Cartan torsion 2-forms vs. Affine (closure) torsion

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last update 01/27/2000

(Email response to Bill Page)

Hi
I am very much interested in the email correspondence that I have received from you over the past 2 or 3 months, and had planned to file a detailed response. This is only a portion.

A preliminary set of comments is worthwhile:

I do not agree verbatim with all of that which is written in Shipov's book, but I am indeed impressed by some of the ideas he has presented. There are many places in the text that have either typos or confusing notation when compared with what I understand. Chapter 5 is where I had the best contact with Shipov's ideas, but I had to force myself to redo - in my notation - the stuff that Shipov has presented. I will try to detail my thoughts below.

I applaud the Shipov conjecture that the vacuum is an A4 space of absoute parallelism. IMO, a space of absolute parallelism is defined as one where the Cartan structural equations (in my notation) are homogeneous in the sense that

$$|\Sigma\rangle := |d\sigma\rangle + [C]\hat{}|\sigma\rangle \Rightarrow 0$$

and

$$[\Theta] := [dC] + [C]\hat{}[C] \Rightarrow 0$$

$|\Sigma\rangle$ is the vector array defined as the "Cartan torsion 2-forms" and $[\Theta]$ is the matrix array defined as the Cartan curvature 2-forms, constructed from the right Cartan matrix, $[C]$. There appears to be no problem with the curvature structure equation, but it is the the first structure equation which involves "torsion" that appears to be contentious. As will be shown below, the problem arises because there are two possibilities to describe the Connection – a right Cartan connection defined the symbol $[C]$ such that for a given Frame matrix $[F]$,

$$d[F] = [F] \circ [C],$$

and a left Cartan connection defined by the symbol $[\Delta]$, such that

$$d[F] = -[\Delta] \circ [F],$$

When I first studied Cartan's ideas, the first structural equation above (with non-zero $|\Sigma\rangle$) always seemed to come "from out of thin air". Later, by looking as subspaces of a euclidean space, or subspaces of a projective space, I believe I understand how the Torsion 2-forms of Cartan arise. In other words I can derive the structural equation, rather than postulate its existence. This has been demonstrated in detail at http://www22.pair.com/csdc/pdf/defects2.pdf However, it should be remembered that:

Cartan's structural equations are defined without imposing metric constraints.

The structural equations stated above are the same as the structural equations presented by Shipov as equations 5.68 and 5.69 on page 195, under the identification that

$$|\sigma\rangle \Rightarrow |e^a\rangle \qquad and \qquad [C_a^b] \Rightarrow [\Delta_a^b].$$

The upper index is the matrix row index and the bottom index is the matrix column index. In my notation $[C_a^b]$ is the matrix of 1-forms constructed as the right Cartan matrix, from the frame matrix $[F_a^b(y^e)]$

$$[C_a^b] = \left[\sum_c C_{ac}^b(y^e) dy^c \right]$$

My notation has the plus sign where Shipov has the minus sign, but that is due to his order of the 1-form products that appear in the 2-forms.

Frame Fields and different concepts of torsion.

I believe that a thorough comprehension of the Cartan concepts should start with the simplest case: That is, consider a C2 mapping ϕ of a space $\{y^a\}$ to a space $\{x^k\}$. (Later on the the ideas will be extended to more complex situations.) The utility of this approach is that it clearly demonstrates on which set of variables the functions are defined. Consider the mapping functions from the set y^a to x^k :

$$\phi : |y^a\rangle \Rightarrow |x^k\rangle = |\phi^k(y^e)\rangle,$$

and note that the mapping induces the differentials

$$d\phi : |dy^a\rangle \Rightarrow |dx^k\rangle = [\partial\phi^k(y^e)/\partial y^a] \circ |dy^a\rangle = [F_a^k] \circ |dy^a\rangle.$$

This equation defines the transformational behavior of a contravariant vector with respect to a diffeomorphic mapping from y^a to x^k. An upper index k is a contravariant index representing a row in matrix notation. A lower index b is a covariant index representing a column in matrix notation. With these definitions, the matrix row column product is equivalent to the Einstein convention of summing over a repeated down up index. When the mapping functions exist, it is evident that the Jacobian matrix of partial derivatives, $[\partial\phi^k(y^e)/\partial y^a]$, with non-zero determinant, is a candidate for a basis frame at the point p. Note that the functions that make up the basis frame are functions of the domain, $\{y^e\}$.

