Topological Parity and the Turbulent state of a Navier-Stokes fluid.

R. M. Kiehn

Emeritus, Physics Dept., Univ. Houston
http://www.cartan.pair.com
first draft 06/01/91, Updated 06/17/2003

Abstract

A global 1-form of Action, A, on a space-time variety not only acts as a Lagrange-Euler integrand for a variational principle, but also induces a fluctuation topology on {x,y,z,t}. When the induced topology is of Pfaff topological dimension 4, the solutions to the variation problem based on such an action, mod fluctuations, lead to a necessary set of partial differential equations of evolution that can be put into correspondence with the Navier-Stokes equations. The Euler characteristic of the topology induced by such extremal vector field solutions is generated by the 4-form K=dA^dA of Topological Parity. As domains of non-zero Euler index are necessary to break parity and time reversal symmetries at a macroscopic level, it follows that $K \neq 0$ is a necessary condition for those vector fields that describe a irreversible or turbulent process of evolution.

1. Introduction

Topological arguments have been used [26] [27] to suggest that those covariant vector fields that represent the turbulent state in a fluid flow must be of Pfaff (topological) dimension 4. These earlier ideas were based upon the fact that when the Euler characteristic of a manifold is not zero, then every contravariant vector field on the manifold, which is assumed to be continuous in the forward direction, must have at least one singular point or self intersection. If that specific forward trajectory is reversed in an evolutionary sense, then, as the singularity is approached in the opposite sense, the return bifurcation branch cannot be chosen unambiguously. Hence, the inverse of the continuous mapping generated by the vector field is not continuous, the mapping is not a homeomorphism, and the associated vector field is irreversible in a deterministic sense.

More recent ideas based on thermodynamic arguments have demonstrated that continuous thermodynamic irreversible processes in general [39] also require that the 1-form of Action be of Pfaff dimension 4 [34]. For hydrodynamic systems that admit encoding in terms of a differential 1-form of Action, A, on a space time variety, then Pfaff dimension 4 requires that the Topological Parity 4 form does not vanish: $K = dA\hat{}dA \neq 0$. Such an argument also implies that the Euler characteristic of the 4D domain is not zero. The topological parity 4 form has an exact preimage, defined as the Topological Torsion 3-form, $H = A\hat{}dA$, such that $K = dH$. The Topological Torsion must be non-zero if the Topological Parity is to be non-zero. A non-zero value for the 3-form of Topological Torsion implies that the topology induced by the 1-form of Action (representing the details of the physical system) is a *disconnected* topological defect in the turbulent state. However a disconnected topology is not sufficient to produce turbulence. It is now appreciated (2003) that Pfaff dimension 3 and 4 implies that the physical system is not in thermodynamic equilibrium (which imposes the constraint that the Pfaff dimension of A is 2 or less

As all researchers in hydrodynamics will agree that irreversibility is a key feature of a turbulent flow, it follows that the three concepts are necessary conditions for thermodynamic irreversibility and, therefor, turbulent flow:

1. The Euler characteristic of a space-time domain is not zero.

2. The Topological Parity of the space-time domain is not zero.

3. The domain is of Pfaff dimension 4 ($H \neq 0,\ K \neq 0$).

Hydrodynamic flows can be laminar and streamline (associated with a Pfaff dimension 2), reversibly chaotic (associated with a Pfaff dimension 3) and irreversibly turbulent (associated with Pfaff dimension ≥ 4). For a physical system encoded by a 1-form of Action, A,on a 4 dimensional variety, $\{x, y, z, t\}$,a non-zero Euler characteristic and a Pfaff dimension 4 all require that the 4-form $K = dA\hat{}dA$, defined as Topological Parity, cannot vanish. Hence there is an intimate correspondence between the three necessary conditions for for turbulence given above, and the irreversible properties of a turbulent flow. The details of that correspondence make up the bulk of this article.

Recall that a given variety $\{x, y, z, t\}$ can support many topologies, and the question arises as to the source of a physically useful topology. A key premise of this article is that the very existence of a differential 1-form, representing a physical system, on a 4D variety $\{x, y, z, t\}$ generates a useful topology on the set [2]. The Topological Parity and hence the Euler characteristic of that induced topology may be computed in several different ways. For hydrodynamic purposes, an evolutionary vector direction field will be taken to be the flow field defined by some solution to the Navier-Stokes partial differential equations. It will be demonstrated below how to chose an appropriate 1-form of Action that will generate the Navier-Stokes partial differential equations. In addition the properties of a solution to the Navier-Stokes equations which can be thermodynamically irreversible and therefor turbulent, will be demonstrated. This result lays to rest the conjecture that the Navier-Stokes equations may not be capable of describing turbulent flows.

As any 1-form of Action may be used to generate a line bundle, and a connection, then the classical methods of differential geometry also may be used to compute the associated adjoint curvature, and ultimately the Euler characteristic. However, certain topological and intrinsic properties for a constrained variety can be more simply evaluated using the methods of Cartan's exterior calculus based upon the use of the Lie derivative acting on differential forms.

These simpler methods do not depend *explicitly* on a metric or a connection, or the now fashionable methods of fiber bundles. As it is the Cartan's simpler methods which are exploited in this article, and as it is anticipated that many researchers in hydrodynamics do not yet have much experience with the Cartan connection techniques, only an abreviated version of the Cartan approach to fiber bundles and connections is presented in Appendix A. It is important to point out, however, that the fiber bundle methods are of utility to researchers in the theory of defects, for it is possible to develop a projective theory of defects that incudes both dislocations and disclinations on an equal footing. Surprisly both species of defects can occur in fluids [31],[32]. In particular it may be shown that the usual contravariant affine treatments cannot include the shears of rotation that often dominate hydrodynamics. The present author has long been interested in the production and destruction of defects, and other evidences of topological evolution, in hydrodynamics [30]. This interest stems from the realization [23][24] that dissipative irreversible processes must be associated with a change in topology from the initial to final state.

It is assumed that evolutionary processes can be encoded in terms of contravariant vector fields, $\mathbf{V}$, and physical systems permit description in terms of a 1-form of Action, A, with covariant coefficients. The evolutionary processes on a variety $\{x, y, z, t\}$ can be put into four general equivalence classes, each class having different topological properties. These topological properties will be defined in terms of the Pfaff class, or Pfaff dimension, of the 1-form A, and the induced 1-forms, $W = i(\mathbf{V})dA$. The idea is that an exterior differential form of a given equivalence class of Pfaff dimension, n, requires a minimum of n functions, and their derivatives, for its description. The concept is at the basis of the Darboux reduction theorems, and the theory of Clebsch functions. For a four dimensional variety, the Pfaff dimension of a domain is either 1, 2, 3 or 4. Only 1-forms, A, that belong to the equivalence class of Pfaff dimension 4 will induce a non-zero Gauss curvature on the four dimensional variety, $\{x, y, z, t\}$. This equivalence class is characterized by the non-null property defined as Topological Parity. For 1-forms of this equivalence class, there does not exist a mapping to a variety of less than four functions which will adequately describe the topological properties generated by the 1-from. In particular, if the integral of the topological parity is non-zero, then all such contravariant vector fields that can exist on a manifole of Pfaff dimension 4 must have singularities and self-intersections. This conclusion implies that an unambiguous retrace of the lines generated by the vector field is impossible, hence the vector field is not reversible past the singularity or self-intersection. Moreover it can be demonstrated that Pfaff dimension 4 is a necessary criteria for continuous thermodynamic processes.

Note that the classic representation of a vector field in terms of three Clebsch functions does not produce a vector field of Pfaff dimension 4, but can be a vector field of Pfaff dimension 3. The argument of this article is that vector fields of Pfaff dimension 3 on a space time variety can be chaotic, but not turbulent. The objective of this article is to classify the vector field solutions to the Navier-Stokes equations, and in particular to determine criteria when such flows are of Pfaff dimension 4. Pfaff dimension 4 is a necessary requirement for intrinsic thermodynamic irreversibility, and therefore a necessary requirement for the existence of the turbulent process.

It is not generally appreciated that the existence of a either a contravartaint or covariant global vector field induces a topology on a variety. In particular, the methods of how to evaluate the Euler characteristic of the induced topology get bogged down by historical techniques that rely heavily on coordinate geometry, and tensor analysis. Fortunately, a simpler method can be developed in terms of Cartan's calculus of exterior differential forms, a method which is not only coordinate free, but does not depend explicitly on a metric or connection constraint imposed upon the variety. In fact, the existence of a global 1-form, A, on the variety $\{x^1, x^2, ..., x^n, s\}$ may be used to construct a basis frame, and a set of compatible connection coefficients on the variety, for which the classical methods may be used to deduce the Gaussian curvature and its integral. Preliminary portions of such methods are described briefly in Appendix A. However, if the 1-form A (with co-variant coefficients) is "renormalized" (or as Chern says, projectivized) then the components of the 1-form may be used to define an adjoint surface of N dimensions in an N+1 dimensional space. The important geometrical quantities, including the Gaussian curvature, of this adjoint N dimensional subspace, are (by the Cayley-Hamilton theorem) determined by the similarity invariant functions of the Jacobian matrix, constructed from the first partial derivatives of the components of the normalized covariant vector field, A. For N=2, this Jacobian matrix is equivalent to the shape matrix of classical surface theory. The mean curvature of this 2D subspace is related to the trace of the Jacobian, or more vividly, is equal to the divergence of the projectivized adjoint vector field. A variational principle on the N+1 space produces a minimal surface in the space of N dimensions, when the divergence of the normalized adjoint field is set to zero. For N=2, the Gauss curvature is related to the determinant of the Jacobian matrix, and its integral yields the Euler characteristic.

The method can be extended to higher dimensional varieties, $\{x^1, x^2, ..., x^n, s\}$. The existence of a global Cartan 1-form of Action, A, can also be used to construct a line bundle [Chern, 1957] over the base, $\{x, y, z, t\}$. Hence, following Chern, a Top Pfaffian may be constructed from algebraic products of the matrix of curvature 2-forms. Chern has proved [Chern, 1954] that the integral of this form yields the Euler characteristic. The matrix of curvature 2-forms is computed classically by differentiating a set of connection 1-forms which constrain the variety and the derivatives of the vector field. However, comparing the projective line bundle and Cartan's methods, it may be shown that the matrix of curvature 2-forms can be computed in an alternate algebraic way that does not depend upon a second differentiation of the vector field. These methods are given outlined in Appendix A. The result is that for the normalized 1-form, A, and its exterior derivatives, $F = dA$, the forms, F and $K = F^\wedge F$, define the elements of the first and second Chern classes for this line bundle. For a four dimensional variety, the Top Pfaffian is equal to $F^\wedge F$, and its integral is associated with the Euler characteristic induced by the 1-form A. As $F^\wedge F$ must not be zero if the Euler characteristic is not to vanish, it follows that the vector field must be of Pfaff dimension 4. In this article, the complex machinery of fiber bundles is not needed, for the Cartan procedure is much more easy to use in engineering applications. An unusual result discovered by Chern and Simons [8] indicates that the integral of $K = F^\wedge F$ over the domain is related by Stokes theorem to the integral of the three form $A\hat{}dA$ over the boundary of the integration domain.

If the top Pfaffian is equal to zero everywhere, then the associated contravariant vector field, interpreted as a hydrodynamic flow vector field, is without self intersections and therefore is unambiguously reversible; i.e., not turbulent. For hydrodynamics, the problem is to determine simple expressions which classify those vector fields that are solutions to the Navier-Stokes equations. The final result is that the Top Pfaffian, or Topological Parity, for flow fields constrained to be solutions of the Navier-Stokes equations, is equal to

$$K = -2v(\boldsymbol{\omega} \cdot curl\, \boldsymbol{\omega})dx^\wedge dy^\wedge dz^\wedge dt. \tag{1.1}$$

When $K = 0$, the Euler characteristic vanishes (at least over compact oriented 4D manifolds) and it follows that such flows are not turbulent. As $\boldsymbol{\omega} = curl\, \mathbf{v}$ is the vorticity of the fluid flow, and v the kinematic viscosity, it follows that Navier-Stokes solutions for which the vorticity vector field is completely integrable in the sense of Frobenious can never be turbulent. It is to be remarked that if the enstrophy of the flow has a gradient orthogonal to the vorticity, then the integrable vorticity field is proportional to the normal field of a minimal surface [33]. This minimal surface, like a Julia set, can bound an interior in which the velocity field is chaotic.

The four-form of topological parity, K is always exact, and can be generated from the 3-form, $H = A^\wedge dA$,defined as the topological torsion. On the set $\{x, y, z, t\}$ generates a contravariant vector field of four components (defined herein as the torsion current $\{\mathbf{T}, h\}$) that may or may not have a zero divergence. The exploitation of this vector field in hydrodynamics is almost non-existent. Mathematicians have known about this vector field and its applications to problems where the solutions to systems of differential equations are not unique for more than 100 years [12].

