Object of Anholonomicity

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Abstract

Notes and comments on Holonomic and Anholonomic Constraints, Affine Torsion, Object of Anholonomicity, Gaps, Defects, Frobenius integrability, and Torsion of Various types

1. Introduction

1.1. Non singular linear Mappings $\Delta[M] \neq 0$

A matrix $[M]$, whose elements are functions on a domain ξ of independent variables, defines at each point p a linear mapping carrying vector arrays into vector arrays. (For most of the discussion herein, the matrix elements of $[M]$ usually will be restricted to C2 functions). The subset q where the linear mapping has a non-zero determinant divides the domain of independent variables ξ into topological components. (The dimension of ξ and q are -at first- presumed to be the same). Recall that the determinant is a highly non-linear function, whose zero sets define a first level of topological defect structures in the domain ξ. These defect structures will be discussed later on on this article. At this time, it will suffice to recall that the concept of a non-zero real determinant separates the linear mappings into two components, those for which $\Delta[M] > 0$ and those for which $\Delta[M] < 0$. The constrained linear mapping, $[F] = [M]_{restricted\,to\,q}$, can be used as a Basis Frame for a vector space over the reduced domain of definition, q. The restriction implies that the matrix has no eigenvectors with eigen values equal to zero.

Further constraints can be imposed. For example, if a point p is constrained to move along some space curve in q, then the Basis Frame, as a set of C2 functions whose arguments depend on the point p, also "moves", with its matrix elements changing values as p changes. The concept thereby defines what is known as Cartan's Repere Mobile (the moving basis frame). To define the curve C requires N-1 Pfaffian equations, or 1-forms (consisting of functions and differentials) set equal to zero. A typical example in the space on N-1 coordinates x and 1 variable called time, t, is given by N-1 Pfaffian constraints, $dx^k - V^k(x,t)dt = 0$. These Pfaffian differential equations of constraint may be holonomic (unique solution to the differential equation exists over the entire domain) or non-holonomic (unique solutions do not exist). The motion of a point along the curve is typical of the particle point of view used in classical physics or of the ray point of view used in optics. The point p need not be constrained to a curve, but might be constrained to, say, a N-1 hypersurface. Then only 1 Pfaffian equation of constraint is needed. This point of view is more representative of the wave front concept used in optics. A wave-particle duality concept requires that the "normal" to the wave front behaves as a ray on a space curve, which is not universally true. Other forms of constraint, typical of string theories, could appear as the Pfaffian expressions, $dx^k - V^k(x,s,t)dt - Z^k(x,s,t)ds = 0$. Typical interpretations presume that s is the string length parameter and t is the time parameter.

1.1.1. Non-Holonomic and Holonomic Linear mappings

There are several ways to utilize the Basis Frame. The most general situation is to presume that the Frame matrix acts as map between vector arrays of differential forms:

$$\left[F_a^k(q^b)\right] \circ |\omega^a(q,dq)\rangle \Rightarrow \left|\sigma^k(q,dq)\right\rangle . \tag{1.1}$$

The vector arrays of p-forms may be arrays of functions (0-forms), 1-forms, 2-forms, ... N-1 forms, etc. A special case occurs if the vector arrays are assumed to be vector arrays of perfect exact differentials, for then the equation

$$Holonomic \qquad \left[F_a^k(q^b)\right] \circ |dq^a\rangle \Rightarrow \left|dx^k\right\rangle \tag{1.2}$$

implies that there exist unique function mappings (solutions to the differential equations) such that $x^k = f^k(q^b)$. When true (not true), the Frame matrix is defined as a holonomic linear mapping (non-holonomic linear mapping). A more common situation found in practice for 1-forms is given by the non-holonomic expression, where exact 1-forms are mapped into non- exact 1-forms:

$$Non-Holonomic \qquad \left[F_a^k(q^b)\right] \circ |dq^a\rangle \Rightarrow \left|\sigma^k\right\rangle. \tag{1.3}$$

Each induced 1-form σ^k has to be tested to determine its Pfaff dimension or class. For example, if the Pfaff dimension, or class, of all induced 1-forms is one, then the domain of support for the determinant of the Frame matrix is such that the induced right Cartan connection matrix (see below) is free from Affine Torsion.

1.1.2. The Relative Velocity formula and Curvature-Coriolis Theorem

The Frame matrix has properties that are independent from the choice of the vector arrays of p-forms, $|\omega^a(q,dq)\rangle$. The constraint on the domain of definition, such that the determinant of $[F]$ is non zero, implies that an inverse matrix, $[G]$, exists, such that $[F] \circ [G] = [1]$. It follows that the differentials of the Frame matrix define a linear connection:

$$\begin{aligned} d[F] &= -[F] \circ [dG] \circ [F] && (1.4)\\ &= +[C_L] \circ [F] && (1.5)\\ &= +[F] \circ [C_R], && (1.6)\end{aligned}$$

where $[C_R]$ is defined as the right Cartan matrix of connection 1-forms, and $[C_L]$ is defined as the left Cartan matrix of connection 1-forms. The two forms of Cartan connections are:

$$\begin{aligned} [C_L] &= -[F] \circ [dG] = +[dF] \circ [G], && (1.7)\\ [C_R] &= -[dG] \circ [F] = +[G] \circ [dF]. && (1.8)\end{aligned}$$

A second exterior differention leads to

$$\begin{aligned} dd[F] &= +\{d[C_L] - [C_L]\,\hat{}\,[C_L]\} \circ [F] = [\Theta_L] \circ [F] && (1.9)\\ &= [F] \circ \{d[C_R] + [C_R]\,\hat{}\,[C_R]\} = [F] \circ [\Theta_R] && (1.10)\end{aligned}$$

The matrices of 2-forms $[\Theta_L]$ and $[\Theta_R]$ are defined as the Left and Right Cartan matrix of Curvature 2-forms. The right and left connnection matrices, as well as the right an left curvature matrices are similarity transforms of one another.

(The right Cartan matrix representation is prefered, as it keeps the order of the functions and the exterior derivatives in proper sequence. Its use also strengthens the idea that the invertible frame matrix implies that exterior differentiation is a closure concept, such that the exterior derivative of any basis vector is a linear combination of the set of basis vectors.)

Now return to the general mapping formula written as applied to vector arrays of p-forms,

$$[G]\circ|\sigma\rangle = |\omega\rangle\,. \tag{1.11}$$

Exterior differentiation produces the vector array of p+1 forms,

$$[G]\circ\{[-C_L]\,\hat{}\,|\sigma\rangle + |d\sigma\rangle\} = |d\omega\rangle\,. \tag{1.12}$$

A second exterior differentiation produces the equation of p+2 forms.

$$\begin{aligned} [G]\circ\{[C_L]\,\hat{}\,[C_R] + [dC_R]\}\hat{}\,|\sigma\rangle + |dd\omega\rangle\} &= \\ [F]\circ\{[\Theta_R]\,\hat{}\,|\omega\rangle + |dd\omega\rangle\} &= \\ [F]\circ\{Curvature\ 2-forms\} &= |dd\sigma\rangle \Rightarrow 0. \end{aligned} \tag{1.13}$$

The "second derivative" equation dd(map) expresses the closure concept (mod singularities) of the basis frame in terms of the vector of induced Cartan "Curvature-Acceleration" 2-forms, $[\Theta_R]\,\hat{}\,|\omega\rangle$, and the vector of Coriolis-Torsion-acceleration 2-forms, $2\,[C_R]\,\hat{}\,|d\omega\rangle$. The nomenclature follows from the historic notation, when the formulas are applied to the special case of dynamic discriptions of a 3D vector in a fixed frame of reference being compared to the dynamic description in a rotating orthonormal frame of reference (see the example below).

The result can be expressed as a theorem:

Theorem 1.1. ***For C2 forms and Frame fields*** $[F]$***, without Poincare singularities, Curvature effects are balanced by Coriolis effects***

Remark 1. *The conjecture that gravity - Curvature can be balanced (somehow) by torsion-Coriolis and singularities is tempting. How to exploit this conjecture is an open question.*

1.1.3. Example Application: The classic formulas of Deformation about a fixed point.

The fundamental dd(map) formula can be used to interpret the case where a vector, defined relative to a distorting frame $[F]$, is mapped into a vector defined with respect to a fixed frame, $[I]$. The origins of the two frames are presumed to be coincident (fixed points). The formulation is typical of the map from a rotation frame into an "inertial" frame. The map formula relates a position vector in the expanding-rotating system and the position vector in a "fixed" frame (using the basis which is the identity matrix.)

$$[F] \circ |q\rangle = |x\rangle = [1] \circ |x\rangle \,. \tag{1.14}$$

Note that when $|q\rangle \Rightarrow 0, \;\; |x\rangle \Rightarrow 0$.

The d(map) formula yields

$$[F] \circ \{[C_R] \,\hat{}\, |q\rangle + |dq\rangle\} = |dx\rangle \,. \tag{1.15}$$

while the right Cartan matrix has matrix elements of the form

$$\left[\sum_b C_{ab}^c(q) dq^b\right] \tag{1.16}$$

If the system is now constrained by the possibly inexact Pfaffian forms (known as the kinematic assumption) such that $dq^b - v^b(q,t)dt = 0$, and $dx^k - V^k(q,t)dt = 0,$ then using the formula

$$[\Omega_R]\, dt = \left[\sum_b C_{ab}^c(q) v^b\right] dt \tag{1.17}$$

the d(map) equation becomes

$$[F] \circ \{[\Omega_R] \circ |q\rangle + |v\rangle\} dt = |V\rangle\, dt. \tag{1.18}$$

Canceling the dt factor on each side yields the "velocity" composition formula of deformation about a fixed point

$$[F] \circ \{[\Omega_R] \circ |q\rangle + |v\rangle\} = |V\rangle \,. \tag{1.19}$$

If the original Basis frame $[F]$ is chosen to be an element of the orthonormal group, then there is no expansion distortion, only a rotation. The term $[\Omega_R] \circ |q\rangle$ in 3D becomes equal to the Gibbs cross product expression, $\mathbf{\Omega} \times \mathbf{q}$. A second exterior differentiation of the distorsion formula yields.

$$[F] \circ \{[\Omega_R] \circ [\Omega_R] \circ |q\rangle \, dt + d\,[\Omega_R]\} \circ |q\rangle + 2\,[\Omega_R] \circ |v\rangle + |dv\rangle\} = |dV\rangle \,. \qquad (1.20)$$

If the further kinematic "acceleration" constraints are made, such that $dv^b - a^b(q,t)dt = 0$, $dV^k - A^k(q,t)dt = 0,$ and $d\,[\Omega_R] = [\boldsymbol{\alpha}]\,dt$, and if the original frame is presumed to be orthonormal, then the classic Coriolis equation is obtained:

$$[F] \circ \{\mathbf{\Omega}\times(\mathbf{\Omega} \times \mathbf{q}) + \boldsymbol{\alpha} \times \mathbf{q} + 2\mathbf{\Omega} \times \mathbf{v} + |\mathbf{a}\rangle\} = |\mathbf{A}\rangle \,. \qquad (1.21)$$

1.1.4. Matrix methods

As shown below, holonomic Affine Torsion is a myth. Holonomic Linear mappings never produce Affine Torsion. A holonomic domain need not be global over the domain of independent variables x. The holonomic domain is "global" only over the subset q where the determinant of the Frame matrix $[F]$ is not zero. (Think space mod particles.)