When the Frame matrix is not generated from a JAcobian matrix, the Frame formula does not necessarily map perfect differentials into perfect exact differentials. Instead, the Frame matrix maps perfect differentials into 1-forms which need not be exact.

Connections

For square matrices $[F_a^k(y^e)]$ whose elements are functions of independent variables, y^e, the differentials of the matrix elements may be computed. On the domain of support (defined as where the determinant of $[F_a^k(y^e)] \neq 0$) the formula defining the inverse matrix $[G_m^a]$

$$[F_a^k] \circ [G_m^a] = [\delta_m^k]$$

leads to the differential expression,

$$d[F_a^k] \circ [G_m^a] + [F_a^k] \circ d[G_m^a] = [0]$$

such that

$$d[F_b^k] + [F_a^k] \circ d[G_m^a] \circ [F_b^m] = [0].$$

The result has two interpretations, with $[C_b^a]$ defined as the right Cartan Connection matrix of differential 1-forms, and with $[\Delta_m^k]$ defined as the left Cartan Connection matrix of differential 1-forms.

$$d[F_b^k] - [F_a^k] \circ [C_b^a] = [0] \quad or \quad d[F_b^k] + [\Delta_m^k] \circ [F_b^m] = [0].$$

The first representation $[C_b^a]$ could be called a passive representation, where the second representation

$[\Delta^k_m]$ could be called an active representation. The matrix elements of the right cartan matrix consist of differential 1-forms

$$[C^b_a] = -d[G^a_m] \circ [F^m_b] = [G^a_m] \circ d[F^m_b] = \left[\sum_c C^b_{ac}(y^e)dy^c \right].$$

The triple indexed coefficients $C^b_{ac}(y^e)$ are functions of the variables (y^e) which in the simple case are mapped to the the variables, x^k. The minus sign is by convention such that $d[F^k_b] = +[F^k_a] \circ [C^a_b]$ has the appearance of a closure concept. That is, the differential of any contravariant basis vector (a column in $[F^k_b]$) is a linear combination of the basis vectors (columns) of the set. This idea of differential closure is repleat throughout Cartan's works.

It is apparent that the two connections are (negative) similarity (collineatory) transforms of one another. That is,

$$[G] \circ [-\Delta] \circ [F] = [C]$$

The representation of the left Cartan matrix is given by the expression

$$[\Delta^k_m] = [F^k_a] \circ d[G^a_m] = -d[F^k_a] \circ [G^a_m] = \left[\sum_c \Delta^k_{mc}(y^e)dy^c \right].$$

Note that the left Cartan matrix has indices on the target x^k, but is composed of functions and differentials on the domain y^e. The right Cartan matrix is intrinsic, for both its indices and arguments are on the domain space y^e.

Given the integrable mapping, a Frame matrix can be deduced as the Jacobian matrix of partial derivatives of the map. However, given a Frame matrix, the deduction of an integrable map may not be possible. The Frame matrix, considered as a matrix (linear) mapping of 1-forms into 1-forms, may not be integrable. If it is assumed that the the 1-forms on the domain are perfect exact differentials, then there are 4 cases to consider. The Frame matrix either transforms the perfect exact differentials into (1) perfect exact differentials, (2) closed but non-exact 1-forms, (3) not closed, but integrable 1-forms, and (4) non-integrable 1-forms. If the map is to a euclidean space, cases (1) and (2) are Affine torsion free; cases (3) and (4) exhibit Affine torsion. In all cases the Cartan torsion 2-forms for spaces of absolute parallelism are zero.

Exact differentials to exact differentials

In the simple case where the mapping is given, the matrix $[F]$ may be viewed as a Frame matrix that transforms differentials $|dy^a\rangle$ into differential forms $|dx^k\rangle$.

$$|dy^a\rangle \Rightarrow |dx^k\rangle = [F^k_a] \circ |dy^a\rangle$$

Exterior differentiation leads to the expression

$$d|dx^k\rangle = 0 = d[F^k_a]\hat{}|dy^a\rangle = [F^k_b] \circ [C^b_a]\hat{}|dy^a\rangle = 0$$

The matrix elements of the vector array of 2-forms $[C^b_a]\hat{}|dy^a\rangle$ when expanded yield

$$[C^b_a]\hat{}|dy^a\rangle \Rightarrow \sum_c C^b_{ac}(y^e)dy^c\hat{}dy^a = \{C^b_{ac} - C^b_{ca}\}dy^c\hat{}dy^a = C^b_{[ac]}dy^c\hat{}dy^a$$

These anti-symmetric components of the three indexed symbol, $C^b_{[ac]}$, are the usual definitions of what is called Affine Torsion. Note that when the mapping functions are explicit, the Affine torsion is zero.

