Response to Page 02/29/2000

(this reponse was started about 02/06 and has had several augmentations before sending)

I can never seem to finish this! Every time I re-read this note I add something else.

Is there a chance that I could get you to use Scientific Notebook. It is excellent (and inexpensive) for constructing equations that can be transmitted as pdf files. The limitations of email for mathematical notation forces me to reconstruct most of what you send, and I just hope I do not make a mistake in translating your emails.

I have a Maple program computing right and left Cartan matrices, for the map from spherical coordinates to cartesian coordinates, and from cartesian coordinates to spherical coordinates that has some interesting features.

see

http://www22.pair.com/csdc/pdf/3dsp2eu.pdf

I will study your email of 03/01/2000 and comment later.

First some comments

1. I understand the desire for, and the utility of, a covariant derivative as a method of defining a differential process which transforms as a tensor, but I prefer to use the Lie derivative (and differential forms instead of tensors). Why? Well in simple terms it is my opinion that the covariant derivative is a transplantation process that preserves as invariant the ”intrinsic distance” between any pair of points. To me, this implies that deformations (where the distance between a pair of points is not constant) cannot be described by the covariant derivative processes. As my goal has been to understand topological evolution, and continuous deformation leading to changes in topology, I convinced myself a number of years ago that the concept of the covariant derivative was not the proper tool. The covariant derivative is fine for bending but not for stretching.

For topological evolution I prefer the concept of the Lie derivative *acting on differential forms*. I believe that the quest for ”covariant” derivatives is an obsession of the current physics literature, and although useful, the constraints inhererent in its definition are not fully justified in nature. As the theories of physics are often ”equations of motion” I do not wish to limit the ”motions” that are possible by an apriori assumption that prevents irreversibility. A restriction to diffeomorphisms is just such a restriction. A restriction to covariant transplantation is another. I believe that the Lie derivative as a transplantation process is more general then the others mentioned above. It is relatively easy to prove that both Hamiltonian and Symplectic ”motions” are special cases of the Lie derivative acting on a 1-form of Action. L(V)A = Q where Q is exact generates all Hamiltonian and Symplectic processes. L(V)dA = 0 generates the Hemlholtz theorems. L(V)A = Q with Q^dQ= 0 generates all thermodynamically reversible processes.

2. The Lie derivative with respect to a contravariant vector field X acting on differential forms (not acting on contravariant vector fields) is not equivalent to a ”covariant derivative” except in constrained circumstances. In contradistinction to Ark, I believe that the Lie derivative with respect to X acting on forms, and expressed by Cartan’s magic formula (Marsden and Riatu) is fundamental.

$$L_X A = i(X)dA + d(i(X)A) = W + dU = Q.$$

The formula is an expression of cohomology in the sense that the difference $Q - W = d(U)$. {When the difference between two forms is exact is essentially the definition of cohomology}. Note that the Lie derivative, as used by Cartan, does NOT depend upon two vector fields, X and Y, only X, but formulas involving 2 vector fields similar to those described in your email can be constructed as:

$$L_X \{i(Y)A\} - i(Y)L_X A = i([X,Y])A$$

which entails the Lie bracket field $[X,Y],$ and similarly,

$$L_{[X,Y]}A = L_X L_Y A - L_Y L_X A$$

From a hydrodynamics point of view, the Lie derivative on a 1-form of Action generates the Navier-Stokes as well as the Eulerian fluid concepts, a result that has impressed me. Also, the concept of the Magnus or Lorentz force appears naturally from the definition of virtual work, $W = i(V)dA \Rightarrow f_k^{LORENTZ} dx^k - (f \circ V)dt.$ (Ark thinks this point if view is nonsense, but I feel he is a bit prejudiced. I know no other way to *derive* the Navier-Stokes equations. I know of no other non-statistical way to make contact with thermodynamics and dynamical systems. I know no other way to so clearly denote the difference between heat and work : Work is transverse: $i(V)W = 0$ Heat is NOT transverse, $i(V)Q = i(V)d(U) \neq 0$)

3. In my opinion differential forms A are entities in themselves and have coefficients that have tensorial *invariant* behavior relative to diffeomorphisms, as they are composites of co and contravariant things. That is, a 1-form, $A = A_k dx^k,$ is composed of a covariant part, $A_k(x)$, and a contravariant part, $dx^k,$ summed over the updown repeated index. The concept that these objects are "intrinsic" or "independent from a choice of coordinates" is due to formulas

$$A = A_k(x)dx^k = A_m(x)\delta_k^m dx^k = \{A_m(x)\left[G_j^m(x)\right]\} \circ \{\left[F_k^j(x)\right]dx^k\} = \{B_j(x)\}\{\left[F_k^j(x)\right]dx^k\} \Rightarrow \bar{B}_j(y)dy^j$$

However, the last step requires two assumptions.