The Basis Frame $\left[F_a^k(q^b)\right]$ is defined herein as a row column square matrix of C2 functions whose arguments are the independent variables q. Over a restricted domain, the rows (upper index) of the matrix often can be put into correspondence with components of contravariant vectors for a given column (lower) index. Similarly for a fixed row index, the matrix columns can be put into correspondence with covariant vectors. A fiber space allusion can be based on the notion that the set q is a subset of x produced by the constraint $det[F] = 0$. However the fibers are "points" as the dimension of x and the dimension of q are the same.

1.1.5. Singularities of the induced 1-forms

Recall that any particular induced 1-form, σ^k, can be used to generate its Pfaff sequence,

$$\{\sigma^k, d\sigma^k, \sigma^k\hat{}\,d\sigma^k, d\sigma^k\hat{}\,d\sigma^k, ...\} \qquad (1.22)$$

which will contain $j \leq N$ non-zero entries. The number j determines the minimum number of functions (in a limited, finite, but local, domain) that are required to express the exterior differential properties of the given form. The number j is defined as the class, or Pfaff dimension, of the 1-form, σ^k. Note that if the

1-form is exact, then its Pfaff dimension is 1; only one function (the minimum number) is required to define the 1-form locally. When the 1-form is exact, the minimum number of functions becomes a global property. However, if the 1-form is closed but not exact, then $d\sigma^k = 0$, and the Pfaff dimension is still 1; but now the 1-form will have "singularities" that will impose further restrictions on the admissable domain q. The classic example is given by the "circulation" 1-form $\sigma^k = \{\psi d\psi^* - \psi^* d\psi\}/\{\psi^*\psi\}$ which has zero curl and is well defined except at the nodes of ψ. The allusion to quantum mechanics is deliberate. Therein the "excited stationary states" - which have nodes - are to be treated as singularities, or topological defects. Closed integrals whose integration chains encircle singularities produce values which have rational ratios. That is, the closed integrals are quantized.

Circulation singularities are often represented by "holes", or sets of missing points, in a surface. The second excited state of the "quantum" example given above would have two holes; the ground state would have zero holes. A decay process to the ground state involves a topological change. It is extraordinary that topological changes at the Pfaff dimension 1 level (changing the number of holes in a surface) can occur continuously or discontinously. That is, it is possible to devise evolutionary processes where both the creation of holes or the destruction of holes can be done continuously! The continuous creation of a hole implies that the domain has a boundary, and the new holes are created as the boundary is deformed and glued together. The continuous destruction of holes does not require the domain to be bounded.

There are other topological defects that depend on the fact that the Pfaff dimension of a particular 1-form is greater than 1. For example when the Pfaff dimension is 3, then it is possible to show that topological evolution to create such 3D defects **cannot** be done continuously, but the destruction of such discontinuities can be accomplished in a continuous manner. Such concepts lead to a non-statistical basis for thermodynamic irreversibility.

2. Pronouncements

Using the Cartan methods of the exterior calculus leads to the following statements and conclusions:

1. **The Cartan connection:** Holonomic Affine Torsion is a myth. The proof of the statement depends upon the definition of a right Cartan matrix

of 1-forms, $[C_a^c(q)]$, called the Cartan connection, which always exists on the domain q, for C1 functions. On the domain, q, the Frame matrix $\left[F_c^k(q)\right]$ has an inverse $[G_m^c(q)]$ such that

$$[G_m^c(q)] \circ [F_a^m(q)] = [1]. \tag{2.1}$$

Differentiation of this matrix expression leads to the equation

$$[dG_m^c(q)] \circ [F_a^m(q)] + [G_m^c(q)] \circ [dF_a^m(q)] = [0]. \tag{2.2}$$

Pre-multiplication by the matrix, $\left[F_c^k(q)\right]$ yields the formulas that define the right $[C_a^c(q)]$ and left $\left[\Delta_m^k(q)\right]$ Cartan connection matrices of 1-forms:

$$\begin{aligned} 0 &= \left[dF_a^k(q)\right] + \left[F_c^k(q)\right] \circ [dG_m^c(q)] \circ [F_a^m(q)] \\ &= \left[dF_a^k(q)\right] + \left[\Delta_m^k(q)\right] \circ [F_a^m(q)] \qquad (2.3) \\ &= \left[dF_a^k(q)\right] + \left[F_c^k(q)\right] \circ [C_a^c(q)] \qquad (2.4) \end{aligned}$$

These equations are also used to define "parallel transport". Focus attention on the right Cartan Connection:

$$d\left[F_a^k(q)\right] = \left[F_c^k(q)\right] \circ [C_a^c(q)] = \left[F_c^k(q)\right] \circ \left[\sum_b C_{ab}^c(q) dq^b\right]. \tag{2.5}$$

The construction is independent from a metric constraint that might be imposed upon the system, and only depends upon the existence of an inverse Frame matrix. There are two possible formats:

$$\left[\sum_b C_{ab}^c(q) dq^b\right] = [C_a^c(q)] = -\,[dG_m^c(q)] \circ [F_a^m(q)] = [G_m^c(q)] \circ [dF_a^m(q)] \tag{2.6}$$

It is important to note that both the indices that denote the elements of the right Cartan matrix, and its arguments, are defined on the initial state, q.

2. **Anholonomic Affine Torsion**: The coefficients of Affine Torsion are defined as $C_{[ab]}^c(q) = C_{ab}^c(q) - C_{ba}^c(q)$. Anholonomic Linear mappings do

not *necessarily* produce Affine Torsion. Holonomic Linear mappings never produce Affine torsion If the induced differential 1-forms are closed, but not exact, the Frame does not produce Affine Torsion. However, there is possibly another level of topological defect structures produced by those induced 1-forms which are closed but not exact. (with applications to understanding the Bohm-Aharanov effect, Flux quantization, Berry phase, Lift of a Joukowski airfoil, etc. via deRham theory). Such defects are defined by "two dimensional" Pfaffian constraints (zero sets of 2-forms). When functions are equal to zero, their induced closures (1-forms) vanish as necessary anholonomic constraints. Note that the vector of closure 2-forms can be written as:

$$\left|d\sigma^k\right\rangle = \left[dF_a^k(q^b)\right] \hat{} \left|dq^a\right\rangle = \left[F_c^k(q^b)\right] \circ \left|C_{[ab]}^c dq^b \hat{} dq^a\right\rangle . \tag{2.7}$$

Hence if the LHS vanishes, implying closure, then the RHS = 0, implying that the affine torsion is zero for any situation where the induced 1-forms σ^k are closed, even if they are not exact.

3. **Closure 2-forms:** Bottom line: The Pfaff dimension of one or more of the induced 1-forms σ^k must be greater than 1 if Affine Torsion is to exist. In other words, some of the induced 1-forms must **not** be closed, if Affine torsion is to exist. This implies that some of the (covariant) rows of a Frame matrix (not the contravariant columns) must have non-zero "curl". The vector of closure 2-forms can be written as:

$$\left|d\sigma^k\right\rangle = \left[dF_a^k(q^b)\right] \hat{} \left|dq^a\right\rangle \Rightarrow \left|\{\partial F_a^k/\partial q^b - \partial F_b^k/\partial q^a\} dq^b \hat{} dq^a\right\rangle . \tag{2.8}$$

If the RHS = 0 (zero curl), then the LHS = 0, implying that the Affine Torsion coefficients, above, must vanish.

4. **The Object of Anholonomicity:** If the induced1-forms σ^k are complete, such that the product, $\sigma^1 \hat{} \sigma^2 \hat{} ...\sigma^N \neq 0,$ then every 2-form $d\sigma^k$ can be expanded as: $d\sigma^k = \Omega_{ba}^k \sigma^b \hat{} \sigma^a$. The vector of "closure" 2-forms becomes:

$$\begin{aligned} \left|d\sigma^k\right\rangle &= \left|\Omega_{mn}^k \sigma^m \hat{} \sigma^n\right\rangle = \Omega_{mn}^k \{(F_b^m(q) dq^b) \hat{} (F_a^n(q) dq^a)\} \\ &= \left|\Omega_{mn}^k \{(F_b^m F_a^n - F_a^m F_b^n\} dq^b \hat{} dq^a\right\rangle \end{aligned} \tag{2.9}$$

The 3-indexed coefficients of this expansion, Ω_{ba}^k, define the "Object of Anholonomicity" and are different from the "3 indexed Object of Affine

Torsion", $C^c_{[ab]}$. In summary, there are three distinct, but equivalent, ways to write the vector of closure 2-forms:

$$\begin{aligned} Closure\ \ 2-forms \quad \left|d\sigma^k\right\rangle &= \left|\Omega^k_{mn}\{(F^m_b F^n_a - F^m_a F^n_b\} dq^b\hat{}dq^a\right\rangle \\ &= \left|\{\partial F^k_a/\partial q^b - \partial F^k_b/\partial q^a\} dq^b\hat{}dq^a\right\rangle \\ &= \left[F^k_c(q^b)\right] \circ \left|C^c_{[ab]} dq^b\hat{}dq^a\right\rangle \end{aligned} \tag{2.10}$$

Due to linear independence, a zero of any of the three representations implies a zero for the other representations. It follows that a zero closure condition (all elements of $\left|d\sigma^k\right\rangle$ are of Pfaff dimension 1) implies zero "curl", zero Affine Torsion and Zero Object of anholonomicity.