Example: Spherical Coordinates to Euclidean Coordinates

Example 1 Frame field = Jacobian matrix

In order to set the stage, to get rid of notational inconsistencies, and to make the understanding of spaces of absolute parallelism a bit more transparent, consider the special case of a space of absolute parallelism defined by a parametric map, ϕ, from n variables (or parameters) $\{y^b\}$ of the initial state into a space of n variables $\{x^k\}$ of the final state.

$$\phi : |y^b\rangle \Rightarrow |x^k\rangle = |\phi^k(y^b)\rangle$$

realized by the expression:

$$\left|\begin{matrix} x \\ y \\ z \end{matrix}\right\rangle = \left|\begin{matrix} r\sin(\theta)\cos(\varphi) \\ r\sin(\theta)\sin(\varphi) \\ r\cos(\theta) \end{matrix}\right\rangle$$

The differentials are related by the equation

$$d\phi : |dy^a\rangle \Rightarrow |dx^k\rangle = [\partial\phi^k(y^b)/\partial y^a] \circ |dy^a\rangle = [F^k_a] \circ |dy^a\rangle$$

$$Induced\_1forms \quad \left|\begin{matrix} dx \\ dy \\ dz \end{matrix}\right\rangle = \begin{bmatrix} \sin(\theta)\cos(\varphi) & r\cos(\theta)\cos(\varphi) & -r\sin(\theta)\sin(\varphi) \\ \sin(\theta)\sin(\varphi) & r\cos(\theta)\sin(\varphi) & r\sin(\theta)\cos(\varphi) \\ \cos(\theta) & -r\sin(\theta) & 0 \end{bmatrix} \circ \left|\begin{matrix} dr \\ d\theta \\ d\varphi \end{matrix}\right\rangle$$

Note that a set of perfect differentials is mapped into a set of perfect differentials by the Jacobian mapping, and the induced differentials are closed in the exterior differential sense of Cartan.

$$d\left|\begin{matrix} dx \\ dy \\ dz \end{matrix}\right\rangle = d\left|\begin{matrix} \sin(\theta)\cos(\varphi)dr + r\cos(\theta)\cos(\varphi)d - r\sin(\theta)\sin(\varphi)d\phi \\ \sin(\theta)\sin(\varphi)dr + r\cos(\theta)\sin(\varphi)d + r\sin(\theta)\cos(\varphi)d\phi \\ \cos(\theta)dr - r\sin(\theta)d\theta \end{matrix}\right\rangle = \left|\begin{matrix} 0 \\ 0 \\ 0 \end{matrix}\right\rangle$$

The matrix of partial differentials is the Jacobian matrix of functions with arguments on the initial state. No metric and no domain of support has been specified as yet.

In that which follows the domain of support is usually defined as that set of values y^b on the initial state, where the Jacobian determinant does not vanish ($r^2\sin(\theta) \neq 0$). The singular points are the origin and the z axis. The Jacobian matrix can be viewed as a matrix of contravariant vectors (on the final state, x^k) in columns, and can be used as a basis frame (with arguments on the initial state y^b) on the domain of support (where det$[\partial\phi^k(y^b)/\partial y^a] \neq 0$). That is, assume the basis frame is given by a set of contravariant columns with row index k and column index a and with arguments on y^b :

$$[F^k_a] = [\partial\phi^k(y^b)/\partial y^a].$$

The Frame matrix formed by the Jacobian matrix for the spherical example has basis vector (columns) that are orthogonal to one another under the matrix product:

$$[F^k_a]^{transpose} \circ [F^k_a] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2(\theta) \end{bmatrix}.$$

However, the Frame Matrix constructed from the Jacobian is not an element of the orthonormal group. The determinant of the Frame is not unity, so that differential volume elements are not preserved under the transformation.

