Two torsion structural equations

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(Email to Bill Page)

Consider a domain that supports a Frame Field $[F^k_a(y)]$ and inverse $[G^a_m(y)]$ where the matrix elements are well defined functions on the independent variables y^a of the domain (or initial state). Define column vector arrays of differentials on the initial state by the symbol $|\delta y^b\rangle$. Each element of the column array can be a differential 1-form, not a perfect differential, and not necessarily a closed 1-form. Use the Frame Field to map the "initial state" $|\delta y^b\rangle$ to a new column array of 1-forms defined as the "final state" $|\delta x^k\rangle,: |\delta y^b\rangle \Rightarrow |\delta x^k\rangle$ or:

$$|\delta x^k\rangle = [F^k_a(y)] \circ |\delta y^a\rangle$$

The final state forms are well defined only as functions of the independent variables of the *initial* state. How the final state variables x^k are defined is not known unless the system is integrable.

Note that there is a dual or reciprocal format given by the equation:

$$|\delta y^b\rangle = [G^b_m(y)] \circ |\delta x^m\rangle$$

Apply the exterior derivative to both equations, to obtain

$$\{[C^b_{ac}(y)dy^c]\hat{}|\delta y^a\rangle + d|\delta y^b\rangle\} = [G^b_m(y)] \circ d|\delta x^m\rangle = |\Sigma^b_{right}\rangle \text{ torsion 2-forms}$$

and

$$\{[\Delta^k_{jc}(y)dy^c]\hat{}|\delta x^j\rangle + d|\delta x^k\rangle\} = [F^k_a(y)] \circ d|\delta y^a\rangle = |\Sigma^k_{left}\rangle \text{ torsion 2-forms}$$

where $[C^b_{ac}(y)dy^c]$ is the right Cartan connection and $[\Delta^k_{jc}(y)dy^c]$ is the left Cartan connection.

These are two different forms of the Cartan torsion structural equations.

The first is more appealing to me for it is well defined in terms of the independent variables on the "initial" state, y^a.

There are 4 cases to consider:

1. Both $|\delta x^j\rangle$ and $|\delta y^a\rangle$ are closed :

$$d|\delta x^j\rangle = 0 \quad and \quad d|\delta y^a\rangle = 0$$

2. $|\delta x^j\rangle$ is closed :

$$d|\delta x^j\rangle = 0 \quad and \quad d|\delta y^a\rangle \neq 0$$

3. $|\delta y^a\rangle$ is closed :

$$d|\delta x^j\rangle \neq 0 \quad and \quad d|\delta y^a\rangle = 0$$

4. Both $|\delta x^j\rangle$ and $|\delta y^a\rangle$ are not closed:

$$d|\delta x^j\rangle \neq 0 \quad and \quad d|\delta y^a\rangle \neq 0$$

The formulas demonstrate that the left Cartan connection plays a "torsion" role in an equation of structure that is distinct from the "torsion" role played by the right Cartan connection matrix.