5. **Cartan Torsion 2-forms:** In addition to the three ways of producing a vector of closure 2-forms, it is possible to construct other useful sets of vector valued 2-forms, with arguments and indices both assocated with the initial state. One possibility is consider the vector valued 1-forms constructed by pre-multiplying the fundamental mapping equation by the inverse matrix to yield

$$[G^c_k(q)] \circ \left|\sigma^k\right\rangle = |dq^c\rangle \tag{2.11}$$

Exterior differentiation followed by pre multiplication by $[F^m_a(q)]$ yields a vector valued set of 2-forms

$$\left|\Sigma^k\right\rangle = \left[\Delta^k_m(q)\right]\ \hat{}\ |\sigma^m\rangle + |d\sigma^m\rangle \Rightarrow 0 \tag{2.12}$$

which involves the Left Cartan connection, $\left[\Delta^k_m(q)\right]$. The vector valued 2-forms, $\left|\Sigma^k\right\rangle$, could be used to define a Cartan vector of Cartan torsion 2-forms, which like the Cartan matrix of curvature 2-forms, must vanish on the parallelizable domain q. It is apparent that the vector of Cartan torsion 2-forms is not the same as the Closure 2-forms used to define Affine Torsion. Exterior multiplication of each element of the vector of Cartan torsion 2-forms by the appropriate 1-form produces a vector of 3-forms that must vanish necessarily:

$$\left|\sigma^k\hat{}\Delta^k_m(q)\hat{}\sigma^m\right\rangle + \left|\sigma^k\hat{}d\sigma^k\right\rangle = 0. \tag{2.13}$$

The last term is precisely the vector of Topological Torsion 3-forms for each induced 1-form, σ^m. So there are two cases: (a) Either each term vanishes, which implies that the topological torsion vanishes, or (b) the two terms in each element of the Cartan torsion 3-forms cancel, and the topological torsion is not zero.

In the more general situation, where

$$[F] \circ \{[C_R] \hat{} \, |\omega\rangle + |d\omega\rangle\} = |d\sigma\rangle \,. \tag{2.14}$$

the vector of Cartan Torsion 2-forms could be defined as

$$\left|\Sigma^k\right\rangle = \{[C_R] \hat{} \, |\omega\rangle + |d\omega\rangle\} = [G] \circ |d\sigma\rangle \tag{2.15}$$

6. **Intrinsic Closure Gaps** Both Affine Torsion and sectional Gauss Curvature will produce "closure Gaps". A differential closure gap is defined as nonzero result obtained following a displacement along dq,1 followed by a displacement along dq2, followed by a displacement along -dq1, followed by a displacement along -dq2. These specific closure gaps are intrinsic in that they reside in the 2-plane defined by the two exact differential displacements, dq1 and dq2, used in their construction. There are other types of curvature gaps (induced by topological torsion) which are not intrinsic. The Affine torsion gap, as physically induced by translational shears to produce edge and screw dislocations, is intrinsic. The Gauss curvature gap, often associated with "rotations" and disclinations is intrinsic. In crystalline structures, the energy to produce curvature gaps is much larger than the energy to produce Affine torsion gaps. The opposite is true in fluids. In an *ideal* crystal, it might be said the translational shears are permitted, but rotational shears are not. In an *ideal* fluid, translational shears are not permitted, but rotational shears are permissible. The nomenclature "screw dislocation" is unfortunate for such differential displacements form a piecewise curve whose endpoints reside in the "tangent plane" to a helical surface. The screw dislocation closure gap does not exhibit "helicity" which would require that the gap has components orthogonal to the tangent plane.

7. **Non-Intrinsic Topological Torsion Gaps**: There are other types of torsion closure gaps (topological torsion) which are **not** intrinsic. A gap is defined as not intrinsic if the closure gap is not in the 2-plane defined by the two exact differential displacements. The non-intrinsic torsion gap has components perpendicular to the 2-plane defined by the two exact differential displacements. Such gaps have helicity (are intrinsically 3 dimensional)

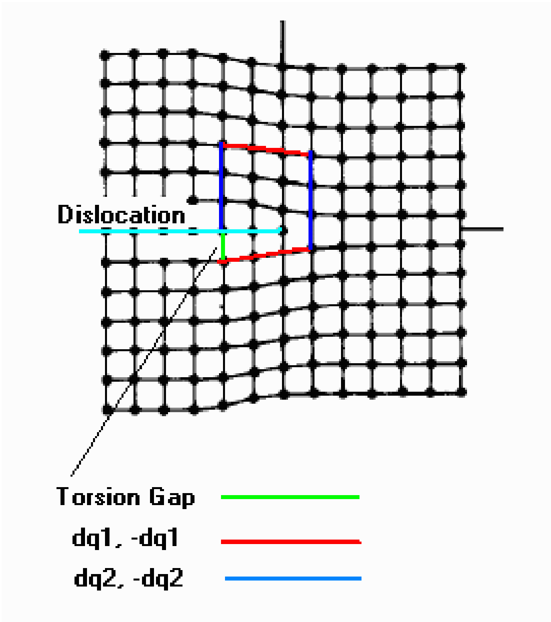

Figure 2.1:

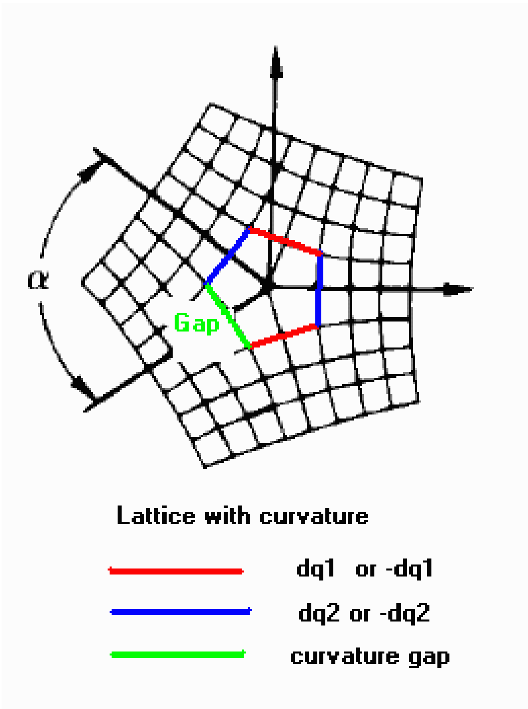
α
Gap
Lattice with curvature
dq1 or -dq1
dq2 or -dq2
curvature gap

Figure 2.2:

and cannot be approximated by a gap in the plane. These gaps are not formed by translational shears, but are more to be associated with rotational shears in combination with translations. If the non-holonomic linear mapping of interest is an element of the Affine group, then Affine torsion is possible, and the Affine torsion gap is intrinsic. If the non-holonomic linear mapping is not an element of the affine group, then there can exist non-intrinsic torsion. For example, the linear mapping could be a projective mapping (not affine) and it would be expected that there existed "torsion" gaps that are orthogonal to the 2-plane defined by the two exact differentials. This concept of (topological) torsion (not affine torsion) is closest to the concept of Frenet Torsion of a space curve. Frenet Torsion equal to zero means the space curve in 3D resides on a 2-surface (the plane in euclidean space). Non-intrinsic Torsion and Torsion gaps out of the 2-plane exist only if the Topological Torsion is non-zero. If the non-holonomic linear mapping of interest is an element of the projective (but not affine) group, then the induced (non-Affine) torsion can be non-intrinsic. Cartan torsion, for example, is not necessarily intrinsic, but depends upon the embedding. How does one know that such things exist without Cartan's intuition? Mathematically the trick is to embed the manifold into a suitably large Whitney space, and then partition the Whitney space into a subspace of interest. Then all of the features described above can be evaluated on the subspace. Another method is to presume the existence of a global 1-form on a domain of dimension n+1. Then compute the n associated vectors that annihilate the 1-form. Use these n vectors, and the components of the initial 1-form (multiplied by some non-zero function) to define a projective basis frame in dimension n+1. The n dimensional subspace can have non-zero curvature and affine torsion as well as non-affine non-intrinsic Cartan torsion, or topological torsion. The embedding space is a space of absolute parallelism. This has been done in detail at :

http://www22.pair.com/csdc/pdf/defects2.pdf

8. An A4 space is a 4 dimensional space that has Zero Cartan torsion, and Zero Cartan curvature. This means that the curvature gaps are zero, and the Affine torsion gaps need not be zero. The domain is parallizeable. The Shipov idea that the vacuum has both possibilities (zero Affine torsion with zero curvature, or non-zero Affine torsion with zero curvature) is what I applaud. (It is not clear that Shipov considers the case of Topological

Torsion)

His work stimulates me to conjecture that the Shipov vacuum may be extended to a space that has Zero curvature, but non-zero topological torsion. That is, the vacuum may have no mass (implying zero curvature) but both affine (intrinsic) and topological (non-intrinsic) torsion. Such effects can occur in a space of absolute parallelism, but the topological torsion is then constrained. Without the constraint, the space-time vacuum would have to be embedded in a higher dimensional space. It could not be an A4 space of absolute parallelism.

9. An Anholonomic linear mapping can be elastic (reversible but perhaps chaotic) even if the topological torsion is not zero. That is, there exist elastic helical deformations which are elastic.

10. For multi-sheeted solutions, envelopes, edges of regression, various types of catastrophes, the topological torsion of

    one or more of the induced 1-forms cannot vanish.

11. A plastic (irreversible) deformation requires that the topological torsion is **not** zero, and is not closed in an exterior differential sense. The minimum dimension for (continuous) irreversibility is 4.

12. Almost all Frame matrices are not Unitary.

13. A unitary matrix is normal and therefor the decomposition into RH and LH parts is degenerate. Chirality is where the RH and LH parts are distinct. (Hence such matrices are not normal in the sense that they do not commute with their adjoint and therefore are not unitary)

14. A choice of anholonomic constraints refines the topology of the domain of interest. IMO the presence of dynamical matter not only generates a metric field with metric curvature, but also generates torsion of both the intrinsic and not intrinsic types.

15. Distant parallelism that is holonomic implies independence from path. Distant parallelism which is anholonomic is path dependent. The concept of independence from path is related to the closed loop integral of a 1-form.

If the 1-form is exact, all closed loops give zero value for the integral. If the 1-form is not exact, the closed loop integral on a boundary is zero, but a closed loop on a cycle which is not a boundary is not zero (see deRham).

