag{2.14}$$

This solution is a Beltrami-like solution, has zero divergence, and can be made time harmonic by multiplying the velocity field by any function of t. The flow exhibits Eckman pumping and has a superficial resemblance to a hurricane. The time independent steady flow is a strictly Beltrami ($curl\ \mathbf{v} = a\ \mathbf{v}$)with the vorticity proportional to the velocity field. In all cases the helicity is given by the expression,

$$Helicity := (r^2 + 4a^2\cos(z/a)^2)/a. \tag{2.15}$$

The kinetic energy is a/2 times the helicity, which is a times the enstrophy. The Pressure generated from the Navier Stokes equation is given by the expression

$$P = 1/2(r^2 + (r^2(\nu/a^2) - 4\nu)sin(z/a) + 4a^2\sin(z/a)^2) \tag{2.16}$$

2.2.2. Example 5. A Saddle Node Hopf Solution

$$\Phi(r,z) = \omega r^2, \quad \Theta(r,z) = r(a+bz), \quad \Lambda(r,z) = U - dr^2 + Bz^2 \tag{2.17}$$

The components of the advective force and dissipative force are given by the expressions,

$$\begin{aligned}\mathbf{f}_V \;=\;& \{r(a+bz)^2 + (U - dr^2 + Bz^2)rb\}\mathbf{u}_r + \\ & \{0\}\mathbf{u}_\phi/r + \\ & \{-2r^2(a+bz)d + 2(U - dr^2 + Bz^2)Bz\}\mathbf{u}_z\end{aligned} \tag{2.18}$$

and

$$\mathbf{f}_D = \{0\}\mathbf{u}_r + \{0\}\mathbf{u}_\phi/r + \nu\{-4d + 2B]\}\mathbf{u}_z \tag{2.19}$$

The divergence of the velocity field is given by the expression:

$$div\,\mathbf{V} := 2\{a + (b+B)z\} \tag{2.20}$$

The helicity of the flow depends upon the rotation, ω,

$$Helicity : \omega(+r^2 b + 2U - 2bz^2) \tag{2.21}$$

but remarkably changes for finite values of r and z, depending on mean flow speed, U.

Note that when $b = 0$, $B = 0$, $a = 0$, the solution is equivalent to the standard incompressible Poiseuille solution for flow down a pipe. The vector velocity field is not harmonic, but vector Laplacian of the velocity field is a constant.

Without these constraints, it is remarkable that the ordinary differential equations that represent the components of the velocity field are in one to one correspondence with the saddle node - Hopf bifurcation of Langford. That is, the ODE,s representing the Langford format for the SN-Hopf are given by the expressions:

$$\begin{aligned} dz/dt &= \Lambda(r, z) = U - dr^2 + Bz^2 \\ dr/dt &= \Theta(r, z) = r(a + bz) \\ d\theta/dt &= \omega \end{aligned} \tag{2.22}$$

.This first order system which exhibits tertiary bifurcation is associated with an exact solution of the Navier Stokes partial differential system in a rotating frame of reference. In principle, the method also relaxes the constraint on incompressibility, and allows a density distribution, or integrating factor, to be computed for an exact solution to the compressible Navier-Stokes equations which can be put into correspondence with saddle node-Hopf bifurcation process.

This example exhibits isochoric ($divV = 0$) flow for $B + b = 0, a = 0$. The steady isochoric pressure is then determined by quadrature, and is given by the expression,

$$P = b(dr^4/2 - (r^2 - 2z^2)U - bz^4)/2 - \nu(4d - 2b)z \tag{2.23}$$

where the constant U can be interpreted as the mean flow down the pipe. Part of the pressure is due to geometry, and part is due to the kinematic viscosity. Note that the pressure is independent from the viscosity coefficient when the velocity field is harmonic; e.g. when $(2d - b) = 0$. As the vector Laplacian of the velocity field determines the dissipation in the system, intuition would say that the harmonic solution is some form of a limit set for the otherwise viscous flow.