(a.) The first presumption is that dx maps differentially into exact differentials dy. That is

$$\left[F_k^j(x)\right]dx^k = \sigma^j(x) \Rightarrow dy^j$$

For the 1-forms $\sigma^j(x)$ to be exact, the implication is that there exists a map between x and y.

$$x \Rightarrow y = \phi(x),$$

such that

$$\left[F_k^j(x)\right] = [\partial\phi^j(x)/\partial x^k] \text{ and } \sigma^k = [\partial\phi^j(x)/\partial x^k]dx^k$$

The Jacobian matrix is defined as functions on the initial state, $x.$

(b.) The second presumption is that arguments of the covector $B_j(x)$ can be converted into the covector $\bar{B}_j(y).$ That is, a functional mapping supposedly exists between y and x.

$$x = \varphi(y) \Leftarrow y.$$

The direction of the arrows is important. They are in opposite directions. One map is the inverse of the other. In this sense (relative to diffeomorphisms where differentiable maps and inverses exist) differential forms are independent from (invariant with respect to) the choice of coordinates. However as discussed below in 4., differential forms have a larger "coordinate free" property, that does not depend upon the constraint of a diffeomorphism. All that is required is presumption (b), not presumption (a). The differential forms are functionally well defined on the initial state $\{y\}$ if the differential form is given on the final state $\{x\}$. The operation is defined as the pullback

In differential geometry, intrinsic implies that there are properties that do not depend upon the embedding. For example the Gauss curvature of a surface is intrinsic. However, note that the intrinsic Gauss curvature of the cone, the cylinder, and a flat piece of paper are zero. Yet they are different in shape. The process of bending does not change the Gauss curvature when equal to zero, but does change the shape. The shapes are artifacts of the embedding and they are not intrinsic properties.

My reading of Cartan has led to me to subsume that the covariant space and differential forms were more important than the contravariant space and contravariant vector fields. This seems to be the perceptions of others besides myself (Arnold says in effect (semi-quote using my imprecise memory) ..you cannot understand mechanics without a knowledge differential forms). P Libermann demonstrates that second order jets do not lead to a natural vector space on the tangent space, but all orders are well behaved on the cotangent space. The Cup product can be defined on the cotangent space but not on the tangent space. ...etc. From a physical point of view the tangent space is (IMO) the space of particles and the cotangent space is the space of waves.

4. For maps between initial and final state which are not diffeomorphisms (which immediately extends the realm of interest to that outside the constrained domain of classical tensor analysis) I know that initial contravariant and covariant tensors do not behave in a similar manner, and lose their alias-alibi status. Where relative to maps that preserve topology in a smooth manner, contravariant and covariant things are hard to distinguish experimentally. When the process involves changing topology, the two species of tensors behave differently, and measurably. In particular, for continuous differentiable maps, determinism is lost for both covariant and contravariant tensor fields. That is, given the functional form of either tensor on the initial state, the functional form of the tensors on the final state cannot be determined. Values at a point can be computed, but the functional form describing the neighborhood in terms of variables on the final state cannot be determined.

On the other hand differential forms have are functionally well defined in terms of the pullback. That is, the differential forms specified on the final state in terms of independent variables on the final state have well defined functional preimages on the initial state with arguments in terms of the independent variables on the initial state. Such can not be said for contravariant vector fields. Push forward is not the same as pullback.