3. NOTES

3.1. Preliminary: differential forms

The topics described below deal with things called Pfaffian equations and differential forms on a domain. Perhaps the simplest way to think of differential forms as to note that they are objects that form the integrands of multi-dimensional integrals. The most well known differential forms are 1-forms and 2-forms (related to 1 dimensional line integrals and 2 dimensional surface integrals) and N-1 form and N forms (related the integrands of current densities and density volume elements). An extraordinary feature of differential forms is that they are well behaved in a functional sense as invariants relative to admissable coordinate transformations called diffeomorphisms (which are maps that do not change topology). Perhaps even of greater importance is the fact that differential forms are well behaved in a functional sense with respect to C2 differentiable maps which are not homeomorphisms. Such maps describe *continuous topological evolution.*

When defined on a differential variety of coordinate functions, the differential forms can separate the domain into two sets. Regions of support (where the differential form is not zero) and the zero sets (regions where the differential form vanishes.) Pfaffian equations are of the latter category; they are essentially differential forms constrained to zero. Pfaff's problem is to find solutions to Pfaffian equations. Differential forms may be exact, closed, integrable or non-integrable. For exact forms, there exists a single global pre-image whose exterior derivative reproduces the differential form. For 1-forms, an exact 1-form is represented by the exterior derivative of a unique global zero form, or function. The components of the exact 1-form behave as the components of a gradient field. A Pfaffian equation as an equation of constraint between coefficient functions and differentials of a 1-form may or may not be exact. If the constraint is exact, then the constraint is defined as a holonomic constraint. All other constraints are called non-holonomic (or equivalently anholonomic) constraints. Pfaffian equations as anholonomic constraints appear throughout physical theories. Sometimes they appear in subtle and unappreciated ways. For example:

Kinematics as an anholonomic constraint A common set of anholonomic constraints (and a set that is usually unappreciated as an anholonomic constraint) is given by the zero set of the kinematic (particle) 1-forms (on the domain $\{q^k, t\}$). That is, consider a C2 map ϕ from $\{q^k, t\} \Rightarrow \{x^k\}$. It is useful to recognize that the map is from a domain (or initial state) to a range (or a final state). The arguments of the map are the independent variables of the initial state. Then consider the definitions.

$$\begin{aligned} \sigma^k &\triangleq dx^k - V^k(q,t)dt = \{\partial\phi^k(q,t)/\partial q^m\}dq^m = \Delta^k \\ With &: \qquad V^k(q,t) \triangleq \{\partial\phi^k(q,t)/\partial t\}. \end{aligned} \tag{3.1}$$

There are 5 cases that will be considered herein.: The 1-forms σ^k are zero, exact, closed, integrable, or non-integrable.

The kinematic hypothesis is that the 1-forms σ^k vanish on the domain of interest, which implies that the "anholonomic fluctuations", Δ^k, must vanish. The constraint implies that

$$\{\partial\phi^k(q,t)/\partial q^m\}dq^m = 0 \tag{3.2}$$

From which it follows that either all of the displacements vanish, $dq^m = 0$,such that the $q^m =$ constant "initial conditions", or the matrix $\{\partial\phi^k(q,t)/\partial q^m\}$ has a zero determinant, and the non-zero displacements define the Null eigenvector of the given matrix. In the latter case the initial conditions are not constants, but are constrained to a singular set. The kinematic constraint is satisfied if the anholonomic fluctuations vanish (the classic case).

A less restrictive constraint is that the 1-forms σ^k are closed, in the sense that $d\sigma^k = d(\Delta^k) = 0$, but $\Delta^k \neq 0$.

$$If\ d\sigma^k = 0,\ then\ dV^k\hat{}dt = (\partial\Delta^k/\partial t)\hat{}dt = \{\partial^2\phi^k(q,t)/\partial t\partial q^m\}dt\hat{}dq^m \Rightarrow 0 \tag{3.3}$$

Then either $t =$ constant, or the "Velocity" functions, V^k, can be functions of t alone. That is there exists a single parameter map from t into the functions, $V^k(t)$. The fluctuation 1-forms Δ^k are not zero, but are independent from time, t. Such is the foundation of the Langevin approach to fluctuations.

The 1-forms σ^k need not be exact perfect differentials, and contain information about certain (1-dimensional) topological properties (or defects) of the domain. Such domains of induced coordinate 1-forms do not support Affine torsion. A necessary requirement for Affine Torsion is that some or all of the induced coordinate 1-forms, σ^k, are not closed.

A third situation and even less restrictive constraint may be posed by the requirement that the Pfaffian equations are integrable. The implication is that there is a set of 3-forms that must vanish, $\sigma^k\hat{}d\sigma^k = 0$. Such a rank 3 Pfaffian anholonomic constraint is required if there exists and integrating factor $\lambda^k(x,t;\beta)$

for each σ^k, such that although $d\sigma^k \neq 0$, the product of the integrating factor and the 1-form is closed: $d\{\lambda^k(x,t;\beta)\ \sigma^k\} = 0$. If $\sigma^k \hat{}\, d\sigma^k \neq 0$, then the Pfaffian equations are not uniquely integrable integrable, and if the 3-form is closed ($d\sigma^k \hat{}\, d\sigma^k = 0$), the 3-forms carry another set of topological properties (or defects) different from the defects generated by a closed, but not exact, system of 1-forms. When $\sigma^k \hat{}\, d\sigma^k \neq 0$, the domain is said to support "Topological Torsion".

It will be demonstrated below that both (1) holonomic coordinates (where each coordinate is a gradient) and (2) closed but not exact coordinates (typical of multi-valued functions) are NOT to be associated with Frame matrices that produce Affine Torsion.

Thermodynamics as an anholonomic constraint Cohomology is the study of forms whose difference is exact. The most famous such physical system is that given by the first law of thermodynamics: $Q - W = dU$. Neither the 1-form of heat, Q, nor the 1-form of work, W is (necessarily) closed, but their difference is exact! The first law is another example of a non-holonomic constraint:

$$Q - W - dU = 0. \tag{3.4}$$

Rolling without Friction as an anholonomic constraint When a ball rolls on a surface without slipping of skidding, there is an anholonomic constraint imposed upon the dynamics. The constraint is often formulated by the Pfaffian equation $\lambda d\theta - dx = 0$.

3.1.1. Exact 1-forms

A differential form, $\sigma(q) = A_b dq^b$, has a Pfaff dimension, or class, represented by an integer, m, that defines the minimum number of independent functions that are required (over a limited domain) to define the differential form. If the differential form has a global definition in terms of a single function $\phi(q)$ over the entire domain, then the form is said to be exact.

$$\text{An Exact 1-form:} \qquad \sigma(q) = d\phi(q) = \{\partial\phi(q)/\partial q^b\}dq^b \tag{3.5}$$

The coefficients of the exact 1-form have the components of a gradient field. Exact forms are globally integrable.

3.1.2. Closed 1-forms

A closed form is a form where the exterior derivative of the form vanishes. All exact 1-forms are closed (for C2 functions),

$$d\sigma(q) = d(d\phi(q)) = \{\partial^2\phi/\partial q^a \partial q^b - \partial^2\phi/\partial q^b \partial q^a\} dq^a \hat{}\, dq^b = 0. \tag{3.6}$$

However, there are 1-forms that are closed, but not exact. Closed, but not exact, 1-forms are integrable over a restricted domain, which may be finite but not global. As an example, consider the 1-form constructed from two independent functions:

$$Example : \sigma(q) = \{\phi(q)d\chi(q) - \chi(q)d\phi(q)\}/\{a\phi^p + b\chi^p\}^{2/p} \tag{3.7}$$

The example is closed (has zero "curl") but is not exact. $\quad d\sigma(q) = 0 \quad (3.8)$

The Pfaff dimension of a closed but not exact 1-form is 1, but the form cannot be globally represented by a gradient of a single function. The domain contains a topological defect(s), not typical of a Cartesian space. There are closed paths (cycles) that are not shrinkable and are not boundaries. A typical example is given by a planar disc with a central hole. Each hole is represented by a unique closed but not exact (deRham) that says something about the fact that the domain of definition is not simply connected. Note that if the two functions in the example are taken to be $\phi = \Psi$ and $\chi = \Psi^\dagger$ then the closed (but not exact) 1-form becomes (for $a = 1, b = 1, p = 2$)

$$\sigma(q) = \{\Psi d\Psi^\dagger - \Psi d\Psi\}/(\Psi^\dagger\Psi) = \{\Psi\nabla_m\Psi^\dagger - \Psi\nabla_m\Psi\}dq^m/(\Psi^\dagger\Psi), \tag{3.9}$$

which is recognized as being proportional to the "probability current" in quantum mechanics. The integrals around closed cycles of such closed but not exact 1-forms are the basis for the Bohm Aharanov effect, and the deRham formulas are the basis for what has been popularized as "flux quantization" or as the "Berry phase".

Closed forms which are not exact are often related to multi-valued functions. The closed 1-form given above is not well defined when the denominator goes to zero. If the domain of definition excludes these regions, then the form is well defined, and there exist closed integration chains called cycles (which are not necessarily boundaries) for which the integral of the form along the closed chain is NOT zero. If the integration is performed about a boundary cycle(s) the value of the integral is zero. An exact form integrated about a closed cycle is zero, whether it is a boundary or not. These closed but not exact 1-forms carry topological information (in terms of the excluded points which are interpreted as defects or singularities). As mentioned above. these closed but not exact 1-forms have physical significance as the flux quantum. Similar closed, but not exact 2-forms, have physical significance as the charge quantum. Closed but not exact 3-forms have physical significance as the spin quantum.

See http://www22.pair.com/csdc/pdf/periods.pdf or R. M. Kiehn, J. Math Phys **18** 4 (1977)

3.1.3. Not closed but integrable 1-forms

Even if a 1-form σ is not closed, such that $d\sigma \neq 0,$ it still may be true that there exists an integrating factor λ that makes the form closed. That is

$$d(\lambda(q)\,\sigma) = d(\lambda(q)\,A_b dq^b) = d\lambda\hat{}\sigma + \lambda\,d\sigma \Rightarrow 0 \tag{3.10}$$

It follows that a necessary condition for the existence of an integrating factor is that

$$\sigma\hat{}\{d\lambda\hat{}\sigma + \lambda\,d\sigma\} = \sigma\hat{}d\lambda\hat{}\sigma + \lambda\,\sigma\hat{}d\sigma = 0 + \lambda\,\sigma\hat{}d\sigma = 0 \tag{3.11}$$

Hence, the 3-form $\sigma\hat{}d\sigma$ must vanish if an integrating factor exists. The condition is known as the Frobenius condition of integrability. The integrating factor does not necessarily convert the non-closed 1-form into an exact 1-form.

3.1.4. Non-integrable 1-forms

If the associated 3-form $\sigma\hat{}d\sigma$ does not vanish, then an integrating factor does not exist, and the system is said to be non-integrable. The Frobenius condition of integrability fails.

