$$\int_{z1} d\phi = 0 \qquad but \qquad \int_{z1} \gamma = n\ 2\pi$$

Gauge conditions as exact differential additions to a 1-form, are trivial. Gauge conditions of the closed but not exact type are NOT trivial. They contain topological information (such as the hole count in a non-simply connected domain. (Bohm-Aharanov, Joukowski airfoil, Meissner expulsion, Sommerfeld quantum conditions, etc.)

These same concepts work for differential forms that are not linear in the differentials. Hence the postulate of electromagnetism that $F - dA = 0$ is a strong topological anholonomic constraint, that says over the domain of support, the 2-form of F (E and B) is exact; that is, the 2-form does not have harmonic parts (although the 1-form, A, can have harmonic parts which are the "flux quanta"). The second postulate of Maxwell electrodynamics is the statement $J - dG = 0.$ The idea again is that the 3-form J is exact without harmonic parts. The 2-form G can have harmonic parts, which serve as the charge quanta.

Now consider the topological torsion for the 1-form A which is defined as the 3-form $H = A\hat{}F = A\hat{}dA.$ If $dH = 0,$ then the question arises: Does H have harmonic parts? If the answer is yes then the harmonic parts serve as "topological" torsion quanta. A necessary condition for existence of such quanta is that the second Poincare invariant must be zero.

Similarly for the 3-form of topological spin $S = A\hat{}G.$ The necessary condition for existence of EM spin quanta is that the $dS = 0,$ or in other words that the First Poincare invariant must vanish. These points are exemplified at

http://www22.pair.com/csdc/car/carhomep.htm

For an interesting solution to the Maxwell postulates, further constrained by the Lorentz vacuum conditions. See

http://www22.pair.com/csdc/maple/reed21.html

Here, the Torsion field is not closed, but the Spin field is closed. In fact the Spin field in the example is the torsion field multiplied by an integrating factor. One would be led to say torsion is source of spin. However, the solution is a special case and the conclusion is not general, for the next example demonstrates that you can have finite topological torsion with zero topological spin. See

http://www22.pair.com/csdc/maple/reed31.html

Equations of Motion

Given an arbitrary 1-form of Action, which is not closed, Cartan has shown that the equations of motion generating a vector field V are of a Hamiltonian form if the Lie derivative of the Action, A, with respect to the vector field V is exact.

Theorem: Solutions V to the equation $L_{(V)}A = d(\Theta)$ are Hamiltonian vector fields.

This theorem is equivalent to the statement that the closed integrals of the harmonic components of A are constants of the motion. The number of holes does not change. The 1-dimensional topological property expressed as the harmonic 1-form is an invariant of processes that are generated from a Hamiltonian function.

Writing out the theorem shows that it is a statement in the form of an anholonomic differential constraint

$$i(V)dA - d(\Theta - i(V)A) = 0$$

Use the symbols $W = i(V)dA$ defined as the work 1-form, and $U = i(V)A$ defined as the "internal energy". The work 1-form W is closed and exact. Hamiltonian systems are systems where the Pfaff dimension of the Work 1-form is 1.

Equations of motion for Non-Hamiltonian dynamics

It is apparent that to find equations of motion for non-Hamiltonian systems, the fundamental anholonomic constraint

$$i(V)dA = d(\Theta - i(V)A)$$

must be modified to include harmonic parts, and non-closed parts.

$$W = i(V)dA = d(\Theta - i(V)A) + \gamma + Z$$

$$dW = dZ \neq 0$$

The last equation destroys the Helmholtz theorem, and the Poincare even dimensional integrals are no longer evolutionary invariants. An example of such a non-Hamiltonian mechanics was suggested in 1974. See

http://www22.pair.com/csdc/pd2/pd2fre5.htm

The formula is the anholonomic constraint

$$W - \Gamma A = i(V)dA - \Gamma A = 0.$$

It is know that this equation requires that the Pfaff dimension of the Action 1-form be even (2n+2). Hence the Pfaff space supports the topological torsion 3-form. Moreover, a unique solution vector V does exist for this problem. In a space of 4 variables this vector is equivalent to the Torsion current (with components proportional to those of the 3-form of non-zero topological torsion, A^dA). Evolution in the direction of the Torsion current is thermodynamically irreversible, as the heat 1-form, Q, does not satisfy the Frobenius integrability theorem, and therefor does not admit an integrating factor.