It is important to keep in mind that that Frame matrix is either a matrix on a domain space of independent variables, say $\{y^a\}$, or on a range space of independent variables, $\{x^k\}$. These Frames (in a dynamical sense) are presumed to be related to a map $\phi^k(y^a)$ into at target space of variables, say $\{x^k\}$. The map is presumed to be differentiable and continuous, but not necessarily invertible. There could be points or regions of the domain space where the Jacobian matrix $[J_a^k(y)]$ of the C1 map $\phi^k(y^a)$ has a zero determinant. The existence of the C1 map permits the differentials $|dx^k\rangle$ on the target space to be determined in terms of the differentials on the domain via the Jacobian matrix of partial differentials:

$$|dx^k\rangle = [J_a^k(y)] \circ |dy^a\rangle$$

However, suppose that a Frame matrix of functions is defined (by some arbitrary rules of construction) **on the target space** as $\left[\overline{F}_m^k(x)\right]$ and construct the vector valued Frame induced 1-forms $\omega^k(x)$ on the target space:

$$|\omega^k(x)\rangle = \left[\overline{F}_m^k(x)\right] \circ |dx^m\rangle$$

This frame field on x induces by the pullback a functionally well defined set of functions that form a Frame field on the domain space y

$$|\omega^k(y)\rangle \Leftarrow [F_a^k(y)] \circ |dy^a\rangle \Leftarrow \left[\overline{F}_m^k(x(y))\right] \circ [J_a^m(y)] \circ |dy^a\rangle \Leftarrow |\omega^k(x)\rangle.$$

The induced differential forms on the domain space may or may not generate a complete basis for all forms on the domain, but they are well defined functionally. The induced matrix is $[F_a^k(y)]$ is also well defined, without the need for an inverse mapping.

(*The procedure to determine the Frame Field* $\left[\overline{F}_m^k(x)\right]$ *on the target space from the Frame field* $[F_a^k(y)]$ *given on the domain space, however, is not possible, unless the inverse Jacobian and the inverse map exist and are known.*

There are several points to consider:

1.) A function on x can be used to induce a function on y by the operation of functional substitution.

$$f(y) \Leftarrow \bar{f}(x(y))$$

It might be thought that a matrix of functions on x would induce a matrix of functions on y, and indeed that takes place. However, the vector of differential 1-forms involves both a Frame field, $\left[\bar{F}^k_m(x)\right]$, on x and the differentials, $|dx^m\rangle$. The pullback operation produces an induced (and perhaps restricted) Frame Field $[F^k_a(y)]$ on y, but it is not equivalent simply to the pullback of the matrix of functions $\left[\bar{F}^k_m(x(y))\right]$. There is a difference. The pull back Frame matrix of functions is given by the expression,

$$[F^k_a(y)] = \left[\bar{F}^k_m(x(y))\right] \circ [J^m_a(y)],$$

but the pullback matrix may not be a Frame matrix on all of $\{y^a\}$. There may be points where the det of the Jacobian vanishes. These points of the domain must be excluded if the pullback of the frame on x is to be a Frame on $\{y^a\}$. (A frame matrix must have $\det[F^k_a(y)] \neq 0$).

However, the criteria of the Frame matrix being invertible is used to construct an expression for the differential of the Frame matrix, as a linear connection. The same technique cannot be used to define the connection for the pullback of the Frame matrix, unless the determinant of the pullback matrix is non-zero.

Although the pullback of a Frame matrix is not necessarily a Frame matrix, the exterior differential of the pullback of a Frame induced 1-forms is well defined even though the INVERSE of the pulled back Frame matrix may not exist.

$$d|\omega^k(x)\rangle = d\left[\bar{F}^k_m(x)\right]\hat{}|dx^m\rangle = \left[\{\partial\bar{F}^k_m(x)/\partial x^j\}dx^j\right]\hat{}|dx^m\rangle = \left|\{\partial\bar{F}^k_m(x)/\partial x^j - \partial\bar{F}^k_j(x)/\partial x^m\}dx^j\hat{}dx^m\right\rangle$$

$$= \left[\bar{F}^k_n(x)\right] \circ [CR^n_m(x)]\hat{}|dx^m\rangle = \left[\bar{F}^k_n(x)\right] \circ \left|CR^n_{[mj]}dx^j\hat{}dx^m\right\rangle$$

$$= \left[\bar{F}^k_n(x)\right] \circ \left|CR^n_{[mj]}\{J^m_a(y)dy^a\}\hat{}\{J^n_b(y)dy^b\}\right\rangle = \quad \text{if the inverse Jacobian exists–}$$

$$= \left[\bar{F}^k_p(x)\right] \circ [J^p_e(y)] \circ ([J^e_n(y)]^{inverse} \circ \left|CR^n_{[mj]}\{J^m_a(y)dy^a\}\hat{}\{J^n_b(y)dy^b\}\right\rangle)$$

$$= [F^k_e(y)] \circ C^e_{[a,b]}dy^a\hat{}dy^b$$

$$= d\{[F^k_a(y)] \circ |dy^a\rangle\} = d|\sigma^k(y)\rangle$$

It is to be recognized that the coefficients $CR^n_{[mj]}$ define the "Affine torsion" coefficients. These coefficients determine if the induced 1-forms $|\omega^k(x)\rangle$ are closed in an exterior differential sense. If the induced 1-forms are integrable to exactness, then the "Affine torsion" coefficients must be zero. That is if $|\sigma^k(y)\rangle = |dy^k\rangle$, the Affine Torsion terms must be zero.