In this article, this topological perspective is given further credence by developing a theory of topological evolution, in which it may be demonstrated that the Navier-Stokes equations form a subset of necessary conditions equivalent to constraints that refine the Cartan topology induced by the global 1-form, A. The point of departure is to recognize that the usual kinematic statement of rigid body dynamics, $\Delta\mathbf{x} = d\mathbf{x} - \mathbf{v}dt = 0$, is a topological constraint on a domain that need not be true for the evolution of deformable media. The non-zero values of $\Delta\mathbf{x}$ will be defined as deformation-fluctuations of position. Higher order fluctuations also may be defined. In this article, the space time variety of dimension 4 will be prolonged to higher dimensions to accommodate the possible fluctuations. Then, following Cartan's method, which is equivalent to a variational principle, certain topological constraints, not equivalent to the null fluctuation constraints of rigid body dynamics, can be placed on this higher dimensional space. Finally, these results can be pulled back to space-time by functional substitution, yielding the Navier-Stokes equations as necessary conditions to satisfy the topological constraints. For certain classes of solutions, the resultant geometry is non-Riemannian, but contains many of the features developed by Finsler [6] and Cartan for spaces that support torsion, and non-integrable vector fields.

The two key results of this article are given by engineering formulas that represent the Topological Torsion 3-form, $H = A\hat{}F$, as axial vector current of four components on the domain of space-time, and the 4-form of Topological Parity, K, as a density distribution on the space-time variety, $\{x, y, z, t\}$. The axial vector current has been completely ignored by both experimental and theoretical hydrodynamicists, perhaps because it is evanescent in a deterministic, non-chaotic, integrable regime. If the Pfaff dimension of the domain is less than 4, then the axial vector torsion current satisfies a conservation law, the Topological Parity 4-form vanishes, and the Euler characteristic of the induced Cartan topology is zero. The flow is reversible, not turbulent, but perhaps describes a chaotic process. If the Pfaff dimension is less than 3, then both the 3-form of Topological Torsion and the 4-form of topological parity vanish, and the flow field admits a deterministic and non-chaotic realization. Reversing the argument, in the chaotic regime, the axial current vector cannot be zero. In order for the flow field to be irreversible on the variety $\{x, y, z, t\}$, it is necessary (but not sufficient) that the Pfaff dimension be 4.

In section 2, the features of the Cartan method and the topology induced by a systems of differential forms are described. The concept of Pfaff dimension is used to define equivalence classes of vector fields.. In section 3, the abstract Finsler space and its associated global Cartan-Hilbert 1-form of Action (constructed on a ten dimensional prolongation of the original space in order to accommodate possible kinematic and dynamical fluctuations and dissipation) is described. In section 4, equations capable of describing topological evolution and irreversible phenomena in a non-statistical manner are constructed in terms of the Lie derivative acting on a system of differential forms generated from the Cartan-Hilbert global 1-form of Action. In section 5 it is demonstrated that the dissipative irreversible Navier-Stokes equations are deduced by applying topological constraints to the general equations of topological evolution generated in section 4. In addition, the topological constraints that yield the Navier-Stokes equations are used to determine a necessary criteria for the turbulent state. In Appendix A a the Cartan method for constructing a Line Bundle technology is displayed. In Appendix B the Cartan method is utilized to describe an electromagnetic system.

2. The Euler Index by the Cartan Method

The Euler index is one of the key topological properties of a domain. For example in two-space, the Euler index of a sphere is 2, the Euler index of a torus is 0, and the Euler index of a two hole button is -2. A vector field can exist globally without singularities (or zeros) only on a space of zero Euler index. The Euler index on space-time depends on the integral of a certain 4-form defined to be the Topological Parity (see below).

By studying the neighborhoods of "singular points" of a vector field on a variety, certain features of the underlying topology may be determined. The vector field "diverges" in these neighborhoods in ways that yield topological information. The vector field on a sphere must have two of these "singular" neighborhoods, that can be shrunk to "points". The vector field on a torus (Euler index = 0) can be without singularities. The idea that divV = 0 in the interior of a domain means that the vector lines associated with vector field can never stop or start, except perhaps at "boundary points". These concepts have been used by hydrodynamicists to study certain features of flow separation on airfoils [51]. The general question is: What is the topology induced by those vector fields that represent fluid flows on a domain of space time? In particular, the thesis to be examined herein is that as a turbulent flow is irreversible, it induces a non zero Euler index on a domain of space-time, and this fact implies that the Pfaff dimension of the domain is 4. If the Pfaff dimension is less than 4, Parity and Time-reversal symmetry [Kiehn, 1991c] are not broken, and the system is reversible.

Although the singularities of contravariant vector fields can be used to characterize the induced topology on the variety $\{x, y, z, t\}$, there are some problems that are better handled in terms of covariant vector fields. To quote Arnold:

"The notion of the index of a singular point of a vector field does not generalize directly to the case of manifolds with boundary. However, it turns out that a generalization to the case of boundary singularities becomes possible if vector fields are replaced by differential forms. For a 1-form on a manifold with boundary we define singular points and their indices so that the sum of the indices of all (interior and boundary) singular points of any 1-form with isolated singularities on a compact manifold with boundary is equal to Euler index of the manifold." [1] The idea is that a globally defined 1-form, A, induces a topology on a variety, and its singularities may be used to compute the topological property called the Euler index. This statement by Arnold, and the significance of singularities as stagnation and separation points in fluid flows, is the raison d'être for a development of hydrodynamics in terms of Cartan's exterior differential calculus [11]

In this article the principle object of interest is the Cartan 1-form of Action, A, defined with its singularities globally over a domain of a space-time variety, $\{x, y, z, t\}$:

$$A = A_\mu dx^\mu. \tag{2.1}$$

The functional properties of this 1-form may be used to construct a topology on the set $\{x, y, z, t\}$. Guided by the fact that a differential constraint, $xdx+ydy+zdz = 0$, imposed on the 3 dimensional variety leads to the 2 dimensional topology of a sphere, this 1-form, A, on space time will be prolonged to a space of higher dimensions. Topological constraints on the higher dimensional space will then be used to deduce the Navier-Stokes equations as a set of necessary conditions on the prolonged 1-form. Hence, solutions to the Navier-Stokes equations fall into classes that will generate the Cartan topology.

Suppose that such a 1-form, A, is given. Then, the exterior derivative of A produces a 2-form, $F = dA$, with components,

$$F = dA = F_{\mu\nu} dx^\mu \hat{} dx^\nu \ . \tag{2.2}$$

The 1-form, A, and the 2-form, $F = dA$, form a "closed" exterior differential system, $\{A, F\}$. Using these two elements, a sequence of higher order sets can be constructed algebraically by forming all possible exterior products of A and F: The resulting system of forms is defined as the Pfaff sequence: $\{A, F = dA, H = A\hat{}dA, K = dA\hat{}dA...\}$.

On space-time of 4 dimensions, there are only 4 possibilities for the Pfaff sequence, and these sets are defined as:

TOPOLOGICAL ACTION $A = A_\mu dx^\mu$
TOPOLOGICAL VORTICITY $F = dA = F_{\mu\nu} dx^\mu\hat{}dx^\nu$
TOPOLOGICAL TORSION $H = A\hat{}dA = H_{\mu\nu\rho} dx^\mu\hat{}dx^\nu\hat{}dx^\rho$
TOPOLOGICAL PARITY $K = dA\hat{}dA = K_{\mu\nu\rho\sigma} dx^\mu\hat{}dx^\nu\hat{}dx^\rho\hat{}dx^\sigma$

The largest non-null element of the Pfaff sequence defines the Pfaff dimension of the domain. Certain flows have Pfaff dimension 1; others, Pfaff dimension 2, 3, and 4. Examples are given in [27].

The domain of support of each element of the Pfaff sequence may be considered as a "point" of what will be called the Cartan topology. In this sense the "point", A, and its closure, $A \cup F$ may be used as base elements to define "open sets" of the Cartan topology. If the domain is such that H and K are null, then that is the end of the construction, and the Cartan topology is a connected topology. However, suppose that there exists a domain of support for the exterior product of A and F, such that H is not empty. Then construct the closure of H as the union, $H \cup K$, and use these 4 "points", $\{A, A \cup F, H, H \cup K\}$, as a basis of open sets for the Cartan topology. The resulting topology is NOT connected! [2] This choice for a topology is extraordinary in that the Cartan exterior derivative may be interpreted as a "limit point operator" relative to the Cartan topology. Given any set of the topology, if the exterior derivative vanishes, then the set has no limit points. In this sense, the vorticity field, as the exterior derivative of the covariant velocity field, generates the limit sets for the covariant velocity field (relative to the Cartan topology). For currents constructed in terms of contravariant vector fields (or N-1 forms), then the vanishing of the exterior derivative (equivalent to the vanishing of the divergence of the contravariant vector field) implies that no limit points for these currents exist within the domain of support. If the divergence vanishes, then the associated axial vector current satisfies a conservation law. The "lines" associated with this vector field do not stop or start within the domain.

It is of fundamental interest to determine the value of the Euler index for this induced Cartan topology, as the Euler index not only is one of most fundamental of topological invariants, it is also crucial in determining if a flow field is unambiguously reversible. Chern has given an approach to evaluating the Euler index [13] in terms of an integral over a certain 2N form which he calls the "Top Pfaffian". Also see Appendix A. On the 4-dimensional variety $\{x, y, z, t\}$, this "Top Pfaffian" is proportional to 4-form of Topological Parity, $K = dA\hat{}dA$. Hence, for those domains where $K = 0$, an immediate result is that the Euler index must vanish. According to the definition, the Pfaff dimension is less than 4 on domains for which $K = 0$.

Most of the known closed form solutions to dynamical systems are to be associated with domains for which $K = 0$. In fact, most known global solutions are of Pfaff dimension 2, which implies that both H and K are null. If the Cartan-Pfaff 1-form, A, is completely integrable in the sense of Frobenius, then $H = A\hat{}dA = 0, K = 0$ and the Euler index for the space-time domain is zero. Complete integrability means that there exists a unique function which for constant values forms a surface through which the trajectories of the associated vector field pass transversely. This means that the points on neighboring trajectories are synchronizable globally over the whole domain. Such vector fields can admit vorticity, but the vorticity must be orthogonal to the velocity at each point. The Topological Torsion, H, must vanish for a completely integrable 1-form.

Another result of Frobenius complete integrability (Pfaff dimension 2 or less) is that neighboring points are isolated in the Caratheodory sense that they reside on the equivalent of a thermodynamic equilibrium hypersurface [15]. In the hydrodynamic case, a completely integrable action always can be mapped to a space of at most two dimensions, and this representation is the epitome of a laminar (to be in layers) flow. If the system does not satisfy the Frobenius integrability theorem, then there may exist local domains that are laminar, but they are isolated. Examples of such systems with Pfaff dimension 3, that support Topological Torsion, are presented in reference [Kiehn, 1991a]. In this article, systems of non-zero K are of interest, and it follows that such systems necessarily support Topological Torsion, H.

If a 1-form of Action is completely integrable in the sense of Frobenius, then the associated vector field never generates chaotic trajectories [49]. A completely integrable vector field always admits a smooth global transverse hypersurface such that there does not exist "an extreme sensitivity" to initial conditions for neighboring trajectories; the initial conditions, and the subsequent evolution of these points, reside on a sequence of globally transverse hypersurfaces. On the other hand, if the Frobenius integrability condition is not satisfied, and suppose that the Pfaff dimension is 3, then there exists a global submersion to a space of 3 dimensions, and a unique smooth transversal hypersurface does not exist over the whole domain. Although there can exist local sets of hypersurfaces that connect local neighborhoods of trajectories, there exist "nearby" trajectories which cannot be so connected. Points on these nearby, but not connected, trajectories may exhibit that "extreme sensitivity" to initial conditions, that has become the definition of deterministic chaos. Points that start out in a nearby domain will not necessarily evolve into a transversely connected set. In fact, this is one of the obvious results of the Cartan topology, for when H and K are zero it is easy to show that the Cartan topology is connected, but when H and K are not zero, the Cartan topology is disconnected [2].