As yet there has been no metric imposed upon the space, but even without specification of a metric it is possible to use the general formulas given above to define a right Cartan connection

$$d[F] = [F] \circ [C] = [F] \circ [-d[G] \circ [F]] = [F] \circ [[G] \circ d[F]]$$

or a Left Cartan connection,

$$d[F] = [\Delta] \circ [F] = [-[F] \circ d[G]] \circ [F] = [d[F] \circ [G]] \circ [F]$$

In classical tensor analysis, the concept of an affine connection is associated with the (right) Cartan matrix (ref. L. Brand) as follows: (Remember, all the functions have arguments y^c)

$$d[F^k_a] = [F^k_b] \circ [C^b_{ac} dy^c] = [F^k_b] \circ \left[\left[G^b_j \right] \circ d\left[F^j_a \right] \right]$$
$$= [F^k_b] \circ \left[\left[G^b_j \right] \circ [\{\partial^2 \phi^j(y^m)/\partial y^c \partial y^a\} dy^c] \right].$$

As the system is integrable and (assumed) twice differentiable, it follows that the coefficient functions of the connection are symmetric

$$C^b_{ac} \Rightarrow \left[G^b_j \right] \circ [\{\partial^2 \phi^j(\xi^m)/\partial \xi^c \partial \xi^a\} d\xi^c] = C^b_{ca}.$$

By direct computation, the right Cartan matrix of connection 1-forms is equal to

$$C^b_a = \begin{bmatrix} 0 & -r d\theta & -r \sin^2\theta d\phi \\ d\theta/r & dr/r & -\cos(\theta)\sin(\theta) d\phi \\ d\phi/r & \cos(\theta) d\phi/\sin(\theta) & dr/r + \cos(\theta) d\theta/\sin(\theta) \end{bmatrix}$$

The three indexed components of the right Cartan matrix are

$$C^b_{ac} \Rightarrow \begin{bmatrix} 0 & C^1_{22} = -r & C^1_{33} = -r\sin^2\theta \\ C^2_{12} = 1/r & C^2_{21} = 1/r & C^2_{33} = -\cos(\theta)\sin(\theta) \\ C^3_{13} = 1/r & C^3_{23} = \cos(\theta)/\sin(\theta) & C^3_{31} = 1/r \ ,\ C^3_{32} = \cos(\theta)/\sin(\theta) \end{bmatrix}$$

Note that $C^b_{ac} - C^b_{ca} = 0$ so there is zero affine torsion.

In fact, if one computes the pullback metric g_{ab} on the initial domain $\{y^c\}$ induced by the quadratic form on the final state, $\eta_{jk} dx^j dx^k$

$$[g_{ab}(y^c)] = \left[F^j_a \right] \circ [\eta_{jk}] \circ [F^k_b] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2(\theta) \end{bmatrix},$$

and then uses the classic Christoffel formulas for deriving a connection from a metric,

$$\mathit{Christoffel} : \Gamma^b_{ac}(\xi^c) \Rightarrow \left\{ {b \atop ac} \right\} = g^{be}\{\partial g_{ce}/\partial y^a + \partial g_{ea}/\partial y^c - \partial g_{ac}/\partial y^e\}.$$

it follows that, for a Jacobian basis frame, the Cartan connection is the same as the Christoffel connection, and the connection is (affine) torsion free:

$$\mathit{If}\ [F^k_a] = [\partial \phi^k(y^b)/\partial y^a],\ \mathit{then}\ C^b_{ac} = \left\{ {b \atop ac} \right\}.$$

It is of some interest to decompose the Cartan connection into its symmetric, $\Gamma^b_{ac} = +\Gamma^b_{ca}$, and anti-symmetric parts, $\omega^b_{ac} = -\omega^b_{ca}$:

$$C^b_{ac} = \Gamma^b_{ac} + \omega^b_{ac}$$

It is the anti-symmetric parts , $\omega^b_{ac} = -\omega^b_{ca}$, of the (right) Cartan connection that lead to the concept of Affine torsion. Note that this has nothing to do with metric. (Compare Shipov (5.77)). It is also possible to think of the symmetric part of the Cartan connection to be composed of a Christoffel part and a non-Christoffel part.

$$C^b_{ac} = \left\{ {b \atop ac} \right\} + T^b_{(ac)}$$

The combination

$$T^b_{ac} = T^b_{(ac)} + \omega^b_{ac}$$

forms what Shipov defines as the Ricci rotation coefficients.

For the integrable case, the Jacobian formulation of a basis frame does NOT generate affine torsion, and the *Ricci rotation coefficients vanish defined in terms of the right Cartan connection*. A two surface immersed into 3 euclidean dimensions does not have affine torsion. (It must be remembered that the basis frame and the connection are defined in terms of the variables of the initial state, $\{y^b\}$).