Normal and non-normal Frame Matrices

In tensor analysis, and in much of the literature of differential geometry, the assumption is made that there exists an orthonormal basis Frame such that $[F]^{transpose} \circ [F] = [1]$. Such Frame matrices are "normal" in the sense that the matrix and its transpose commute. Such an assumption is natural for tensor analysis where covariants are pushed forward by the Inverse transposed of the Jacobian matrix between the independent variables of the initial and final state.

For the arbitrary Frame Field, the only requirement is that the Frame field of columns of contravariant vectors, has determinant non-zero. The column vectors are linearly independent and can be used as a basis. *However, the arbitrary frame field $[F]$ need not be normal*. For real frame fields (any matrix with determinant $\neq 0$), there are two representations in terms of the product of a symmetric matrix $[S]$ and an orthogonal matrix $[O]$ (Murnaghan – use Hermitean and Unitary for complex matrices).

$$[F] = [O_L] \circ [S_L] = [S_R] \circ [O_R]$$

The dual resolution is unique if the eigenvalues of the symmetric matrix are positive definite. The symmetric matrices are given by expressions

$$[S_L] = \sqrt{[F^{transpose}] \circ [F]} \quad and \quad [S_R] = \sqrt{[F] \circ [F^{transpose}]},$$

and the orthogonal matrices are given by the expressions

$$[O_L] = [G^{transpose}] \circ [S_L] \quad and \quad [O_R] = [S_R] \circ [G^{transpose}]$$

assuming that $[G]$ is the inverse of $[F]$ such that $[G] \circ [F] = [1]$.

IMO the importance of the two representations is (intuitively) related to the concepts of chirality and enantiomorphic pairs, which are physical things. However, I have not been able to make a clear, clean expose of this idea. It is something I am working on. From my work on Electrodynamics, it is apparent that there are two features that are related to the above decompositions: helicity and chirality. I also use the cleaner idea of topological torsion and topological spin.

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If the map is to a Euclidean target space, where the "metric" is presumed to be the unit matrix, then $[S_L]$ is the pulled back or induced metric $[g_{ab}]$ on the domain space. The induced metric is well defined with its functions having arguments in terms of the domain independent variables.

If the map is from a Euclidean domain space, with unit metric, then $[S_R]$ is the pushed forward inverse metric $[g^{mn}]$ on the target space.

If the Frame field is "normal" the two symmetric factors have identical squares

$$[S_L] \circ [S_L] = [S_R] \circ [S_R],$$

but their square roots can be different if the eigenvalues are not positive definite. If the square roots are identical, and $[F]$ is normal, then the two orthogonal matrices are similarity transforms of one another,

$$[O_L] = [S_R] \circ [O_R] \circ [S_L]^{-1} \Rightarrow [S_R] \circ [O_R] \circ [S_R]^{-1}.$$

If the square roots are equal to the identity, and $[F]$ is normal, then the Left and Right orthogonal matrices are the same (the Cartesian case). The usual basis frames for a euclidean space are presumed to be orthonormal, but these are subspaces of spaces that support frames with an inverse. As $[F]^{-1} \circ [F] = [F] \circ [F]^{-1} = [1]$, orthogonal frame fields, where $[F]^{transpose} = [F]^{-1}$, are normal, $[F]^{transpose} \circ [F] = [F] \circ [F]^{transpose}$. *The fundamental difference between a space of absolute parallelism and a euclidean space is that the Frame matrix for the Euclidean space must be normal, where the Frame matrix for the A4 space does not have to be normal*.

As mentioned above, I intuitively believe that the fundamental physical features of chirality and torsion are tied into this notion of a non-normal Frame matrix. I have not completed an analysis of this thought. In particular, I have not worked out all of the details between the two representations above, and the right and left Cartan matrices. I suspect that the two classes of normal and non-normal frame matrices have distinct physical features.

