Optical Vortices, Faraday Rotation, Optical Activity

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Abstract

Characteristic solutions to Maxwell's equations, constrained to equivalence classes by constitutive equations that have certain crystal symmetries, are represented by phase 1-forms, k, such that both 3-forms vanish: $k\hat{}F = 0$ and $k\hat{}G = 0$. Simple examples are given which show the differences between Faraday rotation where the 3-form of Topological Torsion is not zero, $A\hat{}F \neq 0$, and Optical Activity where the 3-form of Topological Spin is not zero, $A\hat{}G \neq 0$. The 1-form, k, whose line integral generates the phase function Θ, is not necessarily equal to the system 1-form of Action per unit charge, A, whose components form the vector and scalar potentials. The characteristic solutions demonstrate that linearly polarized solutions are to be associated with topological spin, and that circularly polarized solutions are to be associated topological torsion. This result is counter to the popular view that "spin" is related to circular polarization. When the 3-forms of $A\hat{}F$ and $A\hat{}G$, are closed in a exterior differential sense, then their integrals over closed domains form deformable invariants with values whose ratios are rational (quantized). There are two types of optical phase defects. The first type of defect is related to Faraday rotation, circular polarization, and topological torsion, $A\hat{}F \neq 0$. The second type of defect is related to Optical Activity, linear polarization, and topological spin, $A\hat{}G \neq 0$. It would appear appropriate to describe the first type of defect as Optical Vortices with defects of rotational shears. The second

type of phase defect is related to dislocations which involve translational shears.

1. Introduction

1.0.1. Transverse Inbound and Outbound Waves

First consider a complex four vector potential solution to the vector wave equation which propagates as a transverse wave in the $\pm z$ direction with a phase $\theta = kz \mp \omega t$. There are 4 possibilities: The $\mathbf{E}$ field rotates about the z axis in a Right Handed manner as viewed by an observer looking towards the positive z direction, or it rotates in a Left Handed manner. Outbound $\theta = kz - \omega t$ and Inbound $\theta = kz + \omega t$ waves are to be distinguished.

$$ORH = \left| \begin{matrix} 1 \\ i \end{matrix} \right\rangle e^{i(kz-\omega t)} \qquad IRH = \left| \begin{matrix} 1 \\ i \end{matrix} \right\rangle e^{i(kz+\omega t)} \qquad ORH = \left| \begin{matrix} 1 \\ -i \end{matrix} \right\rangle e^{i(kz-\omega t)} \quad I\,LH = \left| \begin{matrix} 1 \\ -i \end{matrix} \right\rangle \tag{1.1}$$

For media with the symmetries of the Lorentz vacuum, the phase velocities $v = \omega/k$ are the same for all four modes. Addition or subtraction of ORH and OLH produces a Linearly polarized state outbound. Addition or subtraction of IRH and ILH produces a Linearly polarized state inbound.

Next, recall the experimental differences between Optical Activity and Faraday Rotation:

1.0.2. Optical Activity

Consider an optically active fluid (sugar in water) in a cylindrical tube of length L. For Optical Activity, there are also two distinct phase velocities, ω/k_1 and ω/k_2. Outbound Right Handed (ORH) circularly polarized light propagates with a phase speed equal to the phase speed of Inbound Left Handed (ILH) circularly polarized light. Outbound Left Handed (OLH) polarized light propagates with a phase velocity different from the phase velocity of Outbound Right Handed polarized light (ORH), but with the same speed as that of Inbound Right Handed (IRH) polarized light. In summary,

$$\text{Optical Activity Phase Velocity,} \quad V_{ORH} = V_{ILH} \neq V_{OLH} = V_{IRH} \tag{1.2}$$

The wave solutions for optical activity are of the format:

$$ORH = \left| \begin{matrix} 1 \\ i \end{matrix} \right\rangle exp\, i(k_1 z - \omega t) \quad IRH = \left| \begin{matrix} 1 \\ i \end{matrix} \right\rangle exp\, i(k_2 z + \omega t) \tag{1.3}$$

$$OLH = \left| \begin{matrix} 1 \\ -i \end{matrix} \right\rangle exp\, i(k_2 z - \omega t) \quad ILH = \left| \begin{matrix} 1 \\ -i \end{matrix} \right\rangle exp\, i(k_1 z + \omega t) \tag{1.4}$$

The addition of $|ORH\rangle + |OLH\rangle$ produces a Linearly Polarized state propagating outbound, whose plane of polarization rotates. When the two inbound states are added, $|IRH\rangle + |ILH\rangle$, a linearly polarized state is achieved, and its plane of polarization also rotates *in the same direction as the outbound rotation.* In other words, the round trip (outbound+reflection+inbound) motion does not cause the plane of polarization to return to its initial value. This result defines what is meant by a reciprocal effect. If the plane of polarization of the original linearly polarized light beam suffers a rotation in the amount of θ degrees as it traverses the Optically Active media, when reflected in a mirror, the plane of polarization suffers an negative rotation of θ degrees, as the light beam traverses the media in the reverse direction. The plane of polarization returns to its original state after the round trip. (The sense of Right Handed and Left Handed polarization is determined by an observer looking away from himself.)