The computation of the Left Cartan matrix yields much more complex formulas:
(see http://www22.pair.com/csdc/pdf/sphere3d.pdf

Exact differentials to non-exact differentials (case 1)

When the mapping is not given, but a Frame matrix can be constructed over the domain by some other criteria, then it may be true that exact differentials $|dy^a\rangle$ are NOT mapped into exact differentials $|dx^k\rangle$ by the Frame matrix. To denote this possibility, denote the inexact differentials (1-forms) by the symbol $|\delta x^k\rangle$. (Alternatively, the Frame matrix may map non-exact differential 1-forms into exact 1-forms - see case 2 below.) The Frame formula in the first case can be written as

$$|dy^a\rangle \Rightarrow |\delta x^k\rangle = [F^k_a] \circ |dy^a\rangle$$

When the system is integrable to exactness, it follows that the induced 1-forms are closed, $d|\delta x^k\rangle \Rightarrow 0;$ but $d|\delta x^k\rangle = 0$ does not imply that the components of $|\delta x^k\rangle$ are perfect differentials. For example, the 1-form

$$\delta x^1 = dy^1 + \frac{y^2 dy^3 - y^3 dy^2}{(y^2)^2 + (y^3)^2}, \qquad d(\delta x^1) = 0$$

is closed but not exact (it has no curl component).

Application of the exterior derivative to the vector array of induced 1-forms leads to vector array of 2-forms, which will be defined herein as the "Closure 2-forms" or "Affine Torsion 2-forms". If the Closure_Affine Torsion 2-forms vanish, there is no Affine torsion.

$$Closure\_Affine(torsion)\_2\_forms : d|\delta x^k\rangle = d[F^k_a]\hat{}|dy^a\rangle = [F^k_b] \circ [C^b_a]\hat{}|dy^a\rangle = |\Upsilon\rangle$$

Note that this formula can also be written in terms of the left Cartan connection $[\Delta^k_m]$ as

$$d|\delta x^k\rangle = d[F^k_a]\hat{}|dy^a\rangle = -[\Delta^k_m]\hat{}[F^m_a] \circ |dy^a\rangle = -[\Delta^k_m]\hat{}|\delta x^m\rangle$$

which leads to the Cartans torsion 2-forms based on the left Cartan connection.

$$d|\delta x^k\rangle + [\Delta^k_m]\hat{}|\delta x^m\rangle = |\Upsilon\rangle + [\Delta^k_m]\hat{}|\delta x^m\rangle \triangleq |\Sigma^k_{left}\rangle \Rightarrow 0.$$

This equation establishes the format of the first Cartan structural equation, constructed in terms of the **left Cartan** connection matrix. (IMO this is not the preferred structural equation, for it is not intrinsic; the arguments are on the domain space and the indicies are on the target space). For a space of absolute parallelism, the left Cartan torsion 2-forms $|\Sigma^k_{left}\rangle$ are always zero. (There is a similar equation of structure that can be based on the right Cartan connection matrix which is intrinsic– see below).

The Closure-Affine torsion 2-forms $|\Upsilon\rangle$ are not, in general, the same as the Cartan torsion 2-forms $|\Sigma_{left}\rangle$

$$|\Upsilon\rangle \neq |\Sigma\rangle.$$

Note that when the induced differential 1-forms $|\delta x^k\rangle$ on the target space are closed, then the vector array of Closure - Affine 2-forms is zero, $|\Upsilon\rangle = 0$. However, if the the induced differential 1-forms $|\delta x^k\rangle$ on the target space are NOT closed, then $|\Upsilon\rangle \neq 0$, but $|\Sigma^k_{left}\rangle = 0$ for a space of absolute parallelism.

Example 2: Frame Field = "Normalized" Jacobian Matrix (Jacobian multiplied by a function such that det = 1.)

It was noticed in the previous section that the spherical coordinate mapping generated a Frame field that had a determinant that was not equal to unity. The Jacobian Matrix of the spherical coordinate mapping can be modified to yield a unity determinant just by dividing every matrix element by the factor equal to the cube root of the Jacobian determinant, $\sqrt[3]{r^2\sin(\theta)}$. The new frame field so created does not map perfect differentials into perfect differentials. This "normalized" Jacobian frame field induces inexact differentials, or 1-forms, on the target domain.

$$d\phi \;:\; |dy^a\rangle \Rightarrow |\delta x^k\rangle = (\det F_a^k)^{-(1/3)}[F_a^k]\circ|dy^a\rangle$$

$$Induced\_1forms \quad \left|\begin{matrix}\delta x\\ \delta y\\ \delta z\end{matrix}\right\rangle = (r^2\sin(\theta))^{-(1/3)}\begin{bmatrix}\sin(\theta)\cos(\varphi) & r\cos(\theta)\cos(\varphi) & -r\sin(\theta)\sin(\varphi)\\ \sin(\theta)\sin(\varphi) & r\cos(\theta)\sin(\varphi) & r\sin(\theta)\cos(\varphi)\\ \cos(\theta) & -r\sin(\theta) & 0\end{bmatrix}\circ\left|\begin{matrix}dr\\ d\theta\\ d\varphi\end{matrix}\right\rangle$$