2.2.3. Example 6. A Transcritical Hopf Bifurcation

$$\Phi(r,z)=\omega r^2,\quad \Theta(r,z)=r(A-a+cz),\quad \Lambda(r,z)=br^2+Az+Bz^2 \tag{2.24}$$

$$\begin{aligned} \mathbf{f}_V &= \{r(A-a+cz)^2+(br^2+Az+Bz^2)rc\}\mathbf{u}_r + \\ &\quad \{0\}\mathbf{u}_\phi/r + \\ &\quad \{2r^2(A-a+cz)b+(br^2+Az+Bz^2)(A+2Bz)\}\mathbf{u}_z \end{aligned} \tag{2.25}$$

$$\mathbf{f}_D=\{0\}\mathbf{u}_r+\{0\}\mathbf{u}_\phi/r+\nu\{4b+2B\}\mathbf{u}_z \tag{2.26}$$

This example exhibits isochoric ($divV=0$) flow for $a=3A/2$ and $B=-c$. The steady isochoric pressure is then determined by quadrature, and is given by the expression,

$$P=-1/4cbr^4-1/8A^2r^2+1/2A^2z^2-Az^3c+1/2c^2z^4-\nu(4b-2c)z. \tag{2.27}$$

Again it is apparent that the pressure splits into a viscous and a non-viscous component, and when the flow is harmonic ($2b-c=0$), the pressure is independent from viscosity, and there is no dissipation in the flow.

The transcritical Hopf bifurcation is represented by the Langford system

$$\begin{aligned} dz/dt &= \Lambda(r,z)=br^2+Az+Bz^2 \\ dr/dt &= \Theta(r,z)=r(A-a+cz) \\ d\theta/dt &= \omega \end{aligned} \tag{2.28}$$

2.2.4. Example 7. A Hysteritic Hopf Bifurcation

$$\Phi(r,z)=\omega r^2,\quad \Theta(r,z)=r(a+bz),\quad \Lambda(r,z)=U-dr^2+Az+Bz^3 \tag{2.29}$$

$$\begin{aligned} \mathbf{f}_V &= \{r(a+bz)^2+(U-dr^2+Az+Az^3)rb\}\mathbf{u}_r + \\ &\quad \{0\}\mathbf{u}_\phi/r + \\ &\quad \{-2r^2(a+bz)d+(U-dr^2+Az+Az^3)(A+3Az^2)\}\mathbf{u}_z \end{aligned} \tag{2.30}$$

$$\mathbf{f}_D = \{0\}\mathbf{u}_r + \{0\}\mathbf{u}_\phi/r + \nu\{-4d + 6Az\}\mathbf{u}_z \tag{2.31}$$

This system has the remarkable property that the vector Laplacian changes sign at a position $z = 2d/3A$ down stream. There is no global way of making this solution isochoric, for the divergence is equal to

$$div\mathbf{V} = (A + 2a) + 2bz + 3Az^2. \tag{2.32}$$

The hysteretic Hopf bifurcation exhibits what has been called intermittency. The Langford system is

$$\begin{aligned} dz/dt &= \Lambda(r, z) = U - dr^2 + Az + Bz^3 \\ dr/dt &= \Theta(r, z) = r(a + bz) \\ d\theta/dt &= \omega \end{aligned} \tag{2.33}$$

2.2.5. Example 8:

In Figure 3, a time history showing the intermittent bursts of torsion wave energy is demonstrated. These puffs are not periodic in time and exhibit a pre-chaotic Poincare section. Of most interest is the idea that these flows model vortical structures associated with whirlpools, tornadoes, and rotational solitons. In fact, a general solution can be obtain in the form,

$$\Phi(r, z) = \omega r^2, \quad \Theta(r, z) = rG(z), \quad \Lambda(r, z) = (U - dr^2 H(z)) + F(z) \tag{2.34}$$

The zero divergence condition requires that $dF/dz = 2G$, and $dH/dz = 0$. These and other bifurcation solutions are constrained to produce no sound ($divV = 0$). Yet the generation of torsion waves permits the bifurcation process to proceed. If the condition of zero divergence is relaxed, then more options are permitted to stimulate the bifurcation process. Both longitudinal and torsional waves may be generated. Of particular interest to pipe flow and the onset of turbulence is