1.0.3. Faraday Rotation

Consider a gas of He-Ne in a cylindrical tube of length L. Surround the tube with a coil of wire that will produce a coaxial magnetic field that partially aligns the spins of the gas atoms. For Faraday media, there are two distinct phase velocities, ω/k_1 and ω/k_2. Outbound Right Handed (ORH) circularly polarized light propagates with a phase speed equal to the phase speed of Inbound Right Handed (IRH) circularly polarized light. Outbound Left Handed (OLH) polarized light propagates with a phase velocity different from the phase velocity of Outbound Right Handed polarized light (ORH), but with the same speed as that of Inbound Left Handed (ILH) polarized light. In summary,

$$\text{Faraday Effect Phase Velocity,} \quad V_{ORH} = V_{IRH} \neq V_{OLH} = V_{ILH} \tag{1.5}$$

The wave solutions for the Faraday effect are of the format:

$$|ORH\rangle = \left| \begin{array}{c} 1 \\ i \end{array} \right\rangle exp\, i(k_1 z - \omega t) \qquad |IRHO\rangle = \left| \begin{array}{c} 1 \\ i \end{array} \right\rangle exp\, i(k_1 z + \omega t) \quad (1.6)$$

$$|OLHO\rangle = \left| \begin{array}{c} 1 \\ -i \end{array} \right\rangle exp\, i(k_2 z - \omega t) \qquad |ILH\rangle = \left| \begin{array}{c} 1 \\ -i \end{array} \right\rangle exp\, i(k_2 z + \omega t) \quad (1.7)$$

The formulas represent circularly polarized waves. The addition of $|RHO\rangle + |LHO\rangle$ produces a Linearly Polarized state propagating outbound, whose plane of polarization rotates. When the two inbound states are added, $|RHI\rangle + |LHI\rangle$, a linearly polarized state is achieved, and its plane of polarization also rotates *in the same direction as the outbound rotation.* In other words, the round trip (outbound+reflection+inbound) motion does not cause the plane of polarization to return to its initial value. This result defines what is meant by a non-reciprocal effect. If the plane of polarization of the original linearly polarized light beam suffers a rotation in the amount of θ degrees as it traverses the Faraday media, when reflected in a mirror, the plane of polarization suffers an additional rotation of θ degrees, as the light beam traverses the media in the reverse direction. The plane of polarization does not return to its original state, but instead ratchets by 2θ degrees upon completing the round trip. (The sense of Right Handed and Left Handed polarization is determined by an observer looking away from himself.)

1.0.4. Polar and Axial vectors

Following Schouten [2], Post points out that Faraday Rotation is "generated" by a "W vector", while Optical Activity is generated by a "vector". Under certain constraints, the W vector plays the role of an "Axial" vector, while the "vector" becomes a "polar" vector. Upon reflection, a polar vector changes its sense (determined by the arrow head). Point your finger into a mirror. The image points back at you. The sense of the image is opposite to the sense of the object. For polar vectors with a line of action parallel to the mirror surface, the opposite result is obtained. The sense of the image is the same as the sense of the object. Note the differences of orthogonal and parallel reflections.

A reflected axial vector does not change its sense if the line of action is orthogonal to the mirror. Curl you fingers and align your thumb in a direction orthogonal to the mirror. It does not matter whether the thumb points into or away from the mirror. The sense of the "axial vector" is determined by the curl of the fingers. The sense of the reflected image is the same as the sense of the object. The opposite effect occurs when the line of action of the axial vector is parallel to the reflection surface. The sense, as determined by the curl of the fingers, is opposite to that of the reflected image.

The magnetic field **B** and the angular velocity are examples of spatial "W vectors". On the other hand, the **D** field is a spatial "polar vector" in the sense used by Post. The anti-symmetric spatial components of the covariant field intensity tensor 2-form, $F = dA$, are formed by the spatial "W vector" field **B**. The anti-symmetric spatial components of the tensor density, N-2 form, G, where $J = dG$, are formed by the spatial "polar vector" field **D**. These facts yield a clue for distinguishing Faraday Rotation and Optical Activity on topological grounds. As will be shown below, Faraday Rotation is to be associated with the concept of Topological Torsion, and Optical Activity is to be associated with the concept of Topological Spin.

