Periods on Manifolds, Quantization, and Gauge

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Abstract

It is suggested that the quantization of flux, charge, and angular momentum be interpreted as a set of independent natural concepts which physically exhibit certain topological properties of the fields on a space-time manifold. These quantum, or topological, properties may be described in terms of one-, two-, and three-dimensional periods, respectively. In terms of this viewpoint, topological constraints between the one-, two-, and three-dimensional periods can be put into correspondence with various gauge theories. If a dynamical system is to be nondissipative, in the sense that its one-, two-, and three-dimensional-topological periods are reversible invariants of the motion, then it is proved herein that the dynamical field V must be a Hamiltonian vector field, thc field currents must be proportional to V, and the Lagrangian difference between the elastic and.inertial energy density must be twice the interaction energy density, respectively.

1. INTRODUCTION

It is the purpose of this article to display certain topological properties of physical theories and to demonstrate that the notions of quantized flux, charge, and spin can be interpreted in terms of independent topological ideas of fields built on manifolds. The fact that these quantities are in relation to the integers then follows from Brouwer's theorem about closed manifolds. Brouwer's theorem effectively states that values of the closed integrals on a closed oriented manifold are integer multiples of some smallest value (see Appendix A).

A manifold is a nice mathematical object in that the difficulties of singularities, in a sense, have been removed. The regions of difficulty, which lead to phrases like "fields which are zero almost everywhere" can be eliminated, but at the expense of changing the topology of the domain over which the fields are defined. Topology of a base space can be defined in terms of a system of constrained differential forms. - (See Appendix B). Each form can be split into two parts, one of which is closed in the-exterior derivative sense. This closed piece can further be decomposed into a part which is exact and integrable and another part which is not. The closed integral of an exact form is zero, but the closed integral of a closed, but not exact, form is not necessarily zero on a non-Euclidean manifold (see Appendix C). (In fluids, this effect implies the existence of circulation without vorticity.) The total closed portion of a form is defined to be the gauge of the form. This definition widens the usual concept of gauge to include more than just an exact contribution. The number of distinct closed integrals over the gauge parts of a form (the periods) depends upon the manifold topology. In this theory it appears that the quantization of flux concerns the periods of a fundarnental 1-form, $\mathcal{A}$; the quantization of charge concerns the periods of a fundamental 2-form density, $\mathcal{H}$; the quantization of spin concerns the periods of a 3-form density, $\mathcal{S}$. The distinction between forms and form densities (as well as pair and impair forms) becomes evident with respect to the discrete symmetry transformations such as parib and time reversal.

Each quantization concept is independent, and this observation is a point of departure from the Dirac and Schwinger magnetic monopole theories, which, in a sense, are conditional theories, in that they claim that charge is quantized if flux, or angular momentum, is quantized. Any such relationship between the one- two-, and three-dimensional periods must be considered to be a topological constraint, and it is demonstrated below that-these constraints form the basis of various gauge theories. The manifold approach indicates that flux and charge can be quantized even though $div\mathbf{B} = 0$ everywhere on the manifold. (In a sense, these ideas lead to the geometrodynamic-like concept of a mono-pole without a monopole).

Of additional interest ls the clarification of the fact that physical content is to be associated with potentials, counter to the usual historical claim that the potentials have no physical significance. Physical significance is to be associated not with an individual potentiaI, but with the topological equivalence class to which it belongs. The quantum numbers of flux, charge, and spin relate to the one-, two-, and three-dimensional equivalence classes of periods, e. g., of integrals of closed forms over closed chains. In Euclidean space (or better, Euclidean topology) there exist no nontrivial periods and therefore, from the topological point of view, no quantization. The hypothesis of a Euclidean base space for classical field theories therefore precludes any notion of topological quantization. Classical theories contain within them the germs of quantization which become nontrivially evident when the domains of definition are not based on Euclidean topology. In that which follows, the topology of fields built on a four-dimensional spacetime will be studied in terms of a set of fundamental differential forms. For each physical system it is assumed that there exists such a set of fundamental forms, and the ensuing analysis then is applicable to all physical systems which are usually described in terms of a field theory. The presentation of the abskact theory will be given first in the more familiar language of electrodynamics; this presentation then will be followed by a hydrodynamic application which demonstrates the relationship between topological constraints and the NavierStokes equations.

2. THE FUNDAMENTAL FORMS

For a rapid expose of the theory, consider the formulation of electromagnetism as given by Toupin [1] in which it is assumed that a four-dimensional spacetime supports a 2-form $\mathcal{F}$ of electric intensity and an (N-1 =3)-form density, J, of electric current. The arguments that J is conserved imply that J is closed. The idea of charge neutrality implies that J is exact, for then integrals of J over compact, oriented three-dimensional manifolds must vanish. It follows that, from de Rham's theorem, there exists an (N- 2)-form potential density, H, such that $dH = J$,which yields the second pair of Maxwell's equation involving D, H, J, and ρ. However, the potential H consists of a part which is not closed and a gauge part which is closed. That is,

$$H = H_0 + \widehat{H} \tag{2.1}$$

where $dH_0 = J$ and $d\widehat{H} = 0$. The gauge of H is a closed form density and, if integrated over a closed 2-chain, yields a period of H. The choice of gauge has physical significance, for the closed integrals of $\widehat{H}$, that is, the periods of $\widehat{H}$, are to be identified with the electric charge. Classically, charge is the topological, global, quantity of electric flux. The concept that the periods of $\widehat{H}$ define charge is merely the abstraction of Gauss' law,

$$Q = \iint_{2\,cycle} \widehat{H} = \iint_{2\,cycle} \mathbf{D}.d\mathbf{S} + \iint_{2\,cycle} \mathbf{H}\, dt\hat{}\, dl \tag{2.2}$$