The point is that no longer does the exterior derivative of $|\delta x^k\rangle$ equal to zero. The induced 1-forms are not closed. The components of $|\delta x^k\rangle$ are not perfect differentials. The components of $d|\delta x^k\rangle$ form a vector array of Closure Affine 2-forms which are not equal to zero.

$$d\left|\begin{matrix}\delta x\\ \delta y\\ \delta z\end{matrix}\right\rangle = \frac{\left|\begin{matrix}\sin(\theta)\cos(\phi)r^2\cos(\theta)d\theta\wedge dr + r^3\sin(\theta)\sin(\phi)\cos(\theta)d\theta\wedge d\phi + r^2\sin(\theta)^2\sin(\phi)dr\wedge d\phi\\ \sin(\theta)\sin(\phi)r^2\cos(\theta)d\theta\wedge dr - r^3\sin(\theta)\cos(\phi)\cos(\theta)d\theta\wedge d\phi - 2r^2\sin(\theta)^2\cos(\phi)dr\wedge d\phi\\ r^2(\cos(\theta)^2 + 2\sin(\theta)^2)dr\wedge d\theta\end{matrix}\right\rangle}{3(r^2\sin(theta))^{4/3}}$$

This example frame field exhibits affine torsion. The only difference between the Frame field of this example and the frame field of the previous example is a multiplicative factor used to force the determinant to unity. The point of the exercise is to show that "Affine torsion" for integrable but not exact systems, in certain circumstances, can be mapped away. Each of the components of the induced differential 1-forms (for the example frame field) has ZERO topological torsion, but the frame field exhibits "Affine torsion"

$$Topological\_Torsion\_3forms \quad \begin{matrix}\delta x\wedge d(\delta x)\\ \delta y\wedge d(\delta y)\\ \delta z\wedge d(\delta z)\end{matrix} = \left|\begin{matrix}0\\ 0\\ 0\end{matrix}\right\rangle$$

This example Frame matrix (with determinant unity) can be used to define a right Cartan connection, but now the three index symbols of the right Cartan connection have anti-symmetric parts, and the frame exhibits "affine torsion". By direct computation the Christoffel symbols for the induced metric are not equal to the right Cartan connection symbols. The Ricci rotation coefficients are not zero.

Example 3 Frame Field = Orthonormalized Jacobian Matrix (Each frame column vector is rescaled to have norm unity.)

As another example form the othonormal basis set by dividing the Jacobian matrix through by the appropriate scale factors. The resulting frame field is a member of the special orthogonal group.

$$[F_{orthogonal}] = \begin{bmatrix}\sin(\theta)\cos(\varphi) & \cos(\theta)\cos(\varphi) & -\sin(\varphi)\\ \sin(\theta)\sin(\varphi) & \cos(\theta)\sin(\varphi) & \cos(\varphi)\\ \cos(\theta) & -\sin(\theta) & 0\end{bmatrix}$$

$$Induced\_1\_forms \quad \left|\begin{matrix}\delta x\\ \delta y\\ \delta z\end{matrix}\right\rangle = \begin{bmatrix}\sin(\theta)\cos(\varphi) & \cos(\theta)\cos(\varphi) & -\sin(\varphi)\\ \sin(\theta)\sin(\varphi) & \cos(\theta)\sin(\varphi) & \cos(\varphi)\\ \cos(\theta) & -\sin(\theta) & 0\end{bmatrix}\circ\left|\begin{matrix}dr\\ d\theta\\ d\varphi\end{matrix}\right\rangle$$