It follows that a necessary condition for a chaotic velocity field is that the Pfaff dimension of the 1-form of Action shall be greater than 2. But chaos is not the same as turbulence, for the chaotic flow is reversible along each nearby, but perhaps disconnected, space time trajectory, without ambiguity: the chaotic trajectories never intersect in space time. On the other hand, a turbulent flow is presumed to be dissipative and irreversible. If the flow lines intersect in space time, it is impossible to reverse the evolution unambiguously (uniquely) through the obstruction, or bifurcation singularity; such flows are defined to be irreversible. In order for the flow lines to have irreducible self- intersections in space time, it is necessary that the 4-form of topological parity, K, be non-zero. A non-zero Euler index is a signal that the vector field admits such singularities, and the associated evolution is therefore irreversible. It follows that a necessary condition for irreversibility is that the Pfaff dimension of the domain be 4. The conclusion reached is that reversible chaos is of Pfaff dimension 3, and can be distinguished from turbulence, which is irreversible and must be of Pfaff dimension 4.

These ideas support the definition that the turbulent state must be irreducibly time dependent, irreducibly 3 dimensional in space, and dissipatively irreversible. Although time-dependent two dimensional flows can exhibit chaos, they can never be turbulent, for they are of maximum Pfaff dimension 3. At least 4 parameters are required to produce a domain with Pfaff dimension 4. It is the opinion of this author that the concept of "two" dimensional turbulence is not logical. The Topological Parity must not vanish globally in the turbulent state, a fact that would preclude the concept of turbulence on a two space-one time dimensional domain.

For this author the appreciation that Topological Torsion and Topological Parity are important physical concepts came first from applications of the Cartan technique to problems in electromagnetism. An abstract example of the Cartan technique for electromagnetic systems is given in Appendix B, where the 1-form of Action, A, is based on the electromagnetic potentials. A most influential result of this analysis is the fact that the Topological Parity 4-form for a Maxwell system is given by the expression,

$$K = -2\mathbf{E} \circ \mathbf{B} dx^\wedge dy^\wedge dz^\wedge dt. \tag{2.3}$$

In Lagrangian field theories of microscopic elementary particles, non-zero values of $\mathbf{E} \circ \mathbf{B}$ are signals that parity and time reversal symmetries are not preserved [4]. Examples of Parity and Time-Reversal symmetry breaking at a macroscopic level in electromagnetic systems are presented in [Kiehn, 1991]. These results imply that the associated evolutionary processes are not reversible, and that irreversibility is a topological idea independent from size and scales. The ideas led this author to conjecture that similar properties are to be associated with irreversible, dissipative, turbulent hydrodynamic flows. A first step in this direction was the suggestion that the transition to turbulence involved a topological transition from a laminar state without Topological Torsion, $H = A\hat{}dA = 0$, to a state where $H \neq 0$. This transition was expressed by saying that the flow system went from a state that was completely integrable, in the sense of Frobenius, to a system that was not integrable [Kiehn 1976a]. At that time the possibility of an intermediate non-integrable chaotic state of Pfaff dimension 3 occuring before a irreversible, irreducibly time-dependent and three dimensional non-integrable turbulent state of Pfaff dimension 4 was not appreciated. If the Pfaff dimension is less than 4, the Euler index relative to the Cartan topology must vanish on space time.

The creation of turbulence from a laminar state must involve the topological evolution from a connected Cartan topology (Pfaff dimension < 3) to a disconnected Cartan topology (Pfaff dimension > 2). Relative to the Cartan topology, it may be shown that all C2 functional descriptions of vector fields are continuous. As a consequence it is possible to state that the creation of a turbulent state cannot be accomplished by a continuous transformation, but the decay of turbulence is amenable to analysis by means of continuous but irreversible transformation [30]. The implication is that numerical simulations which impose strict continuity conditions on the evolution are not capable of describing the creation of turbulence. On the other hand, continuous numerical simulations can describe the decay of turbulence.

In this article, the technique will be to define a 1-form of action, A, similar to that variational integrand proposed by Finsler [6] and Cartan in their studies of non-Riemannian geometries that admit torsion. The Finsler methods motivated this author to treat the general problem of topological evolution as an extremal problem on a variety of higher dimensions. In effect, the classic kinematic constraints of rigid body motion are considered to be "overly severe" topological constraints on fluid motion, and are relaxed such that the new topology admits "fluctuations". The fluctuations can be interpreted in many ways; in particular they can represent deviations from the classic kinematic constraints, $d\mathbf{x} - \mathbf{v}dt = 0$. These deviations may be interpreted as variations of the initial conditions, or they may be interpreted as an uncertainty of the origin, but for this author they do not imply that a statistical analysis is necessary. A refinement, or specialization, of this "fluctuation" topology leads to the set of necessary partial differential equations which are recognized as the Navier-Stokes equations for a compressible, viscous flow. Of particular importance is the recognition that the dynamics of a deformable system are to be associated with a field of axial "current", whose 4 components form the 3-form of Topological Torsion on (x, y, z, t). This axial-vector current is a completely anti-symmetric third rank tensor field, and is evanescent in rigid body systems that satisfy the topological constraints of a perfect kinematics. The divergence of this axial current may or may not vanish for deformable systems, and it may or may not be an evolutionary invariant. When the divergence is anomalous, the Torsion Current can stop or start in the interior, thereby generating a topological defect in the domain.

3. The Fluctuation 1-form

The format of the Cartan 1-form, A, will be that of the Cartan-Hilbert invariant integrand,

$$A = L(\mathbf{x}, t; \mathbf{v})dt + \mathbf{p} \cdot (d\mathbf{x} - \mathbf{v}dt). \tag{3.1}$$

Note that the original space-time, $\{\mathbf{x}, t\}$ has been extended to a 10 dimension space of functions, $\{\mathbf{x}, t; \mathbf{v}, \mathbf{p}\ \}$. On this set, it is convenient to define a vector field, V, with 10 components given by the functions, $\{\mathbf{v}, 1; \mathbf{a}, \mathbf{f}\}$. Differential and functional constraints will be imposed on this 10 dimension space thereby defining a topology. Further note that the Cartan-Hilbert action involves a classic Lagrange function, $L(\mathbf{x}, t; \mathbf{v})$, and a linear combination of non-zero position "fluctuation or deformation" 1-forms, defined as:

$$\Delta\mathbf{x} = d\mathbf{x} - \mathbf{v}dt \neq 0\ . \tag{3.2}$$

The covariant array, $\mathbf{p}$, of coefficients of the fluctuation 1-forms in 3.1 may be described as set of Lagrange multipliers. It will be demonstrated below that this covariant field, $\mathbf{p}$, dual to the contravariant velocity field, $\mathbf{v}$, plays the role of the canonical momentum, when subjected to additional, but classic, constraints that are equivalent to the constraint of zero temperature.

The 1-forms, $\Delta\mathbf{x}$, are defined as fluctuation-deformation 1-forms for they represent deviations from the pure kinematic point of view associated with a rigid body dynamics or the evolution of a point particle in terms of a single parameter group of transformations. Although not always true, these deviations are often small corrections to the kinematic constraints, $\Delta \mathbf{x} = 0$, and have the appearance of "fluctuations" about the "kinematic" lines that act as guiding centers for the evolution.

The fluctuation 1-forms given by 3.2 are not necessarily zero for deformable media. If it is assumed that the "points", x, evolve in terms of a map, φ, from a set of initial conditions, y, then the differentiable map

$$x^k = \varphi^k(y^m, t), \tag{3.3}$$
$$dx^k = d\varphi^k(y^m, t) = (\partial\varphi^k/\partial y^m)dx^m + (\partial\varphi^k/\partial t)dt \tag{3.4}$$

leads to the differential expression,

$$\Delta x^k = dx^k - v^k dt = dx^k - (\partial\varphi^k/\partial t)dt = (\partial\varphi^k/\partial y^m)dx^m. \tag{3.5}$$

If the parameters, y^m, which could be interpreted either as initial conditions or as the coordinates of the origin, are not constants, then the RHS of 3.5 is not zero, and the system does not evolve according to the kinematic rules associated with a single parameter group. The basic idea is that the statement, $dx^k - v^k dt = 0$, must be interpreted as a topological constraint, just as the statement $xdx+ydy+zdz = 0$ is a topological constraint on Euclidean 3-space that produces the topology of a spherical surface.

In classical hydrodynamics, the non-zero fluctuations given by 3.5 are usually constrained by topological conditions such that their associated 3-form admits an integrating factor, ρ. This topological constraint means that there must exist a function, $\rho = \rho(x, y, z, t)$ such that the non-zero 3-form,

$$\Omega = \rho(dx - v^x dt)\hat{}(dy - v^y dt)\hat{}(dz - v^z dt) \tag{3.6}$$

has a vanishing exterior derivative,

$$d\Omega = \{div(\rho\mathbf{v}) + \partial\rho/\partial t\}dx\hat{}dy\hat{}dz\hat{}dt = 0. \tag{3.7}$$

This topological constraint is usually called the "equation of continuity" for deformable media. It may be shown that this topological constraint makes Ω an absolute invariant of the evolution. If the flow lines are retraceable, implying that the Jacobian determinant of the assumed mapping is of rank 3, then the topological constraint may be interpreted as the "conservation of mass". However, it is not apparent that nature always insists on the assumed topological constraint among the fluctuation-deformation 1-forms. Such a constraint is a matter for test, especially in the case of a turbulent, irreversible, evolutionary process.

The Cartan 1-form will be used not only to generate the Cartan topology, but also to generate, by means of a procedure equivalent to a variational principle, a set of partial differential equations of evolution with solution vector fields, $\mathbf{V}$. The continuous smooth curves tangent to the vector fields, $\mathbf{V}$, in the higher dimensional geometry of the partial differential system may pullback, or intersect, or project, discontinuously in the lower dimensional geometry. The ordinary kinematic differential equations based on $\mathbf{v}$, without fluctuations, yield solution curves that act as "guiding centers" for the fluctuation fields, $\mathbf{V}$, in the limit that the fluctuations are small. The projections of the continuous curves in the geometry of the higher dimensional space may have gaps and tangential discontinuities on space-time. The discontinuities would be interpreted as defects or fluctuations in an otherwise homogeneous and continuous system. These ideas may be compared to the concept of Poincare sections in the theory of non-linear dynamics. The Cartan method permits the concepts of discontinuous fluctuations to be put on a continuous basis in a space of higher dimension. This topological idea of removing apparent discontinuities by embedding in a space of higher dimensions is similar to the geometric idea where a curved space may be embedded in a higher dimensional euclidean flat space.

Physicists often recognize the Cartan Action in the format,

$$\begin{aligned} A &= L(\mathbf{x},t;\mathbf{v})dt + \mathbf{p}\cdot(d\mathbf{x}-\mathbf{v}dt) \\ &= \mathbf{p}\cdot d\mathbf{x} - (\mathbf{p}\cdot\mathbf{v} - L(\mathbf{x},t;\mathbf{v}))dt \\ &= \mathbf{p}\cdot d\mathbf{x} - \mathcal{H}(\mathbf{x},t;\mathbf{v},\mathbf{p})dt, \end{aligned} \tag{3.8}$$

but do not seem to appreciate that this composition may be interpreted in terms of a fluctuation geometry on a space of 10 dimensions, as given by 3.1. In current physical theories, it is often assumed that the function, $\mathcal{H}(\mathbf{x},t;\mathbf{v},\mathbf{p})$, can be written entirely in terms of the variables $(\mathbf{x},t;\mathbf{p})$ alone. In such cases the function $\mathcal{H}(\mathbf{x},t;\mathbf{p})$ becomes the Hamiltonian function of classical mechanics, and the Lagrange multipliers, $\mathbf{p}$, would be identified as the canonical momentum. This assumption corresponds to a functional relationship or constraint between the variables $\mathbf{v}$ and $\mathbf{p}$ such that $\partial\mathcal{H}(\mathbf{x},t;\mathbf{v},\mathbf{p})/\partial\mathbf{v} = 0$. If the relationship is linear, then there would exist a constitutive or metrical relationship between the dual fields, $\mathbf{v}$ and $\mathbf{p}$. Such assumptions are NOT made a priori in this article.

Consider first a Cartan 1-form of action where the fluctuations vanish over a domain. Then the 2-form of limit points is $F = dA = dL\hat{}dt$. It follows that $H = A\hat{}dA = 0$, and $K = 0$. The Pfaff dimension of such systems is 2 at most. Such systems can have vorticity but are without helicity, or Topological Torsion. However, in this article, systems of non-zero H and non-zero K are of interest. Examples of systems that do support Topological Torsion are presented in reference [27]. From this point of view, both Topological Torsion and Topological Parity are to be associated with the concept of non-null kinematic fluctuations which are not transversal to the system momentum, $\mathbf{p} \cdot \mathbf{\Delta x} \neq 0$.