2.2.6. Example 9:

$$\Phi(r, z) = \omega r^2, \quad \Theta(r, z) = r(a + B\cos(kz-\!\!-\; t)), \quad \Lambda(r, z) = U - dr^2 + A\sin(kz-\!\!-\; t) \tag{2.35}$$

Poiseuille flow with Torsion waves This solution represents a modification to Poiseuille flow to include waves propagating in the both z directions. The coefficients A and B are complex. The waves are therefore transverse and elliptically polarized, if the zero divergence condition is to be satisfied (ikA = 2B). These waves represent spiral or torsion waves propagating down the pipe. In general, these torsion waves are guided by the "lines" of vorticity, and may exchange energy with the mean flow field in the manner of a traveling wave amplifier [Johnson 1957].

2.2.7. Example 10:

Torsion waves in a non-rotating frame This solution represents helical or torsion waves without sound (div V = 0). The wave vector k and frequency, ω, are complex, as is the amplitude A. A solution to the Navier-Stokes equations in a non-rotating frame is given by the dispersion relation: These waves do not depend upon the compressibility of the medium. They have several of the features of the Eckman system, but are rotational rather than translational shear solutions. The solutions exhibit a time dependent Beltrami flow ($curlV = kV$). It is to be noted that the complex frequency can be adjusted to be pure real if the imaginary and real parts of the wave vector are equal. Such a choice leads to exponential growth or attenuation in space of the torsion wave "packets". However such torsion wave "packets" persist in time. On the other hand, if either the real or imaginary part of the wave vector, k, is made to vanish, then the frequency, w is pure imaginary. The solution can exhibit oscillatory behavior spatially, but the mode attenuates or grows in time, and is not a traveling wave. In a rotating frame of reference, the dispersion relation and solution are modified by the rotation rate.

3. Conjugate helical waves

The ability to generate closed form solutions to the Navier-Stokes equations in rotating reference frames focuses attention on the importance of propagating conjugate helical waves in fluids, and their superposition and interference. Consider the velocity field

$$\mathbf{V} = (y\mathbf{i} - x\mathbf{j})/(x^2 + y^2) \tag{3.1}$$

This vector field is harmonic in that it has zero curl and divergence except at those points that make up the z axis. This vector field may be considered to be the superposition of two helical flows:

$$\mathbf{V} = [y\mathbf{i}/(x^2 + y^2) + c\mathbf{k}]+[-x\mathbf{j})/(x^2 + y^2) - c\mathbf{k}] \tag{3.2}$$

In particular it may be demonstrated that the Poiseuille solution may be modified to include conjugate helical or spiral waves that propagate in both directions. The steady state solution consists of the phase coherent superposition of these two oppositely propagating or conjugate waves, forming a steady state. If for any reason the perfect phase coherence is destroyed locally, then the steady state formed as a cojugate wave interference pattern will be destroyed and a puff of a traveling wave packet would be expected to form mixed in with the steady pipe flow. Near the critical value of a bifurcation parameter, these conjugate pairs of helical waves would be expected to lose their phase coherence, and the system would enter the turbulent regime.

A plot of the saddle-node Hopf solution is presented in Figure 1 for a choice of parameters that indicates its close relationship to the Hill spherical vortex [Lamb, 1945]. In fact, the torsional or twisted Hill ellipsoidal vortex is the epitome of a three dimensional torsion obstruction, or bubble, in a fluid flow. It is remarkable that these torsional, irreducibly 3-dimensional, defects may be produced continuously in an evolutionary fluid flow. The saddle-node solution that is divergence free is the kernel of the USTF solution created by Moffatt [Moffatt, 1989].

Another immediate application is the solution that represents hysteretic Hopf bifurcation, a solution which produces intermittent torsion bursts and flow reversal. The recent experimental work of Roesner [Roesner, 1989] on fluid flow in a cylindrical cavity with a rotating lid is qualitatively replicated in some of its essential features by the hysteretic Hopf solution in a rotating frame. Figure 2 exhibits the flow reversal in the z direction given by such a solution, while Figure 3 displays the intermittent torsion bursts in the xy plane.

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