4. FRAME MATRICES

The columns of a matrix of functions for which the determinant is non-zero on a finite domain $\{q^a\}$ can be used as a basis matrix for contravariant vectors. The columns of the frame matrix are treated as contravariant vectors with components designated by the upper (row) index. The rows of the Frame matrix are treated as the components of covariant vectors, with the component index being the lower (column) index. The designations follow in accord with the existence of integral maps from the domain to the target. However, the requirements for a Frame matrix do not require that the Frame be deducible from a map of functions. All that is required is that the Frame matrix form a linearly independent set of basis vectors at every point p. The requirement is that the determinant of the Frame matrix is non-zero. Points on the domain where the determinant of the Frame matrix is not zero are called defects or singularities.

There are 4 major cases to consider:

1. The Frame matrix maps exact differentials into exact differentials (the holonomic case where the map from range to target is known.)

$$\left[F_a^k(q^b)\right] \circ |dq^a\rangle \Rightarrow \left|\sigma^k\right\rangle = \left|dx^k\right\rangle \tag{4.1}$$

2. The Frame matrix maps exact differentials into non-exact differentials (non-holonomic)

$$\left[F_a^k(q^b)\right] \circ |dq^a\rangle \Rightarrow \left|\sigma^k\right\rangle \tag{4.2}$$

   There are three subcases to consider.

   (a). The induced 1-forms $\left|\sigma^k\right\rangle$ are closed, integrable, or not integrable. Note that the term "non-holonomic" is used to imply that the induced 1-forms are not exact. As will be shown below, both holonomic (exact) and special non-holonomic (closed but not exact) Frames are Affine torsion free.

   (b) Non-holonomic is a necessary, but not sufficient, condition to produce Affine torsion. The lack of closure (the 2-form vanishes: $d\sigma^k \neq 0$) is a sufficient requirement to produce Affine torsion. However, in certain circumstances the system may be integrable in the Frobenius sense, such that a new Frame can be constructed in terms of "integrating" factors, and that new frame will be of the types (a) or (b) above. A necessary condition for integrability is that the topological torsion 3-form must vanish: $(\sigma^k\hat{}d\sigma^k = 0)$.

   (c) In the non-integrable cases, $(\sigma^k\hat{}d\sigma^k \neq 0)$.

3. The Frame matrix maps non-exact differentials into exact differentials

$$\left[F_a^k(q^b)\right] \circ |\omega^a\rangle \Rightarrow \left|dx^k\right\rangle \tag{4.3}$$

4. The Frame matrix maps non-exact differentials into non-exact differentials

$$\left[F_a^k(q^b)\right] \circ |\omega^a\rangle \Rightarrow \left|\sigma^k\right\rangle \tag{4.4}$$

Only the first case is holonomic, for in that case there exists a functional map from $\{q^a\}$ to $\{x^k\}$. The other cases are anholonomic. The first three cases are special cases of case 4.

Note that in *every* case there is a right Cartan connection such that

$$d\left[F_a^k(q)\right] = \left[F_c^k(q)\right] \circ \left[C_a^c(q)\right] = \left[F_c^k(q)\right] \circ \left[\sum_b C_{ab}^c(q)dq^b\right] \tag{4.5}$$

This result is deduced from the condition that the Frame matrix $\left[F_c^k(q)\right]$ has an inverse $[G_m^c(q)]$. The result of a connection is not imposed as a definition out of the blue. The connection matrix of differential 1-forms is constructed as

$$\left[\sum_b C_{ab}^c(q)dq^b\right] = [C_a^c(q)] = -\left[dG_m^c(q)\right] \circ [F_a^m(q)] = [G_m^c(q)] \circ [dF_a^m(q)] \tag{4.6}$$

The construction is independent from a metric constraint imposed upon the system. It is important to note that the indices that denote the elements of the right Cartan matrix are defined on the initial state, and the arguments are in terms of initial state variables.

Consider the exterior derivative of the Frame matrix, without the imposition of the constraint that the determinant be non-zero. Then for case 2

$$d\left[F_a^k(q^b)\right] \circ |dq^a\rangle = \{\partial F_a^k/\partial q^b - \partial F_b^k/\partial q^a\}dq^b\hat{}dq^a \tag{4.7}$$

When the inverse matrix exists, another expression for the vector of closure 2-forms is obtained as

$$d\left|\sigma^k\right\rangle = [G_k^c(q)] \circ \{\partial F_a^k/\partial q^b - \partial F_b^k/\partial q^a\}dq^b\hat{}dq^a = \tag{4.8}$$

.

Consider the two definitions: (constructed from the connection generated by the Frame matrix)

The **Vector of Affine Torsion 2-forms** is defined as $: |AT^c\rangle = [C_a^c(q)]\ \hat{}\ |dq^a\rangle = \ \left|C_{[ab]}^c dq^b\hat{}\right.$ (4.9)

The **Matrix of Curvature 2-forms** is defined as $[\Theta_a^c] = \{[dC_a^c(q)]+[C_b^c(q)] \,\hat{}\, \left[C_a^b(q)\right]\}$ (4.10)

4.1. Holonomic systems (Case 1)

The basis assumption is that Frame matrix maps exact differentials into exact differentials.

$$\left[F_a^k(q^b)\right] \circ |dq^a\rangle \Rightarrow \left|dx^k\right\rangle \tag{4.11}$$

Such a situation exists when there exists an explicit mapping from one set of (coordinate) variables $\{q^a\}$ to another set of variables $\{x^k\}$ is given in terms of C2 differentiable functions,

$$\phi : q^a \Rightarrow x^k = \phi^k(q^a). \tag{4.12}$$

The induced exact coordinate differentials $\left|dx^k\right\rangle$ are related linearly to the exact differentials $|dq^a\rangle$ by the Jacobian mapping:

$$d\phi : |dq^a\rangle \Rightarrow \left|dx^k\right\rangle = \left[\partial\phi^k(q^b)/\partial q^a)\right] \; |dq^a\rangle \tag{4.13}$$

The exact coordinate differentials are said to be **holonomic**, as they have a unique functional pre-image. Note that a necessary condition (but not sufficient) for integrability to exactness is that the exterior derivatives of the coordinate differentials must vanish, $d(dx^k) = 0$. Often the target of the map is presumed to be a (pseudo) euclidean space with a diagonal metric of constants equal to ± 1. The reverse problem of mapping from a (pseudo) euclidean space can also be considered.

For the integrable situations, the Jacobian matrix of functions may be used as a basis frame $\left[F_a^k(q^b)\right]$

$$\left[F_a^k(q^b)\right] = \left[\partial\phi^k(q^b)/\partial q^a)\right] \tag{4.14}$$

on subspaces of $\{q^b\}$ where the determinant of the Jacobian matrix does not vanish. As the Frame matrix (in this case the Jacobian matrix) has an inverse on such domains, the exterior derivatives of the Jacobian matrix lead to the concept of a connection $[C]$ that linearly connects the differentials of the basis functions to linear combinations of the basis functions. This concept of a connection is consequence of the linear independence of the column vectors that define any basis Frame matrix, and is not a postulated definition:

$$d\left[F_a^k(q)\right] = \left[F_c^k(q)\right] \circ \left[\sum_{c=1..n} C_{ab}^c(q)dq^b\right]. \tag{4.15}$$

The matrix elements of the right Cartan connection matrix, $[C_a^c(q)]$, are differential 1-forms, $C_{ab}^c dq^b$.

If the functions of the holonomic mapping are C2 differentiable, then by direct computation,

$$\begin{aligned} d(dx^k) &= \left[F_c^k(q)\right] \circ [C_a^c(q)]\,\hat{}\,|dq^a\rangle \\ &= \left[F_c^k(q)\right] \circ \left[\sum_{c=1..n} C_{ab}^c(q)dq^b\right] \hat{}\,dq^a \\ &= \left[F_c^k(q)\right] \circ \left|\{C_{ab}^c - C_{ba}^c\}dq^b\,\hat{}\,dq^a\right\rangle = 0 \end{aligned} \tag{4.16}$$

The vector of two forms based on the anti-symmetric components of the connection has been defined as the vector of Affine torsion 2-forms:

$$\textbf{Vector of Affine Torsion 2-forms}: \quad \left|C_{[ab]}^c dq^b\,\hat{}\,dq^a\right\rangle \tag{4.17}$$

However, this vector of Affine Torsion 2 forms must vanish as d(dx) = 0.

BOTTOM LINE: I*f the exterior differential system has a preimage as a holonomic mapping, the Vector of Affine Torsion 2-forms must vanish.*

(Hence statements about "holonomic affine torsion" are misleading if not false (see Sarfatti Vigier 2000).)

Application of the exterior derivative one more time yields

$$\begin{aligned} d(d(dx^k)) &= d\{\left[F_c^k(q)\right] \circ [C_a^c(q)\}]\,\hat{}\,|dq^a\rangle \\ &= \left[F_e^k(q)\right] \circ \{[C_c^e(q)]\,\hat{}\,[C_a^c(q)] + [dC_a^c(q)]\}\hat{}\,|dq^a\rangle \\ &= \left[F_e^k(q)\right] \circ [\Theta^e]\,\hat{}\,|dq^a\rangle \Rightarrow 0. \end{aligned} \tag{4.18}$$

The result is that the 3-form $[\Theta^e]\,\hat{}\,|dq^a\rangle$ must vanish for the holonomic case. This necessary condition does not imply that the matrix of curvature 2-forms must vanish, although that is a possibility.

In summary *The holonomic case is Affine torsion free, but may or may not admit non-zero curvature*

4.2. Anholonomic systems - Case 2

Even without knowledge of the unique integral solutions (which may not exist), it often possible to construct (or impose by other arguments) over a domain a Frame Field, $[F]$. The Frame field acts as a basis matrix of linearly independent functions with arguments on the variety, y^a. (Methods for constructing $[F]$ are described in "The Many Faces of Torsion"). The domain may be finite, but not global. Consider the anholonomic (non-holonomic) case where

$$\left[F_a^k(q^b)\right] \circ |dq^a\rangle \Rightarrow \left|\sigma^k\right\rangle . \tag{4.19}$$

Multiplying by the inverse frame yields

$$\left|dq^b\right\rangle = \left[G_k^b(q^b)\right] \circ \left|\sigma^k\right\rangle \tag{4.20}$$

Exterior differentiation of the first equation yields

$$\left[F_b^k(q^b)\right] \circ \left[C_a^b(q^b)\right] \hat{} \, |dq^a\rangle \Rightarrow d\left|\sigma^k\right\rangle \tag{4.21}$$

which implies that the vector of Affine Torsion 2-forms is given by the expression

$$\left[C_a^b\right] \hat{} \, |dq^a\rangle = \left[G_k^b(q^b)\right] \circ d\left|\sigma^k\right\rangle \tag{4.22}$$

The important result is that the vector of Affine Torsion 2-forms will vanish even if the induced 2-forms $\left|\sigma^k\right\rangle$ are closed, not merely exact. Affine Torsion requires that the induced 1-forms are **not closed**.