When fluctuations are permitted, then the exterior derivative of the Cartan action on the 10 dimensional space becomes explicitly,

$$\begin{aligned} dA &= \{(\partial L/\partial \mathbf{v} - \mathbf{p}) \cdot d\mathbf{v}\}\hat{}dt + d\mathbf{p} \cdot \mathbf{\Delta x} + \{\partial L/\partial \mathbf{x} \cdot d\mathbf{x}\}\hat{}dt \\ &= \{(\partial L/\partial \mathbf{v} - \mathbf{p}) \cdot \Delta \mathbf{v}\}\hat{}dt + \mathbf{\Delta p} \cdot \mathbf{\Delta x} \end{aligned} \tag{3.9}$$

The term, $\Delta \mathbf{v}$, represents the non-zero 1-forms of velocity fluctuations, defined as,

$$\Delta \mathbf{v} = d\mathbf{v} - \mathbf{a}dt \neq 0, \tag{3.10}$$

and, $\mathbf{\Delta p}$, is defined as the non-zero 1-form of Lagrange multiplier fluctuations,

$$\mathbf{\Delta p} = d\mathbf{p} + \{\partial L/\partial \mathbf{x}\}dt \neq 0. \tag{3.11}$$

The functions, $\mathbf{a}$, are defined to be to the contravariant acceleration vector field (with velocity fluctuations) in the same extremal sense that $\mathbf{v}$ is defined as the contravariant velocity vector field (with position fluctuations).

The Hiesenberg like notation, $\mathbf{\Delta p} \cdot \mathbf{\Delta x}$, stands for the sum of 2-forms,

$$\mathbf{\Delta p} \cdot \mathbf{\Delta x} = \sum_k (dp_k + \{\partial L/\partial x^k\}dt)\hat{}(dx^k - v^k dt) \tag{3.12}$$

which is similar to the dot product of two vectors, but here the combinatorial action is through the exterior product, ^. Although closely related to an expectation value generated by an inner product, or to the integrand of a cross-correlation integral, no statistical or ensemble averaging of (9) is assumed in this article. The beauty of the Cartan analysis is that it is retrodictively deterministic and well defined in a pullback sense, even when unique, deterministic prediction is impossible [24].

The bracket factor, $(\partial L/\partial \mathbf{v}-\mathbf{p})=-\partial\mathcal{H}/\partial\mathbf{v}$ in equation 3.9 will be defined as the scaled covariant vector field, $\mathbf{k}/S$. The topological constraint $\mathbf{k}=0$ permits the Lagrange multipliers to be uniquely determined as the canonical momenta of classical mechanics, $\mathbf{p}=\partial L/\partial\mathbf{v}$. A direct computation of the Topological Torsion, H, on the 10 dimensional space yields,

$$H=A^{\wedge}dA=Ldt\hat{}(\Delta\mathbf{p}\hat{}\Delta\mathbf{x})+\{\mathbf{k}/S\cdot\Delta\mathbf{v})\hat{}(\mathbf{p}\cdot\Delta\mathbf{x})\hat{}dt, \tag{3.13}$$

which may be evaluated in principle on 4 dimensional space time by functional substitution.

A similar direct computation in the higher dimensional geometry of variables $\{\mathbf{x},t;\mathbf{v},\mathbf{p}\ \}$ of the exterior derivative, $K=dH$, produces a 4-form that also can be pulled back to $\{x,y,z,t\}$ by functional substitution. The Topological Parity 4-form becomes,

$$K=dA^{\wedge}dA=2\{(\partial L/\partial\mathbf{v}-\mathbf{p})\cdot\Delta\mathbf{v}\}\hat{}dt\hat{}(\boldsymbol{\Delta}\mathbf{p}\cdot\boldsymbol{\Delta}\mathbf{x})+(\boldsymbol{\Delta}\mathbf{p}\cdot\boldsymbol{\Delta}\mathbf{x})\hat{}(\boldsymbol{\Delta}\mathbf{p}\cdot\boldsymbol{\Delta}\mathbf{x})\} \tag{3.14}$$

On a space-time variety, this 4-form becomes Chern's top Pfaffian whose integral gives information concerning the Euler characteristic of space-time. It is apparent that K depends on the exterior product of the fluctuations of position, Lagrange multipliers, and velocity, as well as the bracket factor,$(\partial L/\partial\mathbf{v}-\mathbf{p})=-\partial\mathcal{H}/\partial\mathbf{v}$, and dt.

The classic first variation of the Action integral, $\int A$, is an extremal principle which for the action specified by 3.1 generates a Finsler geometry. According to Chern, the Finsler variation is equivalent to setting $dA=0$, mod $\boldsymbol{\Delta}\mathbf{x}$. In addition, for Finsler geometries, the Lagrange function is presumed to be homogeneous of degree 1 in the $\mathbf{v}$, and this constraint is used by Chern to construct a "projectivized" tangent bundle. The homogeneity condition implies that the variable, t, can be reparametrized, and the vector $\mathbf{v}$ forms the elements of a projective geometry.

In classical field theory, the Finsler constraint is often imposed arbitrarily:

$$\mathbf{k}/S=(\partial L/\partial\mathbf{v}-\mathbf{p})=-\partial\mathcal{H}/\partial\mathbf{v}=0. \tag{3.15}$$

As mentioned above, such a constraint uniquely defines the Lagrange multipliers, $\mathbf{p}$, as the components of the canonical momentum. The Topological Parity 4-form is then dependent on the exterior product of "fluctuations" in position and momentum only, and has the same physical dimensions as the square of Planck's constant. From a qualitative point of view, fluctuations in velocity correspond to the property of temperature.

4. Equations of Topological Evolution on a 4D space.

In the Cartan method, an evolutionary process relative to a vector field, $\mathbf{V}$, is described by the action of the Lie derivative on the p-forms of interest [50]. The action of the Lie derivative on 1-forms is equivalent to the "convective" derivative in Cartesian hydrodynamics, but the Lie derivative is defined without the geometric constraints of a metric or a connection. In the present article, the p-forms of evolutionary interest are those p-forms that make up the Pfaff sequence. If any p-form is invariant with respect to the evolutionary process, V, then its Lie derivative vanishes: $L_{(\mathbf{V})}A = 0$. If all p-forms that make up the topological base of the Cartan topology are invariant, then the topology is invariant, and the process is a homeomorphism: ($L_{(\mathbf{V})}A = 0$ and $L_{(\mathbf{V})}dA = 0$). Such processes are both continuous and reversible, and are to be ignored in this article.

A more general set of evolutionary processes that can admit topological evolution is described by the topological constraint that the limit sets, $dA = F$, are evolutionary invariants. No such constraint is placed on the evolution of A. Such processes are described by the statements:

$$L_{(\mathbf{V})}A = Q \neq 0, \quad L_{(\mathbf{V})}dA = dQ = 0. \tag{4.1}$$

In terms of the Cartan topology, such processes, or vector fields that satisfy 4.1, are said to be uniformly continuous. Direct computation on C2 functions indicates that 4.1 is equivalent to the constraint that 1-form of work, defined as $W = i(\mathbf{V})dA$, is closed (and therefore has no limit points with respect to the Cartan topology):

$$d(i(\mathbf{V})dA) = 0. \tag{4.2}$$

It may be demonstrated that in a hydrodynamic format such 4.1 is equivalent to Helmholtz' theorem of the conservation of vorticity [20] [53]. In an electordynamic format, 4.2 corresponds to the "master" equation for the "perfect plasma" of magneto-hydrodynamics:

$$\partial\mathbf{B}/\partial t = curl\,(\mathbf{v} \times \mathbf{B}). \tag{4.3}$$

This constraint of uniform continuity is satisfied by the more stringent sufficient condition,

$$i(\mathbf{V})dA = 0, \tag{4.4}$$

an equation of historical significance. For a given Lagrange Action, A, Cartan has demonstrated that the first variation of the Action integral is equivalent to the search for those vector fields, $\mathbf{V}$, that satisfy equation 4.4 given above. Such vector fields are called extremal vector fields [Klein 1962, Kiehn 1975]. Given A, the resultant equations deduced from 4.4 are a set of partial differential equations that represent extremal evolution, $\mathbf{V}$. Note that the extremal conditions are insensitive to any renormalization of the vectors, $\mathbf{V}$. That is, if $\mathbf{V}$ satisfies the equations, then $\rho\mathbf{V}$ also satisfies the equations. Such a result is commonplace in the projective geometry of lines, and does not require the Riemannian or euclidean concept of an inner product or a metric [45]. Cartan [1958] has shown that this projective extremal condition is necessary and sufficient for the dynamical system, $\mathbf{V}$, relative to the action, A, to be Hamiltonian [Kiehn 1974]. Such Hamiltonian systems are not dissipative, and guarantee the existence of a single parameter group for the velocity field generated by the solutions of the necessary partial differential equations that describe the topological constraint. Such evolutionary systems cannot describe the creation of turbulence.

However, on the fluctuation space of 10 dimensions, {x, t; v, p}, contraction of dA with the vector field, $\mathbf{V} = \{\mathbf{v}, 1; \mathbf{a}, \mathbf{f}\}$ explicitly yields,

$$i(\mathbf{V})dA = -\mathbf{F}\cdot(d\mathbf{x} - \ \mathbf{v}dt) - \mathbf{k}/S\cdot(d\mathbf{v} \ - \mathbf{a}\ dt) = -\mathbf{F}\cdot\Delta\mathbf{x} - \mathbf{k}/S\cdot\Delta\mathbf{v} \tag{4.5}$$

where $\mathbf{F} = \mathbf{f} - (-\partial L/\partial\mathbf{x})$ represents the dissipative components of the "force". The RHS of 4.5 is not necessarily zero, so such evolutionary processes are not necessarily Hamiltonian, but are candidates for a turbulent evolutionary process. The RHS of 4.5 depends explicitly upon the fluctuations in position and velocity, and is explicitly independent from fluctuations in momentum. The two covariant vector fields, $\mathbf{F} = i(\mathbf{V})\Delta\mathbf{p}$, and $\mathbf{k}/S$, can be identified with the irreversible dissipative mechanisms of friction and radiation in the system fluctuation dynamics. These dissipative terms are not included explicitly in the usual Lagrange theory, and represent fluctuation interactions with the environment. Note that the Finsler-Chern condition, $dA = 0$, mod $\mathbf{\Delta x}$, implies $\mathbf{k}/S = 0$, creating a constraint that on physical grounds implies that the system does not depend on velocity fluctuations (temperature). Further note that the Cartan-Hamilton constraint given by 4.4 is satisfied by several distinct classes of sufficient conditions, depending on choices for the four factors in the RHS of 4.5.

5. Thermodynamic Irreversibility, and the Torsion vector on a 4D variety

To understand what is meant by thermodynamic irreversibility, realize that Cartan's magic formula [43] of topological evolution is equivalent to the first law of thermodynamics.

$$\begin{aligned} \text{Cartan's Magic Formula } L_{(\mathbf{V})}A &= i(\mathbf{V})dA + d(i(\mathbf{V})A) && (11) \\ \text{First Law of Thermodynamics} &= W + dU = Q. && (5.1) \end{aligned}$$

A is the "Action" 1-form that describes the hydrodynamic system. $\mathbf{V}$ is the vector field that defines the evolutionary process. W is the 1-form of (virtual) work. Q is the 1-form of heat. From classical thermodynamics, a process is irreversible when the heat 1-form Q does not admit an integrating factor. From the Frobenius theorem, the lack of an integrating factor implies that $Q\hat{}dQ \neq 0$. Hence a simple test may be made for any process, $\mathbf{V}$, relative to a physical system described by an Action 1-form, A:

$$If \quad L_{(\mathbf{V})}A\hat{}L_{(\mathbf{V})}dA \neq 0 \; then\ the\ process\ V\ is\ irreversible.$$

This definition implies that symplectic processes are reversible (as symplectic processes are defined such that $L_{(\mathbf{S})}dA = dQ = 0$).

The Torsion vector, defined as

$$\text{Torsion Vector direction field } \; i(T)dx\hat{}dy\hat{}dz\hat{}dt = A\hat{}dA, \qquad (5.2)$$

is obviously related to the concept of Topological Torsion, $A\hat{}dA$. It is extraordinary that evolution in the direction of the torsion vector produces the equations of conformal invariance,

$$L_{(\mathbf{T})}A = \sigma A \quad and \quad i(\mathbf{T})A = 0, \qquad (12)$$

such that

$$L_{(\mathbf{T})}A\hat{}L_{(\mathbf{T})}dA = Q\hat{}dQ = \sigma^2 A\hat{}dA \neq 0. \qquad (13)$$

The coefficient σ is related to the non-zero 4 divergence of the Torsion vector, which is equal to the coefficient of the Topological Parity 4-form.

Hence it is demonstrated that a non-zero Topological Parity, yielding a tpological Pfaff dimension of 4, is a necessary condition the a process be thermodynamically irreversible.