Bottom line:

The imposition of an orthonormal frame field is a useful, ubiquitous, but not a general, idea. In general, the basis frame is not a normal matrix. The reason that Shipov's book was of interest to me was that his conjecture that the vacuum is an A4 space (defined as a space with a basis frame that has an inverse, but where the basis frame need not be normal). There are differences between such an A4 vacuum and a Euclidean-Cartesian Vacuum (or spaces where the basis frame is normal). Shipov in his book does not make clear the difference between normal and non-normal basis frames, but it was his intuition about the vacuum that caused me to think about these differences. (Maybe I "read too much" into the Shipov book) However, Shipov's ideas about inertia are still confusing to me.

Non-integrable Frames (Skip this unfinished part)

In general I believe that it is important to remember that there are two sets of independent variables, domain variables y^a and range variables x^k. The C2 map from domain variables to range variables induces a map between domain differentials and range differentials. I usually assume that the frame $e_a^i = [e_a^i] = [F_a^i]$ is a matrix of contravectors a arranged as columns with component functions k designating rows. It is classic to presume that the functions that make up the matrix elements are functions of the independent variables y^a. Hence the Frame field is defined on the domain (or initial state) as

$$[F_a^i] = [e_a^i(y)].$$

The frame field can be differentiated with respect to y but not with respect to x for the mapping functions have not been defined or computed. The frame field can be used to construct vector arrays of differential 1-forms. For example

$$|\sigma^k\rangle = [F_a^k] \circ |dy^a\rangle$$

is but one possibility. Exterior differentiation of this example leads to the formula

$$d|\sigma^k\rangle = \{d[F_a^k]\} \circ |dy^a\rangle = \{d[F_a^k]\}\hat{}[G_m^a] \circ [F_b^m] \circ |dy^b\rangle = \{d[F_a^k]\}\hat{}[G_m^a] \circ |\sigma^m\rangle$$

If it is assumed that y is limited to the subset such that the Frame matrix has a global inverse, then

$$[F_a^i] \circ \left[G_j^a\right] = \left[\delta_j^i\right] = \left[\widetilde{G}_a^i\right] \circ \left[\widetilde{F}_j^a\right]$$

and

$$\left[G_j^a\right] \circ \left[F_b^j\right] = [\delta_b^a] = \left[\widetilde{F}_j^a\right] \circ \left[\widetilde{G}_b^j\right]$$

The notation is such that $\left[\widetilde{F}_j^a\right]$ is the transpose of $\left[F_a^j\right]$. It is NOT presumed that the Frame matrix is an element of the orthogonal group. It is not assumed that the Frame matrix is "normal". In all cases, the matrix elements are functions of the y^a; the x^k have functions yet to be determined. As I have pointed out many times, these formulas lead to various definitions of a "connection". There is both a left and a right Cartan matrix.

$$d[F_b^i] = -[F_a^i] \circ \{d\left[G_j^a\right]\} \circ \left[F_b^j\right] = -\left[\Delta_j^i\right] \circ \left[F_b^j\right] = +[F_a^i] \circ [C_b^a]$$

All matrix elements are functions of the y^a and the matrix elements of the connections are 1 -forms involving the differentials of dy^a (not dx^k). It is apparent that

$$d|\sigma^k\rangle + [\Delta_m^k] \circ |\sigma^m\rangle \doteq |\tau^m\rangle \Rightarrow 0$$

defines a vector array of two forms, which is *similar* to the formula of Shipov (5.68). The only difficulty is that Shipov's definition (5.67) utilizes the formula for $[C_b^a] = -\{d\left[G_j^a\right]\} \circ \left[F_b^j\right] = +\left[G_j^a\right] \circ \{d\left[F_b^j\right]\}$, and not the formula for $[\Delta_m^k]$.

This is part of my confusion.

However, another construction leads to the 1 forms defined as

$$|\omega^k\rangle = \left[\widetilde{G}_a^k\right] \circ |dy^a\rangle$$

and the formula

$$d|\omega^k\rangle = \{d\left[\widetilde{G}_a^k\right]\}\hat{}\left[\widetilde{F}_j^a\right] \circ \left[\widetilde{G}_b^j\right]|dy^b\rangle = \{d\left[\widetilde{G}_a^k\right]\}\hat{}\left[\widetilde{F}_j^a\right] \circ |\omega^k\rangle$$

The 1-forms $|\omega^k\rangle$ are not equal to the 1-forms $|\sigma^k\rangle$ unless the Frame matrix is an element of the orthogonal group.