Exterior differentiation of the second equation yields

$$d\left|dq^b\right\rangle = 0 = \left[dG_k^b(q^b)\right] \circ \left|\sigma^k\right\rangle + \left[G_k^b(q^b)\right] \circ d\left|\sigma^k\right\rangle \tag{4.23}$$

So if the reciprocal frame field is used as a basis frame, there exists a reciprocal right Cartan connection, $\left[D_k^m(q^b)\right]$, such that

$$d\left|dq^b\right\rangle = 0 = \left[G_m^b(q^b)\right] \circ \{\left[D_k^m(q^b)\right] \circ \left|\sigma^k\right\rangle + d\left|\sigma^k\right\rangle\} \tag{4.24}$$

The term in the curly brackets defines the Vector of Cartan Torsion 2-forms based on $\left|\sigma^k\right\rangle$. It is apparent that this Vector of Cartan torsion 2-forms is zero and consists of two compensating parts, one of which is the Vector of Affine connection 2 forms.

There are four cases to consider depending on the properties of the induced 1-forms:

a. If the non-exact 1-forms are closed, such that $d\left|\sigma^k\right\rangle = 0$,then the vector of affine 2-forms, $[C_a^c(q)]\,\hat{}\,|dq^a\rangle$, must vanish. **This anholonomic domain does not support Affine Torsion**. Moreover the 3-form, $[\Theta_a^c]\,\hat{}\,|dq^a\rangle$ must vanish. A sufficient but not necessary condition is that the matrix of curvature 2-forms is zero. In other words such anholonomic domains could have curvature without torsion.

b. If the non-exact 1-forms are not closed then $d\left|\sigma^k\right\rangle \neq 0$. The constraint implies that this type of anholonomic domain necessarily supports Affine Torsion. A second application of the exterior derivative implies that, again, the 3-form $[\Theta_a^c]\,\hat{}\,|dq^a\rangle$ must vanish. Hence, such anholonomic cases necessarily support Affine Torsion, but may or may not have zero curvature.

c. If the $\left|\sigma^k\right\rangle$ are not closed but integrable, then the vector of 3-forms, $\left|\sigma^k\,\hat{}\,d\sigma^k\right\rangle$ must vanish.

d. If induced 1-forms are not integrable. Such domains are said to support topological torsion. The 3-form $\left|\sigma^k\,\hat{}\,d\sigma^k\right\rangle \neq 0$. In simple terms, Affine Torsion is related to the "curl" of the induced coordinate 1-forms, σ^k. If the 1-forms have zero curl, the domain does not support Affine Torsion. If the 1-forms are integrable, then the topological torsion is zero. Non-zero topological torsion implies that the $\left|\sigma^k\right\rangle$ are not integrable.

4.3. Anholonomic systems - Case 3

Consider the anholonomic (non-holonomic) case where

$$\left[F_a^k(q^b)\right] \circ |\omega^a\rangle \Rightarrow \left|dx^k\right\rangle. \tag{4.25}$$

Exterior differentiation yields

$$d\{\left[F_a^k(q^b)\right] \circ |\omega^a\rangle\} = \left[F_a^k(q^b)\right] \circ \{[C_a^c(q)]\,\hat{}\,|\omega^a\rangle + |d\omega^a\rangle\} = d\left|dx^k\right\rangle \Rightarrow 0 \tag{4.26}$$

It must be true that the vector of Cartan torsion 2-forms based ω^a on must vanish

$$\text{Vector of Cartan Torsion 2-forms:} \quad [C_a^c(q)]\,\hat{}\,|\omega^a\rangle + |d\omega^a\rangle = C_{ab}^c dq^b\,\hat{}\,\omega^a \tag{4.27}$$

A second exterior differentiation leads to the condition:

$$\left[F_a^k(q^b)\right] \circ \{[C_a^c(q)] \,\hat{}\, [C_a^c(q)] \,\hat{}\, |\omega^a\rangle + 2\,[C_a^c(q)] \,\hat{}\, |d\omega^a\rangle + [dC_a^c(q)] \,\hat{}\, |\omega^a\rangle\} = 0 \quad (4.28)$$

or

$$\left[F_a^k(q^b)\right] \circ \{[\Theta_a^c(q)] \,\hat{}\, |\omega^a\rangle + 2\,[C_a^c(q)] \,\hat{}\, |d\omega^a\rangle\} = 0 \quad (4.29)$$

The bracket term 3-form has to vanish under the starting hypothesis, which implies that the curvature component $[\Theta_a^c(q)] \,\hat{}\, |\omega^a\rangle$ can be compensated by the Coriolis term $2\,[C_a^c(q)] \,\hat{}\, |d\omega^a\rangle$, if each component is not identically zero. No conditions are place upon the vector of Affine Torsion 2-forms which is not at all equivalent to the vector of Cartan Torsion 2-forms.

Now the 1-forms $|\omega^a\rangle$ may be composed of linear combinations $|\omega^a\rangle = A_b^a dq^b$ such that

$[C_a^c(q)] \,\hat{}\, |\omega^a\rangle = C_{ae}^c dq^e \,\hat{}\, A_b^a dq^b$

As will be seen (below) when the system is not holonomic, and no unique integral equivalent exists, then the spaces involved can have connections which are not free of affine torsion. The bottom line is that there is a correspondence between integrability and no- torsion. This correspondence will be demonstrated below.

4.4. Suggested Reading

There are three books on Pfaffian systems that cover the topics in some detail. The first is by

Schouten, J. A. and Van der Kulk, W., "Pfaff's Problem and its Generalizations", (Oxford Clarendon Press, 1949)

Another is by

M. Zhitomirski, "Typical Singularities of differential 1-forms and Pfaffian Equations" Mathematical Monographs 113, AMS 1992.

Neither of these texts is easy reading.

A third book of interest in "Exterior Differential Systems" by Griffiths, Bryant, Chern et al. (I will update this reference later.)

A paper by P. Fiziev and H. Kleinert gr-qc/9605046 May 1996 does a detailed job explaining "Anholonomic Transformations of Mechanical Action Principle(s)"

A paper by Manual del Leon and David de Diego "On the geometry of non-holonomic Lagrangian systems" J Math Phys. **37** (7) 1996, p.3389 gives some examples, and a set of references, but is mostly restricted to what are called semi-holonomic constraints below. Also see "Solving non-Holonomic Lagrangian Dynamics...." Extract Mathematica **11**, 2 1996 p.325, by the same authors

Also see David C. Robinson and W. F. Shadwick "The Griffiths-Bryant algorithm and the Dirac theory of Constraints" Fields Institute Communications **7** 1996, p189

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5. Basic Ideas

First a few definitions.

> Definition: An exterior differential form, ϖ, is a function of independent variables (functions) and the differentials of the independent variables. (There are also a few restrictions of anti-symmetry and homogeneity in the differentials that lead to algebraic closure. These conditions need not be discussed now. See Flanders "Exterior Differential Forms")

$$A\ differential\ form: \qquad \varpi = f(x^k, dx^k) \tag{5.1}$$

Example: The classic Cartan Hilbert Action is a (linear) exterior differential form

$$\varpi = f(x^k, dx^k) \Rightarrow \omega = p_k dq^k - H(p_k, q^k, t)dt. \tag{5.2}$$

> Definition: A constraint is an exterior differential form set equal to zero.

$$\varpi = f(x^k, dx^k) \Rightarrow 0. \tag{5.3}$$

> Definition: The zero set of a differential form (which is homogeneous in the differentials) defines a constraint called a Pfaffian equation. (The differential form, especially when linear in the differentials, often is referred to as the "Pfaffian").

$$\varpi \Rightarrow \omega = f(x^k, dx^k) = \sum_k A_k(x^k)dx^k \Rightarrow 0. \quad Pfaffian\ Equation \tag{5.4}$$

Classic examples of Pfaffian equations of constraint are given by the kinematic and dynamic expressions subsumed as axioms of freshman mechanics:

$$dx^k - V^k dt = 0, \quad dV^k - A^k dt = 0, \quad dp_k - F_k dt = 0. \tag{5.5}$$

> Definition: A constraint that does not depend upon differentials is said to be a holonomic constraint. A constraint that involves the differentials is said to be an anholonomic or a non-holonomic constraint. If only one differential is involved, then the constraint is called a semi-holonomic constraint.

Example 1 A Holonomic constraint: $f(x^k) - c \Rightarrow x^2 + y^2 + z^2 - 1 = 0$. This constraint, as a function of coordinate (independent) variables, if applied to a mechanical system, implies the motion must be confined to a sphere of unit radius. Holonomic constraints define hypersurfaces which are oriented, global, and often of more than one component. A Mobius band cannot be represented by a holonomic constraint (submersion) on a three dimensional space (It can be represented by a two dimensional immersion into a three dimensional space.)

Example 2 An Anholonomic constraint: $g(x, y, z, dx, dy, dz) = 0$ is a non-holonomic or anholonomic constraint. Such an equation, if linear in the differentials, is defined as a Pfaffian equation. This definition is a bit more general than that definition used in many control theory texts. (see the definition for semi-holonomic constraints given below where the differentials are replaced by velocity functions). The constraint of rolling without slipping is a anholonomic constraint.

$$\lambda d\theta - dx = 0 \tag{5.6}$$

Often it is not appreciated that the kinematic definitions given above are indeed anholonomic constraints on the domain.