6. The Navier Stokes equations

For the liquid state, guided by the physical idea that fluctuations in velocity are related to the concept of temperature, assume that the equations of topological evolution given by 4.5 are refined by the constraints,

$$\mathbf{k} \neq 0,\ \mathbf{k} \cdot \mathbf{\Delta v} = Sd(kT),\ \mathbf{F} = \upsilon\, curl\, curl\, \mathbf{v},\ \ \mathbf{F} \cdot \mathbf{\Delta v} \neq 0. \tag{6.1}$$

In this example, the constraint of canonical momentum ($\mathbf{k}/S = 0$) is relaxed to the alternate constraint that the deviation from canonical momentum, $\mathbf{k}/S$, be transversal to the fluctuations in velocity. That is, if the system is at constant temperature, then $d(kT) = 0$. The equations of motion 4.5 become

$$i(\mathbf{V})dA = -\upsilon\, curl\, curl\, \mathbf{v} \cdot \mathbf{\Delta x} + d(kT). \tag{6.2}$$

For the case of a fluid, define the Cartan fluctuation 1-form, A, as,

$$A = \mathbf{v}\cdot d\mathbf{x} - \mathcal{H}dt, \tag{6.3}$$

with the "barotropic Hamiltonian" function specified as,

$$\mathcal{H} = \mathbf{v} \cdot \mathbf{v}/2 + \int dP/\rho + \lambda div\, \mathbf{v}. \tag{6.4}$$

Substitution into 4.5 yields a necessary system of partial differential equations for the constrained topological evolution:

$$\partial\mathbf{v}/\partial t + grad(\mathbf{v} \circ \mathbf{v}/2) - \mathbf{v} \times curl\, \mathbf{v} = -gradP/\rho + \lambda div\, \mathbf{v} - \upsilon curl\, curl\, \mathbf{v}. \tag{6.5}$$

Equations 6.5 are exactly the Navier-Stokes partial differential equations for the evolution of a compressible viscous irreversible flowing fluid. In other words, the Navier-Stokes equations of hydrodynamics have been deduced by imposing a set of topological refinements on the general equations of topological evolution.

However, the vector field which is a solution to the partial differential system may not be the generator of a single parameter group of transformations, and thereby may exhibit fluctuations and tangential discontinuities when projected to space time.

For the Navier-Stokes system, and by direct computation, the 2-form $F = dA$ has components

$$F = dA = \omega_x dy\hat{}dz + \omega_y dz\hat{}dx + \omega_z dx\hat{}dy + (\mathbf{a}\cdot d\mathbf{x})\hat{}dt \tag{6.6}$$

where

$$\boldsymbol{\omega} = curl\,\mathbf{v} \quad \text{ and } \ \mathbf{a} = -\partial v/\partial t - grad\,\mathcal{H}. \tag{6.7}$$

The 3-form of Helicity or Topological Torsion, H, is constructed from the exterior product of A and dA as,

$$\begin{aligned} H &= A^\wedge dA = H_{ijk} dx^{i\wedge} dx^{j\wedge} dx^k && (6.8) \\ &= +T^x dy^\wedge dz^\wedge dt - T^y dz^\wedge dx^\wedge dt + T^z dx^\wedge dy^\wedge dt - h\ dx^\wedge dy^\wedge dz, && (6.9) \end{aligned}$$

where $\mathbf{T}$ is the fluidic Torsion axial vector current,

$$\mathbf{T} = (\mathbf{a} \times \mathbf{v}) + \mathcal{H}\boldsymbol{\omega} \tag{6.10}$$

and h is the torsion (helicity) density,

$$h\ = \mathbf{v}\cdot\boldsymbol{\omega}. \tag{6.11}$$

The torsion axial vector, $\mathbf{T}$, consists of two parts. The first term represents the shear of translational accelerations, $(\mathbf{a} \times \mathbf{v})$, and the second part represents the shear of rotational accelerations, $\mathcal{H}\boldsymbol{\omega}$. Tubular domains, or singular lines of $\mathbf{T}$, in a hydrodynamic system can be interpreted as "defect" structures that can be put into correspondence with dislocation and disclination defects in continuous media, depending on whether or not the field is dominated by translational or rotational shears. The topological torsion tensor, $H_{ijk} \equiv \{\mathbf{T}, h\}$, is a third rank completely anti-symmetric covariant tensor field, with four components on the variety $\{x, y, z, t\}$. It transforms as a third rank tensor field with respect to all diffeomorphisms on space-time, including the Galilean transformation.

The Navier-Stokes constraint given by 6.5 may be used to express the acceleration term, $\mathbf{a}$, kinematically; i.e.,

$$\mathbf{a} = -grad\mathcal{H} - \partial\mathbf{v}/\partial t = -\mathbf{v}\times curl\,\mathbf{v} + \boldsymbol{\upsilon}\times curl\,curl\,\mathbf{v}. \tag{6.12}$$

By substituting 6.12 into 6.10, the torsion axial vector current becomes expressible in terms of the helicity density, h, the Lagrangian function, L, and the viscosity as:

$$\mathbf{T} = \{h\mathbf{v} - L\,curl\,\mathbf{v}\} - \boldsymbol{\upsilon}\times curl\,curl\,\mathbf{v}, \tag{6.13}$$

$$with\ h = \mathbf{v}\cdot curl\,\mathbf{v} \tag{6.14}$$

Note that the torsion axial vector current persists even for Euler flows, where $\upsilon = 0$. The measurement of the components of the Torsion axial vector current have been completely ignored by experimentalists in hydrodynamics.

The Topological Parity 4-form can be evaluated by exterior differentiation as,

$$K = dH = dA\hat{}dA = -2(\mathbf{a}\cdot\boldsymbol{\omega})dx\hat{}dy\hat{}dz\hat{}dt. \tag{6.15}$$

The 4 vector of Topological Torsion $[\mathbf{T}, h]$ has a 4 dimensional divergence on $\{x, y, z, t\}$ equal to the coefficient of the Topological Parity, K,

$$div\,\mathbf{T} + \partial h/\partial t = -2(\mathbf{a}\cdot\boldsymbol{\omega}) \tag{6.16}$$

and yields the helicity-torsion current conservation law, if the anomaly, $(\mathbf{a}\cdot\boldsymbol{\omega})$, on the RHS vanishes. It is to be observed that when $K = 0$, the Euler index is zero, and the integral of H over a boundary of support vanishes by Stokes theorem. This idea is the generalization of the conservation of the integral of helicity density in an Eulerian flow. Note the result is true for a viscous fluid, subject to the constraint of zero Euler index. However, if the topological parity 4-form, K, does not vanish, then the lines of the torsion current, $\mathbf{T}$, do not satisfy a conservation law and can start or stop within the fluid interior. The parity 4-form, K, is the source for the production or destruction of anomalous topological defects in the evolutionary process.

By a similar substitution of 6.12 into 6.16, the topological parity pseudo-scalar becomes expressible in terms of engineering quantities for a Navier-Stokes flow as,

$$K = -2\upsilon(curl\ \mathbf{v}\cdot curl\,curl\,\mathbf{v})dx\hat{}dy\hat{}dz\hat{}dt \tag{6.17}$$

$$= -2\upsilon(\boldsymbol{\omega}\cdot curl\,\boldsymbol{\omega})dx\hat{}dy\hat{}dz\hat{}dt \tag{6.18}$$

******Added 04/26/2003

For the case where $A = \mathbf{v} \circ \mathbf{dr} - \{\mathbf{v}.\mathbf{v}/2\}dt$, it follows from dA that

$$\boldsymbol{\omega} = curl\ \mathbf{v} \quad and \quad \mathbf{a} = -\partial\mathbf{v}/\partial t - grad\{\mathbf{v}.\mathbf{v}/2\}\} \tag{6.19}$$

These vector fields always satisfy the Poincare-Faraday induction equations, $dF = ddA = 0, or,$

$$curl\ \mathbf{a} - \partial\boldsymbol{\omega}/\partial t = 0, \quad div\ \boldsymbol{\omega} = 0. \tag{6.20}$$

The 3-form of Topological Torsion, H, constructed from the exterior product of A and dA, has four components, and can be written as,

$$H = A\hat{}dA = H_{\mu\nu\rho}dx^{\mu}\hat{}dx^{\nu}\hat{}dx^{\rho} \tag{6.21}$$

$$= T_x dy\hat{}dz\hat{}dt - T_y dx\hat{}dz\hat{}dt + T_z dx\hat{}dy\hat{}dt - h dx\hat{}dy\hat{}dz, \tag{6.22}$$

where $\mathbf{T}$ is the Topological Torsion axial vector current,

$$\mathbf{T} = [\mathbf{a} \times \mathbf{v} + \{\mathbf{v}.\mathbf{v}/2\}\ curl\ \mathbf{v} = \mathbf{a} \times \mathbf{v} + \{\mathbf{v}.\mathbf{v}/2\}\ \boldsymbol{\omega}, (\mathbf{v} \circ curl\,\mathbf{v}) \tag{6.23}$$

If the Navier-Stokes equations admit a nonbarotropic pressure term, and a dissipation related to compressibility, then the acceleration, $\mathbf{a}$, can be defined as

$$\mathbf{a} = -[grad\{\mathbf{v}.\mathbf{v}/2\} + \partial\mathbf{v}/\partial t] = -\mathbf{v} \times curl\ \mathbf{v} + gradP/\rho - \lambda grad(div\ \mathbf{v}) + v\{curl\ curl\ \mathbf{v}\}. \tag{6.24}$$

By substituting this expression for $\mathbf{a}$ into the formula for the torsion vector current, a simple engineering representation is obtained for a Navier-Stokes fluid:

$$\mathbf{T} = \{h\mathbf{v} - \{\mathbf{v} \circ \mathbf{v}/2\}curl\ \mathbf{v}\} - \mathbf{v} \times gradPl\rho + \lambda grad(div\ \mathbf{v}) - v\{\mathbf{v} \times (curl\ curl\ \mathbf{v})\} \tag{6.25}$$

Note that the torsion axial vector current persists even for Euler flows with zero vorticity, $\omega = 0$. The measurement of the components of the Torsion vector have completely ignored by experimentalists (and theorists) in hydrodynamics.

By a similar substitution, the topological parity pseudo-scalar becomes expressible in terms of engineering quantities as,

$$K=-\{2(a.\omega)\}\Omega_4=\{2\{gradP/\rho-\lambda\ div\ \mathbf{v})\circ curl\ \mathbf{v}+v\{curl\ \mathbf{v}.(curl\ curl\ \mathbf{v})\}\}\Omega_4. \tag{6.26}$$

From this expression it is apparent that even in the limit of zero viscosity (high Reynolds number), it is still possible to produce torsion defects when the pressure gradient and divergence terms are not zero, and are not orthogonal to the the vorticity, ω. Moreover, If the vorticity field is integrable in the sense of Frobenius, then viscosity does NOT contribute to the creation of torsion defects. The integral of K over {x,y,z,t} gives the Euler Index of the flow.

******

The integral of K over $\{x, y, z, t\}$ gives the Euler index of the flow. It is to be observed that the Topological Parity pseudo-scalar, K, is always zero for non-viscous Eulerian flows, and can be zero for viscous Navier-Stokes flows if the vorticity vector, $\boldsymbol{\omega}$, satisfies the Frobenius integrability condition, $(\boldsymbol{\omega}\cdot curl\,\boldsymbol{\omega})$ =0. From the result given above, it is apparent that K, and therefore, the Euler index of the domain of support, is not necessarily zero, unless the vorticity field, $curl\ \mathbf{v}$, admits a foliation of codimension 1. See Arnold [1]. If the flow is to be irreversible, the flowlines must have at least one point of intersection in space-time, and therefore the Euler index of the domain of support cannot be zero. It follows that K for the domain of support cannot be zero, and therefore the Pfaff dimension of the turbulent state must be 4, and the vorticity field of a turbulent Navier-Stokes fluid is not integrable.

If the RHS of 4.5 goes to zero, then the constraint becomes equivalent to the Euler equations of motion of a non-viscous fluid. The criteria that $i(\mathbf{V})dA = 0$ has been shown by Cartan [5] to be necessary and sufficient for the solution vector to have a Hamiltonian representation relative to the Action, A. This result corresponds to the well known Clebsch potential representation for an Euler flow [41]. Note that even in a viscous flow, if $\mathbf{F}_{(viscous)} = \upsilon\, curl\, curl\, \mathbf{v}$ =0, the system has a Hamiltonian representation! If the kinematics are without fluctuations, then the conclusion is again reached that the system is Hamiltonian. The implication is that harmonic velocity fields have no irreversible dissipation.