The induced 1-forms,

$$\left|\begin{array}{c}\delta x\\ \delta y\\ \delta z\end{array}\right\rangle = \left|\begin{array}{c}\sin(\theta)\cos(\varphi)dr + \cos(\theta)\cos(\varphi)d\theta - \sin(\varphi)d\phi\\ \sin(\theta)\sin(\varphi)dr + \cos(\theta)\sin(\varphi)d\theta + \cos(\varphi)d\phi\\ \cos(\theta)dr - \sin(\theta)d\theta\end{array}\right\rangle$$

are no longer exact nor closed in a differential sense. Indeed the vector of Closure Affine 2-forms is not zero

$$Closure\_Affine\_2forms \quad d\left|\begin{array}{c}\delta x\\ \delta y\\ \delta z\end{array}\right\rangle = \circ\left|\begin{array}{c}cos(\phi)cos(\theta)d\theta\hat{}dr - sin(\theta)sin(\phi)d\phi\hat{}dr - cos(\theta)sin(\phi)d\phi\hat{}d\theta\\ sin(\phi)cos(\theta)d\theta\hat{}dr + sin(\theta)cos(\phi)d\phi\hat{}dr + cos(\theta)cos(\phi)d\phi\hat{}d\theta\\ \sin(\theta)dr\hat{}d\theta\end{array}\right\rangle$$

so the frame exhibits Affine torsion.

However, the induced 1-forms are NOT INTEGRABLE and have non zero topological torsion.

$$Topo\log ical\_Torsion\_3forms \quad \left|\begin{array}{c}\delta x\hat{}d(\delta x)\\ \delta y\hat{}d(\delta y)\\ \delta z\hat{}d(\delta z)\end{array}\right\rangle = \left|\begin{array}{l}-\cos(\theta)\cos(\phi)\sin(\phi)dr\hat{}d\phi\hat{}d\theta\\ +\cos(\theta)\cos(\phi)\sin(\phi)dr\hat{}d\phi\hat{}d\theta\\ 0\end{array}\right\rangle$$

The moral of this exercise is to demonstrate the many possibilities inherent in Frame fields. In the next section, another form of non-exact connections will be examined. Another vector array of torsion 2-forms will be defined.

Example 4 The induced diagonal metric as a Frame Field

In many articles, the "square root" of diagonal metric is used a frame field. For the spherical coordinate system the induced euclidean metric on $\{y^e\}$ was given above. Forming the square root yields a candidate for a Frame matrix.:

$$[F_{metric\_sq\_root}] = \left[\begin{array}{ccc}1 & 0 & 0\\ 0 & r & 0\\ 0 & 0 & r\sin(\theta)\end{array}\right],$$

The induced 1-forms,

$$\left|\begin{array}{c}\delta x\\ \delta y\\ \delta z\end{array}\right\rangle = \left|\begin{array}{c}dr\\ rd\theta\\ r\sin(\theta)d\phi\end{array}\right\rangle$$

are no longer exact nor closed in a differential sense. Indeed the vector of Closure Affine 2-forms is not zero

$$Closure\_Affine\_2forms \quad d\left|\begin{array}{c}\delta x\\ \delta y\\ \delta z\end{array}\right\rangle = \left|\begin{array}{c}0\\ dr\hat{}d\theta\\ \sin(\theta)dr\hat{}d\phi + r\cos(\theta)d\theta\hat{}d\phi\end{array}\right\rangle$$

so the frame exhibits Affine torsion.

However, the induced 1-forms are INTEGRABLE and have zero topological torsion. Such is always the case for diagonal Frame fields.

$$Topo\log ical\_Torsion\_3forms \quad \left| \begin{array}{c} \delta x \hat{} d(\delta x) \\ \delta y \hat{} d(\delta y) \\ \delta z \hat{} d(\delta z) \end{array} \right\rangle = \left| \begin{array}{c} 0 \\ 0 \\ 0 \end{array} \right\rangle$$

Non exact differentials to exact differentials (case 2)

In this case, the Frame matrix will be assume to map 1-forms into perfect differentials. The situation is related to the inverse Frames and coordinate mappings studied above. A key feature is that the position vector from the origin to the point p of the target range is specified in terms of the Frame matrix. The differentials of the position vector are specified relative to the basis frame $[F]$.

$$|dx^k\rangle = [I] \circ |dx^k\rangle = [F] \circ [G] \circ |dx^k\rangle = [F] \circ |\delta y^b\rangle$$

The 1-forms on the domain, $|\delta y^b\rangle$, are deduced from the inverse Frame matrix, $[G^a_m]$, as given by the formula:

$$|\delta y^b\rangle = [G^b_k] \circ |dx^k\rangle.$$

This formuyla should be compared with the previous case, where it was assumed that

$$|\delta x^k\rangle = [F^k_a] \circ |dy^a\rangle.$$

To repeat, it is assumed that there exist differential forms, δy^a (linear combinations of exact differentials), on y^a that map to exact differentials dx^k on the target space, x^k.

$$|\delta y^b\rangle \Rightarrow |dx^k\rangle = [F^k_b] \circ |\delta y^b\rangle$$