Example 3 A Semi Holonomic constraint: $g(x, y, z; V^x, V^y, V^z) = 0$. It is usually assumed that V^x, V^y, V^z are defined as differentials of x,y,z with respect to time. Note that this assumption implies that the system admits the additional anholonomic kinematic constraints mentioned above. Under the assumption that $d\theta - \Omega dt = 0$ and $dx - V dt = 0$, the anholonomic constraint of rolling without slipping becomes a semi-holonomic constraint.

$$(\lambda\Omega - V)dt = 0 \supset (\lambda\Omega - V) = 0. \tag{5.7}$$

Particle motion subsumes the kinematic constraints. Wave (or fluid motion) does **not**. Particles and waves are equivalent on domains where the kinematic equations are valid for the velocity field. These domains form a submanifold of the space of coordinates and velocity functions. The space of "waves" is much larger than the space of "particles". (It turns out that when the system is integrable, the two developments are alias-alibi related, and then can be said to be equivalent. A Hamiltonian formulation and a Lagrange formulation are strictly equivalent only when they are integrable. A Lagrange formulation with anholonomic constraints is equivalent to a Hamiltonian formulation if they are of the same class or Pfaff dimension. (see below) As will be shown below, integrability implies that either the differential 1-form can be represented in terms of at most two independent functions of the original independent variables: $\omega = \varphi d\psi$, or $\omega = d\psi$. The associated Pfaffian equations of constraint are the same for either representation. However, the exterior derivative of the first possibility is not zero, where the exterior derivative of the second possibility is zero.

Every differentiable holonomic constraint, $\phi(x^j) - c = 0$, leads (by differentiation) to a non-holonomic Pfaffian equation:

$$d\phi(x^j) = \sum_k (\partial\phi/\partial x^k) dx^k \Rightarrow \sum_k A_k(x^j) dx^k = 0 \tag{5.8}$$

but not conversely. Differential forms, ω, not equal to zero define a module of vector fields such that $i(\mathbf{V})\omega = 1$. Pfaffian equations (differential forms constrained to zero) define a module of vector fields such that $i(\mathbf{V})\omega = 0$

> Definition: A differential 1-form, $\omega = (\sum_k A_k(x^j)dx^k)$ is said to be integrable on domains where either $\sum_k A_k(x^j)dx^k = d\phi(x^j)$ or $\Theta(x^j)(\sum_k A_k(x^j)dx^k) = d\phi(x^j)$. The function $\Theta(x^j)$ is defined as an integrating factor.

A Pfaffian equation of constraint constructed from an integrable 1-form has a unique direction field, in the sense that the coefficient functions, $A_k(x^j)$, are proportional to the gradient of a single function, $\phi(x^j)$. The hypersurface, $\phi(x^j) = 0$, defines the subspace (N-1 dimensional domain) of points which are isolated from the larger (N dimensional) domain.

Theorem: (Frobenius) An integrable differential 1- form $\omega = A_k(x^j)dx^k$ satisfies the exterior differential equation (an equation of constraint!)

$$\omega\hat{}d\omega = 0. \tag{5.9}$$

Definition: The 3 -form $\omega\hat{}d\omega$ is defined as the "topological torsion". Integrable 1-forms have zero topological torsion.

The kinematic constraints mentioned above satisfy the Frobenius theorem. As the differentials dx are presumed to be exact, their exterior derivatives vanish. Hence the exterior derivative of the differential form $\omega = dx - Vdt$, whose zero set represents the Pfaffian constraint, has only one component equal to the 2-form, $d\omega = dV^x\hat{}dt$. If the Pfaffian is integrable, it must be true that $d\omega = 0$. Hence this single 2-form must vanish, which implies that $V^x = V^x(t)$ a function of t alone. That is, the kinematic assumption implies that the velocities admit a representation in terms of a single parameter (usually called time) Consider a more general case, where there exists a differentiable map of more than 1-parameter (string or m-branes) from $\{t, y^m\} \Rightarrow x^k = f^k(t, y^m)$. Then

$$\omega = dx^k - \{\partial f^k(t, y^m)/\partial t\}\, dt = \sum\{\partial f^k(t, y^m)/\partial y^j\}\, dy^j \tag{5.10}$$

This equation has the appearance of the kinematic formula, if the function $\{\partial f^k(t, y^m)/\partial t\}$ is identified with $V^k(t, y^m)$, and if the RHS is zero: $\sum\{\partial f^k(t, y^m)/\partial y^j\}\, dy^j = 0$. If the matrix $[F(t, y^m)] = \{\partial f^k(t, y^m)/\partial y^j\}$ is of maximal rank (det $[F] \neq 0$) then the only possibility is that the y^m are fixed constants in the sense that $dy^m = 0$. If the y^m are not constants (constant initial conditions) then the kinematic equations will have fluctuations where the RHS is not zero.

Definition: A Pfaff sequence of a differential 1-form A consists of a finite number n of ordered non-zero elements, constructed as {A, dA, A^dA, ...}. The class, or Pfaff dimension, of the 1-form A at a point $\{x^k\}$ is equal to n, the number of non-zero elements of the sequence.

Example: The differential form $A = y^1dy^2 + dy^3$ defined on the 4 dimensional domain, $\{y^1, y^2, y^3, y^4\}$ generates the Pfaff sequence,

$$\{A = y^1dy^2 + dy^3, dA = dy^1\hat{}dy^2, A\hat{}dA = dy^1\hat{}dy^2\hat{}dy^3, dA\hat{}dA = 0\}. \tag{5.11}$$

The Pfaff dimension of the example is 3. The first zero element in the sequence is the 4 form $dA\hat{}dA$. The class or Pfaff dimension of an integrable form is 2 or less. The proof follows from the Frobenius theorem. The Pfaff dimension in effect determines the minimum number of independent functions which are required to describe the 1-form. Integrable 1-forms can be written in the format $A = \varphi d\psi$ where $\varphi(x^k)$ and $\psi(x^k)$ are independent functions of the independent variables.

Definition: An Anholonomic Kinematic Fluctuation is defined as

$$dx^k - V^k\,dt = \Delta x^k \neq 0. \quad dV^k - A^k\,dt = \Delta V^k \neq 0. \tag{5.12}$$

The Cartan-Hilbert invariant integral has an Action integrand, A, which is based upon Anholonomic Kinematic Fluctuations. The "momenta" p_k, play the role of Lagrange multipliers.

$$\begin{aligned} Action &= L(x,V,t)dt + p_k\Delta x^k = L(x,V,t)dt + p_k(dx^k - V^k\,dt) &(5.13)\\ &= p_k dx^k - (p_k V^k - L)dt = p_k dx^k - H(p_k,V^k,x^k,t)dt &(5.14)\end{aligned}$$

Note that the "Hamiltonian" function $H(p_k, V^k, x^k, t)$ depends, in general, on the velocity functions, V^k, as well as the Lagrange multipliers, p_k, and the coordinates and time, x^k, t. The Maximum Pfaff dimension of the Action is 2n+2 although though there are what appear to be 3n+1 independent functions used in its construction.

The exterior derivative of the Cartan Hilbert Action is given by the expression

$$dA = (p_k - \partial L/\partial V^k)(\Delta V^k)\hat{}dt + (dp_k - \partial L/\partial x^k dt)\hat{}(\Delta x^k) \tag{5.15}$$

which demonstrates the influence of the fluctuations (in both velocity and position) on the 2-form, dA. It is apparent that the anholonomic fluctuations in velocity, (ΔV^k), are unimportant if the momenta are presumed to be canonical, $(p_k - \partial L/\partial V^k) \Rightarrow 0$. Intuitively, fluctuations in Velocities are attributed to temperature, where fluctuations in position are associated with pressure.

Definition: Topological torsion of a 1-form of Action, A,(on a 4 dimensional domain) is defined as the 3 form $A\hat{}dA$

A differential 1-form with non-zero topological torsion is of Pfaff dimension 3 or more. The 1-form can be expressed in terms of not less than 3 independent functions. Therefore, according to the Frobenius theorem, the1-form A which supports a non-zero Topological Torsion 3 form is not uniquely integrable.

6. Subtleties

6.1. Exactness and Closure

A constrained differential 1-form, A, usually does not have a unique primitive function, ϕ, whose total differential generates a Pfaffian equation to within a factor, Θ. A necessary but not sufficient condition for the existence of the unique global holonomic function is that the exterior derivative of A, or at least $\{\Theta A\}$, must vanish. For $\Theta = 1$, the covariant vector field of components, A_k, must have zero "curl". If A represents a vector potential, then such potentials produce no field intensities, B. In a fluid, one would say that there is no vorticity. (The integrating factor sometimes can be considered as a conformal factor)

When the exterior differential of a form vanishes, $d\omega = 0$, the form is said to be closed, and the union of the form and its closure forms a differential ideal. If the differential form is exact then there is a unique pre-image (in the example, a function ϕ) such that $\omega = d\phi$. What is the difference between and exact form and a closed form? The answer is that on a simply connected domain, there is no difference. However, if the domain is not simply connected (think of holes in a piece of paper) then there is a difference. For each hole there is a closed *but not exact component* to the differential form. As an example consider the 1-form which as it stands is neither exact nor closed.

$$\sigma = (ydx - xdy) \;\; with \;\; d\sigma = 2dy\hat{}dx \tag{6.1}$$

Multiply the form σ by the closure factor $1/(\pm x^2 \pm y^2)$.

$$\gamma = \sigma/(\pm x^2 \pm y^2) = (ydx - xdy)/(\pm x^2 \pm y^2) \;\; with \;\; d\gamma = 0 \tag{6.2}$$

The resulting 1-form is now closed but not exact. The domain of support must exclude a small set where the denominator goes to zero. If the signs are the same, the excluded set is a point at the origin. The original euclidean plane now has a hole, and is no longer simply connected. The form γ is called a harmonic form (in the sense of deRham).

What is remarkable is that the integral of a closed and exact form on a cycle, $z1$, is zero, but the integral of a closed but not exact form on a cycle is an integer multiple of some constant.

$$\int_{z1} d\phi = 0 \quad but \quad \int_{z1} \gamma = n\,2\pi \tag{6.3}$$

Gauge conditions as exact differential additions to a 1-form, are trivial. Gauge conditions of the closed but not exact type are NOT trivial. They contain topological information (such as the hole count in a non-simply connected domain. (Bohm-Aharanov, Joukowski airfoil, Meissner expulsion, Sommerfeld quantum conditions, etc.)

These same concepts work for differential forms that are not linear in the differentials. Hence the postulate of electromagnetism that $F-dA=0$ is a strong topological anholonomic constraint, that says over the domain of support, the 2-form of F (E and B) is exact; that is, the 2-form does not have harmonic parts (although the 1-form, A, can have harmonic parts which are the "flux quanta"). The second postulate of Maxwell electrodynamics is the statement $J-dG=0$. The idea again is that the 3-form J is exact without harmonic parts. The 2-form G can have harmonic parts, which serve as the charge quanta.

Now consider the topological torsion for the 1-form A which is defined as the 3-form $H = A\hat{}F = A\hat{}dA$. If $dH = 0$, then the question arises: Does H have harmonic parts? If the answer is yes then the harmonic parts serve as "topological" torsion quanta. A necessary condition for existence of such quanta is that the second Poincare invariant must be zero.