Analytically, if the RHS of 4.5 vanishes or is closed, then the solution vector fields are uniformly continuous, for the even dimensional elements of the Pfaff sequence, (the limit sets, $F = dA\ and\ K = dH = F^{\wedge}F$) are invariants of the flow. However, the odd dimensional intersections of the Pfaff sequence need not be invariant, implying the possibility of a continuous but changing topology. If topology changes, such flows can not be homeomorphisms, but are uniformly continuous; therefore, they must be irreversible, in the sense that a continuous inverse does not exist. Time reversal symmetry is broken.

The constraint of uniform continuity may be relaxed, for a flow is continuous if the limit points of the flow permute into the closure of the evolving topology [Hocking, 1984]. Such a flow is said to be continuous, but not necessarily uniformly continuous. It is possible to show that relative to the Cartan topology, all evolutionary processes generated by the Lie derivative with respect to C2 vector fields are continuous. The equations of continuous, but not uniformly continuous, evolution must satisfy transversal constraints of the form [Kiehn 1974],

$$i(\mathbf{V})dA = \Gamma A + \{closed\, 1 - forms\}. \tag{6.27}$$

This result implies that the viscous forces, $\mathbf{F}$, in 4.5 must be proportional to the canonical momentum, $\mathbf{p}$, a result typical of empirical viscous friction assumptions. In other words, solutions to the Navier-Stokes equations 6.5, which are of the form , may be continuous but irreversible relative to the Cartan fluctuation topology. It is this class of processes that can be used to describe the decay of turbulence, but not its creation.

7. Summary

It has been argued that a necessary condition for an evolutionary process to be irreversible in space time is that the Euler characteristic of Cartan topology induced on space time by the process is non-zero. It follows that Topological Parity associated with such processes is non zero, and the evolutionary field is of Pfaff dimension 4. For hydrodynamic applications, a flow vector field of Pfaff dimension 4 must be time-dependent, and spatially three dimensional. A turbulent flow is therefore irreversible, time dependent, and irreducibly three dimensional. For a Navier-Stokes fluid, if the vorticity field is integrable in the sense of Frobenious, the flow is not turbulent, but can be chaotic.

8. APPENDIX A The Cartan 1-form of Action as a Line Bundle

This appendix is used to describe the details of how to compute topological and geometrical properties of generalized spaces using the Cartan exterior methods. The objective is to construct Cartan's connection 1-forms, (Repere Mobile), and call attention to the fact that two types of torsion defects (both rotational and translational) can be generated on a projective manifold of dimension n+1. Although the affine translational torsion has a growing literature, the projective rotational torsion has been ignored. Yet, rotational torsion, intuitively, seems to be of more importance for hydrodynamic situations. The idea is to display the rudiments of Frames, Cartan connections, and matrices of local curvature 2-forms, which can be used in the form of Chern classes [7] which in turn are used to evaluate the Euler characteristic on a variety. The methods can be used to display the correspondence between the Euler characteristic and the concept of Topological Parity.

Consider a 1-form of Action on a 2n+2= 4D domain of definition given by the expression,

$$A = \lambda(x, y, z, t)\{v_k(x, y, z, t)dx^k - cdt\}. \tag{8.1}$$

At any point p of the domain, there exists 2n+1=3 vectors $\mathbf{e}_m$ of four components that are orthogonally transversal to the form in the sense that $i(\mathbf{e}_m)A = 0$. These vectors (to within an arbitrary factor) may be used as column vectors of a basis frame at the point p. The coefficient functions of the one form itself (to within an arbitrary factor) form the 2n+2 elements of a basis frame at the point p. A useful but not unique choice for a basis set at the point p is given by the expression,

$$\mathbb{F} = [\mathbf{e}_k, \mathbf{n}] = \mathbb{F} = \begin{bmatrix} 1 & 0 & 0 & -\lambda v_x \\ 0 & 1 & 0 & -\lambda v_y \\ 0 & 0 & 1 & -\lambda v_z \\ v_x/c & v_y/c & v_{/cz} & +\lambda c \end{bmatrix}. \tag{8.2}$$

The determinant of this matrix is equal $\det \mathbb{F} = c\lambda(c^2 + A_x^2 + A_y^2 + A_z^2)$, which is never zero for $\lambda > 0$. Hence this basis frame has an inverse almost everywhere. Note that the components of a 1-form, A, to within a factor, have been used construct a basis frame at almost all points of the manifold. Although the adjoint normal direction field, $\mathbf{n}$, established by the coefficients of the given 1-form, are orthogonal to the other column vectors, $\mathbf{e}_k$, in the frame, it is apparent that the $\mathbf{e}_k$ are not orthogonal to one another. The direction field, $\mathbf{n}$, could be rescaled by appropriate choice of the factor in order to force the Frame matrix to be unimodular. However, this choice, as shown below, is a severe constraint on the geometry.

The existence of the inverse matrix, of course, guarantees that the differential of every basis vector is decomposable into a linear combination of the original elements of the basis set, independent of the partition. The matrix of coefficients of this linear expansion defines the (right) Cartan matrix of connection 1-forms, $\mathbb{C}_r$,

$$d\mathbb{F} = \mathbb{F} \circ \{d\mathbb{F} \circ \mathbb{F}^{-1}\} = \mathbb{F} \circ \{-d\mathbb{F}^{-1} \circ \mathbb{F}\} = \mathbb{F} \circ \mathbb{C}_r \tag{8.3}$$

over the domain of support for the basis frame (where $\mathbb{F}^{-1}$ exists). (An alternate development would use the left Cartan matrix representation, $d\mathbb{F} = \mathbb{C}_l \circ \mathbb{F}$).

It is convenient to partition the (arbitrary) basis frame $\mathbb{F}$ in terms of the *associated* (horizontal, interior, coordinate or transversal) vectors, $\mathbf{e}_k$, and the *adjoint* (normal, exterior, parametric or vertical) field, $\mathbf{n}_p$,

$$\mathbb{F} = [\mathbf{e}_k, \mathbf{n}] = [\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{n}]. \tag{8.4}$$

The corresponding Cartan matrix has the partition,

$$d\mathbb{F} = d \begin{bmatrix} e_1^1 & e_2^1 & ... & n^1 \\ e_1^2 & e_2^2 & ... & n^2 \\ ... & ... & ... & ... \\ e_1^n & e_2^n & ... & n^n \end{bmatrix} = \mathbb{F} \circ \mathbb{C} = \mathbb{F} \circ \begin{bmatrix} \Gamma_1^1 & \Gamma_2^1 & ... & \gamma^1 \\ \Gamma_1^2 & \Gamma_2^2 & ... & \gamma^2 \\ ... & ... & ... & ... \\ h_1 & h_2 & ... & \Omega \end{bmatrix} \tag{8.5}$$

The vector equations for the derivatives of the columns of the frame matrix can be written as

$$d\mathbf{e}_k = \mathbf{e}_m \Gamma_k^m + \mathbf{n} h_k \tag{8.6}$$

$$d\mathbf{n} = \mathbf{e}_m \gamma^m + \mathbf{n}\Omega, \tag{8.7}$$

which indicates a closure concept in the sense that the differentials of basis vectors are composed of linear combinations of themselves. However, it is important to note that in general the differentials of the interior vectors are not linear combinations of interior vectors and the differentials of exterior vectors are also not closed among themselves.

The Cartan matrix, $\mathbb{C}$, is a matrix of differential 1-forms which can be evaluated explicitly from the C1 functions that make up the basis frame. Moreover, the differential of the position vector can be expanded in terms of the same basis frame and a set of Pfaffian 1-forms:

$$d\mathbf{R} = \mathbb{I} \circ \left| \begin{matrix} d\mathbf{x} \\ ... \\ d\mathbf{t} \end{matrix} \right\rangle = \mathbb{F} \circ \mathbb{F}^{-1} \circ \left| \begin{matrix} d\mathbf{x} \\ ... \\ d\mathbf{t} \end{matrix} \right\rangle = \mathbb{F} \circ \left| \begin{matrix} \boldsymbol{\sigma} \\ ... \\ \boldsymbol{\omega} \end{matrix} \right\rangle, \tag{8.8}$$

where the vector $\left| \begin{array}{c} \boldsymbol{\sigma} \\ ... \\ \boldsymbol{\omega} \end{array} \right\rangle$ is a (4 component) vector of 1-forms that can be computed explicitly. The 1-form ω need not be closed, and in many interesting cases is not integrable, $\omega\hat{}d\omega \neq 0$.

By the Poincare lemma, it follows that

$$dd\mathbb{F} = d\mathbb{F}\hat{}\mathbb{C} + \mathbb{F}\hat{}d\mathbb{C} = \mathbb{F} \circ \{\mathbb{C}\hat{}\mathbb{C}+d\mathbb{C}\} = 0, \tag{8.9}$$

and

$$dd\mathbf{R} = d\mathbb{F}\hat{} \left| \begin{array}{c} \boldsymbol{\sigma} \\ \boldsymbol{\omega} \end{array} \right\rangle + \mathbb{F} \circ \left| \begin{array}{c} d\boldsymbol{\sigma} \\ d\boldsymbol{\omega} \end{array} \right\rangle = \mathbb{F} \circ \{\mathbb{C}\hat{} \left| \begin{array}{c} \boldsymbol{\sigma} \\ \boldsymbol{\omega} \end{array} \right\rangle + \left| \begin{array}{c} d\boldsymbol{\sigma} \\ d\boldsymbol{\omega} \end{array} \right\rangle\} = 0. \tag{8.10}$$

The bracket factor

$$[\text{Cartan Curvature 2-forms}] \;= \{\mathbb{C}\hat{}\mathbb{C}+d\mathbb{C}\}, \tag{8.11}$$

defines what is called the (global) matrix of Cartan Curvature 2-forms, and the bracket factor

$$|\text{Cartan Torsion 2-forms}\rangle = \{\mathbb{C}\hat{} \left| \begin{array}{c} \boldsymbol{\sigma} \\ \boldsymbol{\omega} \end{array} \right\rangle + \left| \begin{array}{c} d\boldsymbol{\sigma} \\ d\boldsymbol{\omega} \end{array} \right\rangle\} \tag{8.12}$$

defines what is called the (global) vector Cartan Torsion 2-forms. For any Frame matrix with inverse, the global Cartan Torsion 2-forms, and the global Cartan Curvature 2-forms must vanish. However, as shown below the vectors of Cartan torsion 2-forms and the Cartan curvature 2-forms associated the interior subspace are not necessarily zero. The Torsion 2-forms , and can be decomposed into interior dislocations and exterior disclinations.

As the Frame matrix and the Cartan matrix are partitioned relative to the tangent (or interior) vectors **e** and the normal (or exterior) vectors, **n**, the Poincare lemma breaks up into linearly independent factors, each of which must vanish. The results are:

$$dd\mathbf{R} = \mathbf{e}\{d\,|\boldsymbol{\sigma}\rangle + [\boldsymbol{\Gamma}]\hat{}\,|\boldsymbol{\sigma}\rangle - \omega\hat{}\,|\gamma\rangle\} + \mathbf{n}\{d\omega + \Omega\hat{}\omega + \langle\mathbf{h}|\,\hat{}\,|\boldsymbol{\sigma}\rangle\} = 0 \tag{8.13}$$

$$dd\mathbf{e} = \mathbf{e}\{d[\boldsymbol{\Gamma}] + [\boldsymbol{\Gamma}]\hat{}[\boldsymbol{\Gamma}] + |\gamma\rangle\;\hat{}\;\langle\mathbf{h}|\} + \mathbf{n}\{d\,\langle\mathbf{h}| + \Omega\hat{}\,\langle\mathbf{h}| + \langle\mathbf{h}|\,\hat{}[\boldsymbol{\Gamma}]\} = 0 \tag{8.14}$$

$$dd\mathbf{n} = \mathbf{e}\{d\,|\boldsymbol{\gamma}\rangle + [\boldsymbol{\Gamma}]\hat{}\,|\boldsymbol{\gamma}\rangle - \Omega\hat{}\,|\boldsymbol{\gamma}\rangle\} + \mathbf{n}\{d\Omega + \Omega\hat{}\Omega + \langle\mathbf{h}|\,\hat{}\,|\boldsymbol{\gamma}\rangle\} = 0 \qquad (8.15)$$