Note that by exterior differentiation,

$$d\{[F^k_a] \circ |\delta y^a\rangle\} = [F^k_b] \circ |d(\delta y^b)\rangle + [C^b_a] \hat{} |\delta y^a\rangle = [F^k_b] \circ |\Sigma^b\rangle = d|dx^k\rangle \Rightarrow 0.$$

This formula leads to the Cartan structural equation

$$|d(\delta y^b)\rangle + [C^b_a] \hat{} |\delta y^a\rangle = |\Sigma^b\rangle$$

where the vector array $|\Sigma^b\rangle$ defines the Cartan torsion 2-forms, and the components of the non exact differential forms are identified as:

$$|\sigma^a\rangle = |\delta y^a\rangle = [G^b_k] \circ |dx^k\rangle$$

Of course, for a space of Absolute Parallelism $|\Sigma^b\rangle = 0$, but the partitioned Cartan torsion 2-forms need not vanish on subspaces of a spaced of Absolute Parallelism. The Cartan Torsion 2-forms, $|\Sigma^b\rangle$, are not the same as the Closure Affine torsion 2-forms, $|\Upsilon\rangle$

Example5 Euclidean coordinates mapped to spherical coordinates.

The map from $\{r, \theta, \phi\}$ to $\{x, y, z\}$ can be inverted to yield (the somewhat awkward formulas) for a map from the domain $\{x, y, z\}$ to the target $\{r, \theta, \phi\}$.

$$\left| \begin{array}{c} r \\ \theta \\ \phi \end{array} \right\rangle = \left| \begin{array}{c} +\sqrt{x^2+y^2+z^2} \\ \arccos(\frac{z}{\sqrt{x^2+y^2+z^2}}) \\ \arccos(\frac{x}{\sqrt{x^2+y^2}}) \end{array} \right\rangle$$

and the Jacobian Frame matrix such that

$$\left|\begin{array}{c} dr \\ d\theta \\ d\phi \end{array}\right\rangle = [F]\circ\left|\begin{array}{c} dx \\ dy \\ dz \end{array}\right\rangle = \begin{bmatrix} \frac{x}{\sqrt{x^2+y^2+z^2}} & \frac{y}{\sqrt{x^2+y^2+z^2}} & \frac{z}{\sqrt{x^2+y^2+z^2}} \\ \frac{-zx}{\sqrt{x^2+y^2}\,(x^2+y^2+z^2)} & \frac{-zy}{\sqrt{x^2+y^2}\,(x^2+y^2+z^2)} & \frac{\sqrt{x^2+y^2}}{(x^2+y^2+z^2)} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} & 0 \end{bmatrix}\left|\begin{array}{c} dx \\ dy \\ dz \end{array}\right\rangle$$

$$= \begin{array}{|c|} \hline d\sqrt{(x^2+y^2+z^2)} \\ \hline \{-zd(\sqrt{(x^2+y^2})+(\sqrt{(x^2+y^2}\,)dz\}/(x^2+y^2+z^2) \\ \hline (-ydx+xdy)/(x^2+y^2) \\ \hline \end{array} = \left|\begin{array}{c} \delta x \\ \delta y \\ \delta z \end{array}\right\rangle$$

It is apparent that the non-exact 1-forms $\left|\begin{array}{c} \delta x \\ \delta y \\ \delta z \end{array}\right\rangle$ are transformed into the exact differentials $\left|\begin{array}{c} dr \\ d\theta \\ d\phi \end{array}\right\rangle$. Each of the 1-forms on the domain is differentially closed,

$$d\left|\begin{array}{c} \delta x \\ \delta y \\ \delta z \end{array}\right\rangle = 0$$

but is either a harmonic form or an exact form. The harmonic forms have the classic representation as

$$Harmonic:\gamma = (\alpha d\beta - \beta d\alpha)/(+\alpha^2+\beta^2), \qquad d\gamma = 0, \qquad \int_{closed}\gamma = n\pi$$

It follows that the Closure Affine torsion 2-forms are zero (even though the 1-form mapping is from closed 1-forms to exact 1-forms, and not exact 1-forms to exact 1-forms). The Cartan matrix becomes algebraically lengthy, but can be obtained from Maple. If the domain space is presumed to have a euclidean metric, then the Christoffel symbols are zero, and are NOT equal to the elements of the right Cartan matrix. Hence rotation coefficents (defined as the difference between the right Cartan coefficients and the Christoffel symbols all defined intrinsically on the domain space) will appear and will be equal to the right Cartan coefficients.