1.1. Topological Formulation of Maxwell's Equations.

1.1.1. Exterior Differential Systems

It is known that Maxwell's system of PDE's (without constitutive constraints) can be expressed as an exterior differential system [3] on a variety of independent variables. Exterior differential systems impose topological constraints on a differential variety. For the Maxwell electromagnetic system on a domain {x,y,z,t} the two topological constraints have been called the Postulate of Potentials, and the Postulate of Conserved Currents. [4]. These two topological constraints lead to the system of Partial Differential Equations, known as Maxwell's equations, for any coordinate system so constrained. No metric, no connection, nor other restraints of a geometrical nature are required on the 4 dimensional differential variety of independent variables, typically written as $\{x, y, z, t\}$.

$$\text{Postulate of Potentials (an exact 2-form)} \qquad F - dA = 0 \tag{1.8}$$

$$\text{Postulate of Conserved Currents (an exact 3-form)} \qquad J - dG = 0 \tag{1.9}$$

The method of exterior differential systems insures that the description is not only diffeomorphically invariant in form (natural covariance of form with respect to all invertible smooth coordinate transformations), but also the description is functionally well defined with respect to maps which are C2 continuous, but not necessarily invertible. This statement implies that those exterior differential forms which are defined on a final state variety can be "pulled back" in a functionally well defined manner to an initial state variety, even though the map from initial to final state of coordinate variables is NOT a diffeomorphic coordinate transformation. The inverse mapping need not exist. This result is truly a remarkable property of Maxwell electrodynamics, for it permits the analysis of certain irreversible electrodynamic processes without the use of statistics. The "push forward" process is not functionally well defined when the inverse map does not exist, a fact that demonstrates that topological evolution induces an "arrow of time" [5].

1.1.2. Constitutive Constraints

In practical applications, it is possible to impose constraints on the Maxwell system in the form of constitutive relations between the thermodynamically conjugate variables of field intensity $(\mathbf{E}, \mathbf{B})$ and field excitations $(\mathbf{D}, \mathbf{H})$. Post has demonstrated that the constitutive tensor (density) has many of the properties of the Riemann tensor [6]. These constraints are NOT necessarily equivalent to the Riemann tensor generated by a Riemannian metric imposed upon the variety $\{x, y, z, t\}$. In many circumstances the equivalence classes of such constitutive constraints can be put into correspondence with the geometrical symmetries of the 32 crystal classes that are used to discriminate between the many different observed physical structures. As mentioned above, a *complex* 6x6 constitutive constraint has been used by Post to delineate between Optical Activity, Faraday Phenomena, Birefringence and Fresnel-Fizeau motion induced effects in electromagnetic signal propagation. The complex constitutive tensor cannot be deduced from a real metric tensor. However it would appear that it has a constructive definition in terms of a non-symmetric connection.

Indeed, the work of Post using a complex constitutive tensor has been extended [7] to demonstrate the existence of irreducible "quaternion" solutions to the Maxwell system. Quaternion waves cannot be represented by complex functions, which are the usual choice for describing electromagnetic signals. Complex wave solutions generate a 4th order characteristic polynomial for the phase speed which is doubly degenerate. The wave speeds have only two distinct magnitudes depending upon direction and polarization. For cases where a center of symmetry is not available, and yet the medium supports both Optical Activity and Faraday rotation, the wave solutions can NOT be expressed as complex functions, but can be written as quaternions. The resulting 4th order characteristic polynomial for the wave speeds is not degenerate, and has four distinct root magnitudes. The results indicate that the phase propagation speed of light is different for each direction of propagation and for each mode of polarization. The theory has been used to explain the experimental results measured in dual polarized ring laser apparatus.

In contrast, in a medium with the Lorentz symmetries, the characteristic polynomial is 4-fold degenerate; e.g., all polarizations and all directions have the same propagation speed. The result leads to the ubiquitous statement that the speed of light is the same for all observers, which is incorrect for media that do not have the Lorentz symmetries. For Birefringent, or Optically Active, or Faraday media, the characteristic polynomial for phase velocity is doubly degenerate, implying a relationship exists between for the 4 modes of propagation. There exist only two distinct phase velocity magnitudes. The correlation speeds for direction and polarization pairs have been presented above. Faraday rotation and Optical Activity have different propagation direction-polarization handedness correlations. The Faraday rotation is not reciprocal; the rotation induced by Optical Activity is reciprocal.

1.2. TO be completed

1.3. Optical Activity is a D effect (translational accelerations)

1.4. Faraday effect is a B effect (rotational accelerations and vorticity)