By reasons of linear independence, each of the curly bracket factors must vanish, leading to the results on the interior domain (coefficients of $\mathbf{e}_k$):

$$d\,|\boldsymbol{\sigma}\rangle + [\boldsymbol{\Gamma}]\hat{}\,|\boldsymbol{\sigma}\rangle = \omega\hat{}\,|\boldsymbol{\gamma}\rangle \equiv |\boldsymbol{\Sigma}\rangle = \left|\begin{matrix} \omega\hat{}\gamma^1 \\ \omega\hat{}\gamma^2 \\ \omega\hat{}\gamma^3 \end{matrix}\right\rangle \qquad (8.16)$$

$$\text{with } |\boldsymbol{\Sigma}\rangle = \text{the interior torsion vector of dislocation 2-forms.}$$

$$d[\boldsymbol{\Gamma}] + [\boldsymbol{\Gamma}]\hat{}[\boldsymbol{\Gamma}] = -\,|\boldsymbol{\gamma}\rangle\,\hat{}\,\langle\mathbf{h}| \equiv [\boldsymbol{\Theta}] = \begin{bmatrix} \gamma^1\hat{}h_1 & \gamma^1\hat{}h_2 & \gamma^1\hat{}h_3 \\ \gamma^2\hat{}h_1 & \gamma^2\hat{}h_2 & \gamma^2\hat{}h_3 \\ \gamma^3\hat{}h_1 & \gamma^3\hat{}h_2 & \gamma^3\hat{}h_3 \end{bmatrix} \qquad (8.17)$$

$$\text{with } [\boldsymbol{\Theta}] = \text{the matrix of interior curvature 2-forms.}$$

$$d\,|\boldsymbol{\gamma}\rangle + [\boldsymbol{\Gamma}]\hat{}\,|\boldsymbol{\gamma}\rangle = \Omega\hat{}\,|\boldsymbol{\gamma}\rangle \equiv |\boldsymbol{\Psi}\rangle = \left|\begin{matrix} \Omega\hat{}\gamma^1 \\ \Omega\hat{}\gamma^2 \\ \Omega\hat{}\gamma^3 \end{matrix}\right\rangle \qquad (8.18)$$

$$\text{with } |\boldsymbol{\Psi}\rangle = \text{the exterior torsion vector of disclination 2-forms.}$$

$|\boldsymbol{\Psi}\rangle$ physically seems to represent a different kind of ”torsion” when compared to the torsion 2-forms represented by $|\boldsymbol{\Sigma}\rangle$. The $|\boldsymbol{\Psi}\rangle$ components depend upon $|\boldsymbol{\gamma}\rangle$ and Ω, while the $|\boldsymbol{\Sigma}\rangle$ components depend upon upon $|\boldsymbol{\gamma}\rangle$ and ω. Kondo [17] developed the theory of dislocation defects based on $|\boldsymbol{\Sigma}\rangle$, while it appears that $|\boldsymbol{\Psi}\rangle$ can represent disclination defects.

The first two equations 8.16, 8.17 are precisely Cartan's equations of structure (on an affine domain). It is the last equation 8.18 of exterior disclination 2-forms, $d\,|\boldsymbol{\gamma}\rangle + [\boldsymbol{\Gamma}]\hat{}\,|\boldsymbol{\gamma}\rangle = \Omega\hat{}\,|\boldsymbol{\gamma}\rangle = |\boldsymbol{\Psi}\rangle$, that appears to be a new equation of structure valid on a projective domain, when $\Omega \neq 0$. The 1-form Ω, defined as the abnormality 1-form, can be interpreted in terms of a combined expansion and rotation. The components $|\boldsymbol{\gamma}\rangle$ can be interpreted in terms of rotations, while the components $\langle\mathbf{h}|$ can be interpreted in terms of translations.

Additional constraints can be imposed upon the Frame matrix, limiting the generality and application. When the normalization factor, λ, is chosen in such a way as to force the determinant of the transformation to be unity (or a constant), the abnormality 1-form Ω becomes zero. This single constraint on the determinant can be intepreted as reducing the general Cartan connection matrix to a projective Cartan matrix. In such cases, the disclination 2-forms, $|\mathbf{\Psi}\rangle$, vanish. If the arbitrary Frame matrix is locally constrained such that the Cartan connection matrix is an element of the orthogonal structure group, then Ω vanishes, and the Cartan matrix, becomes antisymmetric, with $\langle\mathbf{h}| = -|\boldsymbol{\gamma}\rangle$. There are two types of Affine Cartan matrices. The first type is an element of the matrix group where $|\boldsymbol{\gamma}\rangle = 0$. The second type of affine transformation is an element of the matrix group where $\langle\mathbf{h}| = 0$. Often the structural group is chosen as a Lie group.

A purpose of this section was to prove constructively the existence of $|\mathbf{\Psi}\rangle$, a vector of "exterior" torsion 2-forms which, it is suggested herein, should be put into correspondence with disclination defects, rotational shears and coherent structures in hydrodynamics. This vector is zero on euclidean orthonormal or affine manifolds. Another purpose was to focus attention on the Cartan matrix of curvature 2-forms.

The matrix of interior curvature two forms, $[\mathbf{\Theta}]$, can be constructed from the knowledge of connection coefficients, $d[\mathbf{\Gamma}] + [\mathbf{\Gamma}]\hat{}[\mathbf{\Gamma}]$, (which requires the use of a second differential process), or algebraically from the "outer-exterior" product of $-|\boldsymbol{\gamma}\rangle\ \hat{}\ \langle\mathbf{h}|$.

$|\mathbf{\Psi}\rangle$ physically seems to represent a different kind of "torsion" which I am trying to put into correspondence with disclination defects. Recall that Kondo has developed the theory of dislocation defects based on $|\mathbf{\Sigma}\rangle$.

There are also three equations of structure on the exterior domain (coefficients of $\mathbf{n}$) which are given by the constructions:

$$d\omega + \Omega\hat{}\omega = -\langle\mathbf{h}|\ \hat{}\ |\boldsymbol{\sigma}\rangle\,, \tag{8.19}$$

$$d\langle\mathbf{h}| + \Omega\hat{}\ \langle\mathbf{h}| = -\langle\mathbf{h}|\ \hat{}[\mathbf{\Gamma}], \tag{8.20}$$

$$d\Omega + \Omega\hat{}\Omega = \theta = -\langle\mathbf{h}|\ \hat{}\ |\boldsymbol{\gamma}\rangle\,, \tag{8.21}$$

where θ represents the exterior curvature 2-forms

A remarkable result of this construction is the fact that the matrix of interior curvature 2-forms, $[\mathbf{\Theta}]$, can be constructed in two ways. The classical method utilizes differential processes $\{d[\mathbf{\Gamma}] + [\mathbf{\Gamma}]\hat{}[\mathbf{\Gamma}]\}$, while the second method is purely algebraic $\{-|\gamma\rangle \hat{} \langle\mathbf{h}|\}$. The order of partial derivatives contained in the algebraic (exterior) expression for the interior curvature $\{-|\gamma\rangle \hat{} \langle\mathbf{h}|\}$ is one less than the classic expression built on the connection coefficients, $\{d[\mathbf{\Gamma}] + [\mathbf{\Gamma}]\hat{}[\mathbf{\Gamma}]\}$.

Exterior differentiation of the matrix of interior curvature 2-forms yields:

$$d[\mathbf{\Theta}] = -d|\gamma\rangle \hat{} \langle\mathbf{h}| = (-|d\gamma\rangle \hat{} \langle\mathbf{h}|) + (|\gamma\rangle \hat{} \langle d\mathbf{h}|) =$$

$$([\mathbf{\Gamma}]\hat{}|\gamma\rangle \hat{} \langle\mathbf{h}|) - (\Omega\hat{}|\gamma\rangle \hat{} \langle\mathbf{h}|) - (|\gamma\rangle \hat{} \Omega\hat{} \langle\mathbf{h}|) - (|\gamma\rangle \hat{} \langle\mathbf{h}|\hat{}[\mathbf{\Gamma}]) = 0$$

The fundamental result is that the matrix of 2-forms that forms the interior curvature matrix is closed! It is this fact that leads ultimately to the idea relating the Euler characteristic and the components of the curvature 2-forms.

It is important to note that due to the partition, the exterior curvature is a closed (in this example a scalar valued) 2-form $\theta = -\langle\mathbf{h}|\hat{}|\gamma\rangle$ with

$$d\theta = -\langle d\mathbf{h}|\hat{}|\gamma\rangle + \langle\mathbf{h}|\hat{}|d\gamma\rangle = +\Omega\hat{}\langle\mathbf{h}|\hat{}|\gamma\rangle + \langle\mathbf{h}|\hat{}[\mathbf{\Gamma}]\hat{}|\gamma\rangle - \langle\mathbf{h}|\hat{}[\mathbf{\Gamma}]\hat{}|\gamma\rangle + \langle\mathbf{h}|\hat{}\Omega\hat{}|\gamma\rangle = 0,$$

is closed. As Ω is a 1-form for a single exterior vector, , then the 2-form θ is exact. The exterior exterior curvature 2-forms can generate a Maxwell-Faraday system of PDE's See Appendix B).

Both the exterior and the interior curvature 2-forms can be matrix valued depending upon the partition of the Frame. Each curvature matrix exhibits a set of similarity invariants deduced from the coefficients of the Cayley-Hamilton characteristic polynomial. It would appear therefore that their are two species of Chern characteristic classes that can be constructed from the Cayley-Hamilton polynomial similarity invariants.

If (in the example) the projective Cartan matrix is constrained to be euclidean, then $\Omega = 1$, and both $\mathbf{h} = 0$,and $\boldsymbol{\gamma} = 0$. Hence both the interior and the exterior curvature vanish. Indeed, then both types of torsion 2-forms vanish.

On the otherhand, if the Cartan matrix is anti-symmetric (as it must be for an orthonormal frame matrix) then $\Omega = 0$, and $\boldsymbol{\gamma} = -\mathbf{h}$. Hence, the exterior curvature vanishes, and $|\mathbf{\Psi}\rangle = 0$, but the domain could support interior curvature and dislocation torsion 2-forms, $|\mathbf{\Sigma}\rangle \neq 0$. If the Cartan matrix is left affine, then $\mathbf{h} = 0, \Omega = 1$. The interior and exterior domains are flat, but the structure could admit both forms of torsion 2-forms.

The moral of this appendix is that there usually is more than one way to do things. In the case of a projectivized line bundle over a variety, the Cartan method of computing the significant quantities is equivalent to the methods of fiber bundle theory, but it is much simpler to use and easier to interpret in physically useful ways for engineering applications.

9. APPENDIX B The Electromagnetic Format

For the electromagnetic case, the Cartan 1-form may be defined in terms of the vector and scalar potentials,

$$A = \Sigma_{k=1}^{3} A_k(x,y,z,t)dx^k - \phi(x,y,z,t)dt = \mathbf{A}\circ d\mathbf{r} - \phi dt. \tag{9.1}$$

Using the classical notation of Sommerfeld, define the **E** and **B** field intensities as

$$\mathbf{E} = -\partial \mathbf{A}/\partial t - grad\phi, \qquad \mathbf{B} = curl\ \mathbf{A} \equiv \partial A_k/\partial x^j - \partial A_j/\partial x^k. \tag{9.2}$$

The components of the Darboux-Cartan-Maxwell field $dA = F$ become,

$$\begin{aligned} F &= dA = \{\partial A_k/\partial x^j - \partial A_j/\partial x^k\}dx^j\hat{}dx^k \\ &= F_{jk}dx^j\hat{}dx^k = \mathbf{B}_z dx\hat{}dy ... + \mathbf{E}_x dx\hat{}dt ..., \end{aligned} \tag{9.3}$$

which may be written as an anti-symmetric matrix (or as a Sommerfeld six-vector) of six functions :

$$F_{jk} = \begin{bmatrix} 0 & B_z & -B_y & E_x \\ -B_z & 0 & B_x & E_y \\ B_y & -B_x & 0 & E_z \\ -E_x & -E_y & -E_z & 0 \end{bmatrix} \tag{9.4}$$

The Topological torsion, H, becomes

$$H = A\hat{}dA = i([T,h])dx\hat{}dy\hat{}dz\hat{}dt \tag{9.5}$$

$$= T^x dy\hat{}dz\hat{}dt - T^y dz\hat{}dx\hat{}dt + T^z dx\hat{}dy\hat{}dt - h\ dx\hat{}dy\hat{}dz, \tag{9.6}$$

with the torsion current defined as,

$$T = -[\mathbf{E}\times\mathbf{A} + \mathbf{B}\phi], \tag{9.7}$$

and the helicity density,

$$h = -\mathbf{A} \circ \mathbf{B}. \tag{9.8}$$

The Topological Parity 4-form becomes the global top Pfaffian on the 4 dimensional space-time variety, and is equal to

$$K = dA^\wedge dA = -\{2\mathbf{E} \circ \mathbf{B}\} dx\hat{}dy\hat{}dz\hat{}dt \tag{9.9}$$

Note that

$$div\,\mathbf{T} + \partial h/\partial t = -2(\mathbf{E} \cdot \mathbf{B}). \tag{9.10}$$

The 3-form of axial current, H, is NOT conserved when $K \neq 0$. This divergence anomaly for electromagnetic systems was demonstrated from other arguments by [3].