Extremals and the Calculus of variations.

For integrals of the Action around closed loops, the values at the ”endpoints” cancel out. Similar constraints are often placed on open integrals, forcing the cancellation of contributions at boundary points. The solutions of the problem are then given by vector fields that generate paths such that

$$L_{(V)} \int_{z1} A = \int_{z1} i(V)dA + \int_{z1} di(V)A \Rightarrow \int_{z1} i(V)dA = 0$$

It is apparent that the if equation is satisfied (giving the equation for an extremal as the ”Lie derivative” of the integral must vanish) then the work 1-form must vanish. The bottom line is that Extremals are associated with anholonomic constraints, $W = f_k dx^k - Pdt = 0.$ Extremal solutions say that there are vector fields such that $f_k V^k - P = 0.$ This equation is the freshman definition of power as the product of force times velocity.

What is even more remarkable is that this equation, $W = 0,$ has solutions only in spaces (as defined by the Pfaff sequence for the Action 1-form) of odd Pfaff dimension, 2n+1. (e.g. State Space).

> **Theorem**: Unique extremals, defined as solutions to the equation i(V)dA=0 for a given A, do not exist on domains of Pfaff dimension 2n+2

The theorem is easy to prove, for if the Pfaff space is a symplectic manifold of even dimension then the 2-form dA has an anti-symmetric matrix representation with no zero eigenvalues. On the other hand if the Pfaff space is an odd dimensional contact manifold, then the anti-symmetric matrix representation of dA has a unique eigen vector with eigen value zero. Hence on 2n+1 Pfaff space, the extremal exists and is unique.

CLOSURE 2-forms

The curl format

Consider the vector array of 1-forms:

$$|\sigma^k\rangle = [F^k_a(q^b)] \circ |dq^a\rangle$$

and the vector of " closure" 2-forms:

$$|d\sigma^k\rangle = [dF^k_a(q^b)]\hat{}|dq^a\rangle \Rightarrow |\{\partial F^k_a/\partial q^b - \partial F^k_b/\partial q^a\} dq^b \hat{} dq^a\rangle$$

The Object of Anholonomicity format

Recall the Cartan-Darboux idea that if the 1-forms σ^k are complete, such that the product, $\sigma^1 \hat{} \sigma^2 \hat{} ... \sigma^N \neq 0,$ then every 2-form $d\sigma^k$ can be expanded as: $d\sigma^k = \Omega^k_{ba}\sigma^b \hat{} \sigma^a.$ The vector of " closure" 2-forms becomes:

$$|d\sigma^k\rangle = |\Omega^k_{ba}\sigma^b \hat{} \sigma^a\rangle.$$

The three index symbols Ω^k_{ba} form the components of the Object of Anholonomicity.

The Affine Torsion format

It the basis frame is complete, then there exists a right Cartan matrix of connection 1-forms

$$[dF^k_a(q^b)] = [F^k_c(q^b)] \circ [C^c_{ab} dq^b]$$

such that the vector of " closure" 2-forms becomes:

$$|d\sigma^k\rangle = [F^k_c(q^b)] \circ \left|C^c_{[ab]} dq^b \hat{} dq^a\right\rangle$$

The three index symbols $C^c_{[ab]}$ are the coefficients of the Affine torsion object.

Summary

Hence there are three equivalent formulations for the vector of closure 2-forms:

$$|d\sigma^k\rangle = |\Omega^k_{ba}\sigma^b \hat{} \sigma^a\rangle = |\{\partial F^k_a/\partial q^b - \partial F^k_b/\partial q^a\} dq^b \hat{} dq^a\rangle = [F^k_c(q^b)] \circ \left|C^c_{[ab]} dq^b \hat{} dq^a\right\rangle$$

The Affine torsion 3 index symbols are not the same as the Object of Anholonomicity, but all three formulations express the **same** vector of closure 2-forms $|d\sigma^k\rangle$

to be completed later

1. Discuss the torsion induced by A as compared to the torsion induced by W.
2. Discuss the affine torsion associated with an integrable but not exact coordinate system and how the idea intertwines with conformal maps and dilatations and chirality.
3. Compare Cartan Torsion 2-forms, affine torsion, topological torsion and anholonomic constraints.