Various definitions of the vector of Torsion 2-forms (unfinished)

Lovelock and Rund p.139, 165

Using the $C^a_{bc}(y)$ notation (instead of $\Gamma^a_{bc}(x)$) for the right Cartan connection, the matrix of connection 1-forms is defined as

$$\omega^a_b(y, dy) = C^a_{bc}(y)dy^c$$

Then the classic vector of Affine torsion 2-forms is defined as

$$\Omega^a = \omega^a_b \hat{} dy^b = \{C^a_{bc}(y)dy^c\}\hat{} dy^b = \{C^a_{[bc]}(y)\}dy^c \hat{} dy^b$$

where the summation of repeated indices follows the ordered pair rule of exterior forms. The Affine torsion components are thereby defined via the non-zero components $C^a_{[bc]}(y)$ of the connection. This equation is equivalent to the formula

$$\Omega^a = d(dy^a) + \omega^a_b \hat{} dy^b$$

when the differentials dy^a are exact, or closed, as $d(dy^a) \Rightarrow 0$. However the vector of Cartan torsion 2-forms defined as

$$\text{Cartan Structural Equation 1:} \quad \Sigma^a = d(\sigma^a) + \omega^a_b \hat{} \sigma^b \neq \Omega^a$$

are not the same as Ω^a when the σ^a are not exact 1-forms. The Σ^a are zero when either 1. both terms in the expression are zero (which is the case of no Affine torsion) or 2. when the first term cancels the second term. If a space of absolute parallelism is defined as $\Sigma^a = 0$ and $\Omega^a_b = 0$, then there are spaces of absolute parallelism without affine torsion and spaces of absolute parallelism with affine torsion. In both cases, the Cartan torsion 2-forms is zero.

The matrix of curvature 2-forms is given by the expression

$$\text{Cartan Structural Equation 2:} \quad \Omega^a_b = d\omega^a_b + \omega^a_c \hat{} \omega^c_b$$

To repeat, IMO, the A4 space of Shipov is defined as:

$$\Sigma^a = 0 \quad \text{and} \quad \Omega^a_b = 0$$

Although the Cartan torsion 2-forms Σ^a vanish for the A4 space, $\Sigma^a \Rightarrow 0$, the right Cartan connection can still have anti-symmetric components in the sense that $\{C^a_{[bc]}(y)\} \neq 0.$

Consider subspaces of an n+1 dimensional space where $\{y^a, s\} \Rightarrow \{x^k\} = \phi^k(y^a, s),\ \{x^0\} = \phi^0(y^a, s).$ It follows that (for $a = 1..n, k = 1..n$)

$$|dx^k\rangle = [F^k_a]|dy^a\rangle + |F^k_0\rangle ds$$

$$dx^0 = \langle F^0_a| \circ |dy^a\rangle + F^0_0 \circ ds$$

From the mapping definitions and C2 functions $\phi^k(y^a, s)$ it follows that

$$d(dx^k) = 0 \quad and \quad d(dx^0) = 0.$$

However, the forms

$$\sigma^k = [F^k_a]|dy^a\rangle \quad \text{and} \quad \omega = [F^0_a]|dy^a\rangle$$

are not *necessarily* integrable, but are a system of 1-forms that at most can be of Pfaff dimension or class 3. Define the kinematic fluctuation1-forms:

$$\Delta x^k = |dx^k\rangle - |F^k_0\rangle ds = [F^k_a]|dy^a\rangle$$

and

$$\Delta x^0 = dx^0 - F_0^0 \circ ds = \langle F_a^0 | \circ |dy^a\rangle.$$

As the LHS is constructed of only 3 independent functions, the expression RHS must be of Pfaff dimension 3 at most. If the map is affine, then $\langle F_a^0| = 0$, the fluctuation $\Delta x^0 \Rightarrow 0$, and the function F_0^0 is either a constant, or a function of s alone.

Schouten p.139, 165

Note that Schouten claims the A4 space is Cartan torsion free, and if the space admits allowable coordinates (which means that the Frame matrix leads to integrable "differentials" or $|d\sigma^k\rangle = 0)$ then the space is affine torsion free. He also claims that for anholonomic coordinates (where $|d\sigma^k\rangle \neq 0)$ the space is Cartan torsion free, but can admit affine torsion. The affine torsion (anti-symmetry in the connection lower indices) is equal to the object of anholonomicity times minus 1 in an A4 anholonomic space.