Similarly for the 3-form of topological spin $S = A\hat{}G$. The necessary condition for existence of EM spin quanta is that the $dS = 0$, or in other words that the First Poincare invariant must vanish. These points are exemplified at

http://www22.pair.com/csdc/car/carhomep.htm

For an interesting solution to the Maxwell postulates, further constrained by the Lorentz vacuum conditions. See

http://www22.pair.com/csdc/maple/reed21.html

Here, the Torsion field is not closed, but the Spin field is closed. In fact the Spin field in the example is the torsion field multiplied by an integrating factor. One would be led to say torsion is source of spin. However, the solution is a special case and the conclusion is not general, for the next example demonstrates that you can have finite topological torsion with zero topological spin. See

http://www22.pair.com/csdc/maple/reed31.html

6.2. Equations of Motion

Given an arbitrary 1-form of Action, which is not closed, Cartan has shown that the equations of motion generating a vector field V are of a Hamiltonian form if the Lie derivative of the Action, A, with respect to the vector field V is exact.

Theorem: Solutions V to the equation $L_{(V)}A = d(\Theta)$ are Hamiltonian vector fields.

This theorem is equivalent to the statement that the closed integrals of the harmonic components of A are constants of the motion. The number of holes does not change. The 1-dimensional topological property expressed as the harmonic 1-form is an invariant of processes that are generated from a Hamiltonian function.

Writing out the theorem shows that it is a statement in the form of an anholonomic differential constraint

$$i(V)dA - d(\Theta - i(V)A) = 0 \tag{6.4}$$

Use the symbols $W = i(V)dA$ defined as the work 1-form, and $U = i(V)A$ defined as the "internal energy". The work 1-form W is closed and exact. Hamiltonian systems are systems where the Pfaff dimension of the Work 1-form is 1.

6.3. Equations of motion for Non-Hamiltonian dynamics

It is apparent that to find equations of motion for non-Hamiltonian systems, the fundamental anholonomic constraint

$$i(V)dA = d(\Theta - i(V)A) \tag{6.5}$$

must be modified to include harmonic parts, and non-closed parts.

$$W = i(V)dA = d(\Theta - i(V)A) + \gamma + Z \tag{6.6}$$

$$dW = dZ \neq 0 \tag{6.7}$$

The last equation destroys the Helmholtz theorem, and the Poincare even dimensional integrals are no longer evolutionary invariants. An example of such a non-Hamiltonian mechanics was suggested in 1974. See

http://www22.pair.com/csdc/pd2/pd2fre5.htm

The formula is the anholonomic constraint

$$W - \Gamma A = i(V)dA - \Gamma A = 0. \tag{6.8}$$

It is know that this equation requires that the Pfaff dimension of the Action 1-form be even (2n+2). Hence the Pfaff space supports the topological torsion 3-form. Moreover, a unique solution vector V does exist for this problem. In a space of 4 variables this vector is equivalent to the Torsion current (with components proportional to those of the 3-form of non-zero topological torsion, $A\hat{}dA$). Evolution in the direction of the Torsion current is thermodynamically irreversible, as the heat 1-form, Q, does not satisfy the Frobenius integrability theorem, and therefor does not admit an integrating factor.

6.4. Extremals and the Calculus of variations.

For integrals of the Action around closed loops, the values at the "endpoints" cancel out. Similar constraints are often placed on open integrals, forcing the cancellation of contributions at boundary points. The solutions of the problem are then given by vector fields that generate paths such that

$$L_{(V)}\int_{z1} A = \int_{z1} i(V)dA + \int_{z1} di(V)A \Rightarrow \int_{z1} i(V)dA = 0 \tag{6.9}$$

It is apparent that the if equation is satisfied (giving the equation for an extremal as the "Lie derivative" of the integral must vanish) then the work 1-form must vanish. The bottom line is that Extremals are associated with anholonomic constraints, $W = f_k dx^k - Pdt = 0$. Extremal solutions say that there are vector fields such that $f_k V^k - P = 0$. This equation is the freshman definition of power as the product of force times velocity.

What is even more remarkable is that this equation, $W = 0$, has solutions only in spaces (as defined by the Pfaff sequence for the Action 1-form) of odd Pfaff dimension, 2n+1. (e.g. State Space).

> **Theorem:** Unique extremals, defined as solutions to the equation i(V)dA=0 for a given A, do not exist on domains of Pfaff dimension 2n+2

The theorem is easy to prove, for if the Pfaff space is a symplectic manifold of even dimension then the 2-form dA has an anti-symmetric matrix representation with no zero eigenvalues. On the other hand if the Pfaff space is an odd dimensional contact manifold, then the anti-symmetric matrix representation of dA has a unique eigen vector with eigen value zero. Hence on 2n+1 Pfaff space, the extremal exists and is unique.

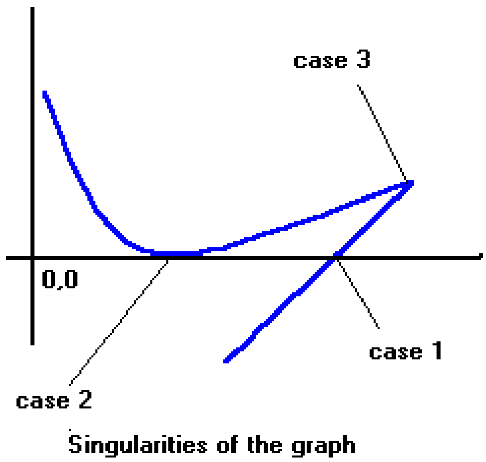

Figure 7.1:

7. DEFECTS

The primitive defect structures are of several types for any object that can be represented by an exterior differential form, be it a 0 form (function) or a n form (density) or a 1-form, or in general a p form. For the linear mapping the most interesting situation is to find the singular points of the determinat of $[M]$. There are several cases to consider.

1. The object itself is zero (as in this case where the determinant function is zero).
2. Both the object and its (exterior) differential vanish. This case is called the characteristic set.
3. The object is not zero, but its exterior differential vanishes. This case is often called the extremal case.
4. Neither the object nor its differential vanish. (The primitive non-singular case)

For a simple graph, the following figure depicts some possible singularities.

The constrained linear mapping, $[F] = [M]_{restricted\, to\, q}$, can be used as a Basis Frame over the reduced domain of definition, q. Note that in the real case there are, in general, two distinct linear mappings; those with determinant positive and those with determinant negative. Special cases of interest are those where $[F]$ contains the identity, where $[F]^2$ contains the identity, and where $[F]^4$ contains the identity. If the determinant of the Basis Frame (which is similarity invariant) is a polynomial function of degree n which can be factored, then each factor, of degree (say) m, when set equal to zero implicitly defines a distinct topological component . (The degree n is usually not the same as the dimension N of q.) The components can behave in a sense as "coherent structures or particles". It is interesting to note that if the factors of the determinant are complex curves (a function of two complex numbers of degree m set equal to zero: $f(z1(x), z2(x)) = 0$), Arnold says that such a factor (component) has the properties of a real surface with (m-1)(m-2)/2 handles and m holes. (I do not know where this theorem is proved.) Complex curves can be arranged to satisfy anti-symmetry under "particle" interchange.

8. CLOSURE 2-forms (a component of Cartan Torsion 2-forms)

8.0.1. The curl format (apparently is favored by Kleinert.)

Consider the vector array of 1-forms: $\left|\sigma^k\right\rangle = \left[F_a^k(q^b)\right] \circ |dq^a\rangle$ and the vector of " closure" 2-forms:

$$\left|d\sigma^k\right\rangle = \left[dF_a^k(q^b)\right] \,\hat{}\, |dq^a\rangle \Rightarrow \left|\{\partial F_a^k/\partial q^b - \partial F_b^k/\partial q^a\}dq^b\,\hat{}\,dq^a\right\rangle \tag{8.1}$$

8.0.2. The Object of Anholonomicity format (apparently is favored by Corum.)

Recall the Cartan-Darboux idea that if the 1-forms σ^k are complete, such that the product, $\sigma^1\,\hat{}\,\sigma^2\,\hat{}\ldots\sigma^N \neq 0$, then every 2-form $d\sigma^k$ can be expanded as: $d\sigma^k = \Omega_{mn}^k \sigma^m\,\hat{}\,\sigma^n$. The vector of " closure" 2-forms becomes:

$$\left|d\sigma^k\right\rangle = \left|\Omega_{mn}^k \sigma^m\,\hat{}\,\sigma^n\right\rangle. \tag{8.2}$$

The three index symbols Ω_{mn}^k form the components of the Object of Anholonomicity.

8.0.3. The Affine Torsion format (apparently is favored by Cartan.)

It the basis frame is complete, then there exists a right Cartan matrix of connection 1-forms $\left[dF_a^k(q^b)\right] = \left[F_c^k(q^b)\right] \circ \left[C_{ab}^c dq^b\right]$ such that the vector of " closure" 2-forms becomes:

$$\left|d\sigma^k\right\rangle = \left[F_c^k(q^b)\right] \circ \left|C_{[ab]}^c dq^b \hat{} dq^a\right\rangle \tag{8.3}$$

The three index symbols $C_{[ab]}^c$ are the coefficients of the Affine torsion object.

8.0.4. Summary

Hence there are three equivalent formulations for the vector of closure 2-forms:

$$\left|d\sigma^k\right\rangle = \left|\Omega_{mn}^k \sigma^m \hat{}\sigma^n\right\rangle = \left|\{\partial F_a^k/\partial q^b - \partial F_b^k/\partial q^a\}dq^b \hat{} dq^a\right\rangle = \left[F_c^k(q^b)\right] \circ \left|C_{[ab]}^c dq^b \hat{} dq^a\right\rangle \tag{8.4}$$

The Affine torsion 3 index symbols are not the same as the Object of Anholonomicity, but all three formulations express the **same** vector of closure 2-forms $\left|d\sigma^k\right\rangle$

8.1. to be completed later

1. Discuss the torsion induced by A as compared to the torsion induced by W.

2. Discuss the affine torsion associated with an integrable but not exact coordinate system and how the idea intertwines with conformal maps and dilatations and chirality.

3. Compare Cartan Torsion 2-forms, affine torsion, topological torsion.

4. The retrodictive point of view would consider the pullback process, where given a Frame matrix on q, and a vector array of p forms on the final state x, use the pullback mechanisim to induce p forms on q. The matricies would be the transpose of F and the adjoint of F, both of which exist for both det non zero and det zero domains.