Following Chern, the Euler index becomes the integral

$$X = constant \iiiint -\{2\mathbf{E} \circ \mathbf{B}\} dx\hat{}dy\hat{}dz\hat{}dt \tag{9.11}$$

When the electric field is orthogonal to the magnetic field, then the Euler index is zero. The idea that this Poincare invariant might have deeper meaning led [10] to state: "It is somewhat curious that the scalar-product of the electric and magnetic forces is of so little importance in classical theory, for ..9.10 .. would seem to be the most fundamental invariant of the field. Apart from the fact that it vanishes for electromagnetic waves propagated in the absence of any bound electric field (i.e., remote from electrons), this invariant seems to have no significant properties. Perhaps it may turn out to have greater importance when the study of electron-structure is more advanced."

The concept of a domain of non-null Euler index, $K \neq 0$, now appears to be useful to the theory of magnetic reconnection in the electromagnetic case [47] and to vortex reconnection [44] in the hydrodynamic case. The correspondence between the bridging and rib structures produced in numerical simulations of turbulent fluid flows and the 4-string interaction of superstring theory is remarkable [16]. The concept ($K \neq 0$) appears to be applicable to the understanding of the stretching of lines and surfaces in turbulent flows where time-reversal symmetry is violated [9]. The appearance of large scale structures in certain flows has been associated with the lack of parity invariance [52]. The concepts of macroscopic violations of P and T symmetries appear to have application to the theory of the quantum Hall effect [54].

10. Acknowledgments.

This work has been supported in part by the Energy Lab and in part by ISSO at the University of Houston.

References

[1] Arnold, V. I. (1981) London Math. Soc. Lect. Note Series 53 (Cambridge University Press) 225-.

[2] Baldwin, P. and Kiehn, R. M. (1991) “Cartan's Topological Structure“, Poster presented at the summer workshop, “Topology in Fluid Dynamics“, Inst. for Th. Phy. UCSB, August 1991. LANL arXiv/math-ph/0101033 (http://www22.pair.com/csdc/pdf/topstru.pdf)

[3] Berger, M. A. and Field, G. B. (1984) “The topological properties of magnetic helicity“, J Fluid Mech. **147**, 133-148

[4] Callan, C. G., Jr., Gilkey, P.B., Hanson, A. J. (1976) Phys. Lett 63B 334-. Note the phrase “This ($\mathbf{E} \cdot \mathbf{B} \neq 0$ in the Lagrangian) clearly breaks P and T invariance.. “ Also see Schultz, A., et. al. (1979) “Lifting of the four fold degeneracy...“, Phys Lett **74**A, 384.

[5] Cartan, E. (1958) Lecons sur les invariants integrauxs (Hermann, Paris 1958).

[6] Chern, S.S. (1948) Science Reports Nat. Tsing Hua Univ. 5, 95-121.

[7] Chern, S.S. (1988) Historical Remarks on Gauss-Bonnet, MSRI 04808-88 Berkeley, CA. Also see ... (1944) “A Simple intrinsic proof of the Gauss-Bonnet formula for Closed Riemannian Manifolds“, Annals of Math. **45**, 747-752.

[8] Chern, S.S. and Simons, J., (1974), ”Characteristic forms and geometric invariants”, Ann. of Math. **99**, p.

[9] Drummond, I. T. and Munch, W. H. P. (1990) “Turbulent Stretching of Lines and Surfaces“, in: H. K. Moffatt and T. S. Tsinober eds, Topological Fluid Mechanics, (Cambridge University Press) 628-636.

[10] Eddington, A.S. (1963) The Mathematical Theory of Relativity (Cambridge University Press) 194.

[11] Flanders, H. (1963) Differential Forms (Academic Press, New York,1963)

[12]

[13] Greub, W., Halperin, S. and Vanstone, R. (1973) Connections, Curvature and Cohomology (Academic Press, New York)

[14] Hocking, J. G. (1984) Topology, Addison Wesley, N.Y., p.2

[15] Hermann, R. (1968) Differential Geometry and the Calculus of Variations (Academic Press, New York)

[16] Kaku, M. (1988) Introduction to Superstrings (Springer-Verlag, New York) 386.

[17] aaaaaaa

[18] Kiehn, R.M. (http://www22.pair.com/csdc/pdf/inttrans.pdf)

[19] Kiehn, R.M.(1974) "An Extension of Hamilton's Principle to Include Dissipative Systems", J. Math Phys. **5**, p. 9-. (http://www22.pair.com/csdc/pdf/hamprinc.pdf)

[20] Kiehn, R.M. (1975) "Intrinsic hydrodynamics with applications to space-time fluids", Int. J. of Eng. Sci. **13**, p. 941-. (http://www22.pair.com/csdc/pdf/inthydro.pdf)

[21] Kiehn, R.M. (1977) (http://www22.pair.com/csdc/pdf/periods.pdf)

[22] Kiehn, R.M. (1991) "Some Closed Form Solutions to the Navier Stokes Equations" LANL arXiv/physics/0102002 (http://www22.pair.com/csdc/pdf/nvsol.pdf)

[23] Kiehn, R.M. (1976a) NASA-AMES research document NCA-2-OR-295-502 (1976). Also, ... (1977) J. Math Phy. 18, 614. (http://www22.pair.com/csdc/pdf/rmkames.pdf)

[24] Kiehn, R.M. (1976b) "Retrodictive Determinism", Int. J. of Eng. Sci. **14**, p. 749-. (http://www22.pair.com/csdc/pdf/retrodic.pdf)

[25] Kiehn, R. M., (1987) The Falaco Effect as a topological defect was first noticed by the present author in the swimming pool of an old MIT friend, during a visit in Rio de Janeiro, at the time of Halley's comet, March 1986. The concept was presented at the Austin Meeting of Dynamic Days in Austin, January 1987, and caused some interest among the resident topologists. The easily reproduced experiment added to the credence of topological defects in fluids. It is now perceived that this topological phenomena is universal, and will appear at all levels from the microscopic to the galactic. LANL arXiv/gr-qc/0101098 (http://www22.pair.com/csdc/pdf/falaco97.pdf)

[26] Kiehn, R.M. (1990) "Topological Torsion, Pfaff Dimension and Coherent Structures", in: H. K. Moffatt and T. S. Tsinober eds, Topological Fluid Mechanics, (Cambridge University Press) 449-458. (http://www22.pair.com/csdc/pdf/camb89.pdf)

[27] Kiehn, R. M. (1991a) "Compact Dissipative Flow Structures with Topological Coherence Embedded in Eulerian Environments ", in: Sagdeev, et al. editors, The Non-linear Dynamics of Structures World Scientific Press, Singapore. (http://www22.pair.com/csdc/pdf/permb.pdf)

[28] Kiehn, R. M. , Kiehn, G. P., and Roberds, (1991b) "Parity and time-reversal symmetry breaking, singular solutions and Fresnel surfaces:, Phys. Rev A **43**, pp. 5165-5671. (http://www22.pair.com/csdc/pdf/timerev.pdf)

[29] Kiehn, R.M. (1991c) "Continuous Topological Evolution", submitted for publication. LANL arXiv/math-ph/0101032 (http://www22.pair.com/csdc/pdf/contevol3.pdf)

[30] Kiehn, R. M. (1992a) "Topological Defects, Coherent Structures and Turbulence in Terms of Cartan's Theory of Differential Topology" in Developments in Theoretical and Applied Mechanics, Proceedings of the SECTAM XVI conference, B. N. Antar, R. Engels, A.A. Prinaris and T. H. Moulden, Editors, The University of Tennessee Space Institute, Tullahoma, TN 37388 USA. (http://www22.pair.com/csdc/pdf/utsi.pdf)

[31] Kiehn, R. M. (1992b) "Instability Patterns, Wakes and Topological Limit Sets" in J.P.Bonnet and M.N. Glauser, (eds) "Eddy Structure Identification in Free Turbulent Shear Flows" Kluwer Academic Publishers (1993) p. 363. (http://www22.pair.com/csdc/pdf/poitier.pdf)

[32] Kiehn, R. M. (1992c), "Hydrodynamic Wakes and Minimal Surfaces with Fractal Boundaries" in J.M. Redondo, O. Metais, (eds) "Mixing in Geophysical Flows" CIMNE Barcelona (1995) p.52 (http://www22.pair.com/csdc/pdf/barcelon.pdf)

[33] Kiehn, R. M. (1993) Talk presented at the EUROMECH 308 meeting at Cortona, Italy. Unpublished.(http://www22.pair.com/csdc/pdf/italy.pdf)

[34] Kiehn, R. M. (1998), "Coherent Structures in Fluids are Topological Torsion Defects", in J.N. Sørensen, et al., eds., "IUTAM Symposium on Simulation and Identification of Organized Structures in Flows", Kluwer Academic Publishers, Dordrecht, (1999) (http://www22.pair.com/csdc/pdf/copen5.pdf)

[35] Kiehn, R. M. (1999), "2 D Turbulence is a myth" (invited speaker IUTAM conf) (http://www22.pair.com/csdc/pdf/hague6.pdf)

[36] Kiehn, R. M. (1999), "Topological evolution of classical electromagnetic fields and the photon" in Dvoeglazov, V., ed, *Photon and Poincaré Group,* Nova Science Publishers, Inc., Commack, New York, 1999, p.246-262. ISBN 1-56072-718-7 (http://www22.pair.com/csdc/pdf/photon.pdf)

[37] Kiehn, R. M., (2002), "Curvature and torsion of implicit hypersurfaces and the origin of charge", Annales de la Foundation Louis de Broglie, vol 27, p.411. LANL arXiv/gr-qc/0101109 (http://www22.pair.com/csdc/pdf/contevol3.pdf)

[38] Kiehn, R. M. (2000) "The Photon Spin and other Topological Features of Classical Electromagnetism" in Amoroso, R., et al., eds., *Gravitation and Cosmology, From the Hubble Radius to the Planck Scale,* Kluwer: Dordrecht, Netherlands, 2002, p.197-206. (http://www22.pair.com/csdc/pdf/vig2000.pdf)

[39] Kiehn, R. M. (2002), "Thermodynmic Irreversibility and the Arrow of Time" in ????? (to be published). (http://www22.pair.com/csdc/pdf/arwfinal.pdf)

[40] Klein, J.(1962) Ann. Inst. AIFUB Fourier, Grenoble 12, 1

[41] Lamb, H. (1945) Hydrodynamics (Dover, New York) 129.

[42] Lipschutz, S. (1965) General Topology (Schaum, New York), 88.

[43] J.E.Marsden and T. S. Riatu, (1994) *Introduction to Mechanics and Symmetry* Springer-Verlag, p.122

[44] Metcalfe, R. W. and Hussain, F. (1990) "Topology of Coherent structures and Flame Sheets in Reacting Mixing Layers", in: H. K. Moffatt and T. S. Tsinober eds, Topological Fluid Mechanics, (Cambridge University Press) 659-678. Also see: Melander, M. V. and Hussain, F. (1990) "Topological Aspects of Vortex Reconnection", in: loc. cit. 485-499.

[45] Meserve, B. E. (1983) Projective Geometry (Dover, New York)

[46] ddd

[47] Otto, A., Hesse, M., and Schindler, K. (1990) "General Magnetic Reconnection in 3D Systems", in: H. K. Moffatt and T. S. Tsinober eds, Topological Fluid Mechanics, (Cambridge University Press) 225-234.

[48] Schultz, A., Kiehn, R. M., Post, E. J., and Roberds, R. B., 1979, "Lifting of the four fold EM degeneracy and PT asymmetry", Phys Lett **74**A, p.384.

[49] Schuster, H, G. (1984) Deterministic Chaos (Physics Verlag)

[50] Sledbodzinsky, W. (1970) Exterior forms and their applications (PWN,Warszawa)

[51] Su, W., Liu W., and Liu Z. (1990), "Topological Structures of Separated Flows.", in: H. K. Moffatt and T. S. Tsinober eds, Topological Fluid Mechanics, (Cambridge University Press) 395-407.

[52] Sulem, P. L., She, Z. S., Scholl, H. and Frisch, U. (1989) "Generation of Large Scale Structures in Three Dimensional Flow Lacking Parity Invariance", J. Fluid Mech **205** 341-352.

[53] Tur, A. V. and Yanovsky, V. V. (1991) "Invariants in dissipationless hydrodynamic media", Accepted for publication in J. Fluid Mech.

[54] Wen, X.G., Wilczek, F. and Zee, A. (1989) "Chiral spin states and super conductivity, Phys Rev B **39** 11413-.