Now the remarkable result is that all periods (integrals on closed manifolds) are integer multiples of some smallest value, by Brouwer's degree of a map theorem.[2] Hence, the idea that charge Q is quantized by the integers is a natural result of the notion that charge is a topological property of a two-dimensional gauge period. No magnetic monopole assumption is needed to provide a reason for the quantization of charge once the topological viewpoint is admitted. The nonintegrable parts of the gauge have physical content, counter to the usual claim that potentials have no physical meaning.[3]

A similar argument goes for $\mathcal{F}$, the electric intensity 2-form; the potential $\mathcal{A}$, such that $\mathcal{F} = d\mathcal{A}$, can be split into a nonclosed part and a gauge part which is closed, but which need not be exact:

$$\mathcal{A} = \mathcal{A}_o + \widehat{\mathcal{A}}, \qquad d\widehat{\mathcal{A}} = 0. \tag{2.3}$$

As $\mathcal{F}$ is exact, $d\mathcal{F} = 0$ by the Poincare lemma; which yields the first Maxwell pair of equations involving **B** and **E** without any "monopole" source. The one-dimen-sional periods of $\mathcal{A}$ define the topological property physically called the flux quantum,

$$\Phi = \oint_{1\,cycle} \widehat{\mathcal{A}} \tag{2.4}$$

Again the Brouwer degree of a map theorem for closed manifolds asserts that these periods are quantized in terms of some smallest value. Again the choice of gauge has physical significance, as do the potentials themselves, for it is the topological properties of the gauge that yield the flux quantum. Note that $\mathcal{A}$ is a pair form, and not a form density.

The quantization of the flux (4) is related to the Bohr Sommerfeld idea which quantized the periodic action-angle variables. Note that here the 1-form, $\mathcal{A}$, is not a single action component, but is a Pfaffian form summed over all spacetime variables. Depending on the topology, many such distinct Pfaffian forms exist, each yielding a different topological period. This author was made aware of the possibilities that charge and flux quanta were related to two-dimensional and one-dimensional periods by E. J. Post.[4] The Toupin specification of electromagnetic theory in terms of an exact 2-form $\mathcal{F}$, and an exact 3-form density, J, was a culmination of ideas put together by Cartan, Weyl, Bateman, Murnaghan, Van

Dantzig and Post, but Toupin's treatment does not consider the periods of the potentials. The relationship of quantization of the periods to an interpretation of Brouwer's theorem appears herein for the first time.

Now on a spacetime of four dimensions, there also should exist a fundamental exact 4-form or 4-form density, which should carry the properties of the three-dimensional periods in space-time. The gauge parts of the associated 3-form should be quantized. Toupin [5] introduced a concept of a 4-form density of "action" but did not exploit the topological features of the idea. The present author found an exact representation for such a 4-form density in terms of the previously described fields, and interpreted the relationship between the divergence of the three potential and the 4-form density as an intrinsic transport theorem.[6] The exact 4-form density is closely related to the Lagrange density of classical field theory, save for a factor of two in the field energy densities. The explicit formula for the action is

$$\mathcal{L} = \mathcal{F}\hat{}H - \mathcal{A}\hat{}J \equiv (\mathbf{B} \circ \mathbf{H} - \mathbf{D} \circ \mathbf{E} - \mathbf{A} \circ \mathbf{J} + \rho\phi)\Omega \tag{2.5}$$

where Ω is the volume element $dx\hat{}dy\hat{}dz\hat{}dt$. This 4-form density is exact with a potential, S,such that $\mathcal{L} = d(S) = d(\mathcal{A}\hat{}H)$, and leads to the three-dimensional period of action,

$$\text{Sp} = \iiint_{3\,cycle} \widehat{S} = \iiint_{3\,cycle} (\mathcal{A}\hat{}H) = \iiint_{3\,cycle} (i(\mathbf{A} \times \mathbf{H} + \mathbf{D}\phi, \mathbf{A} \circ \mathbf{D})\Omega \tag{2.6}$$

It is the periods of the gauge part of S that are of interest herein, for by Brouwer's theorem they are quantized. Note that formally the units of S are joule-sec, while the units Q are Coulombs and the units of Φ are joule-sec/Coulombs for an electromagnetic representation. The electromagnetic representation of the topological theory thereby demonstrates that the periods of a 1-form $\mathcal{A}$ can be put into correspondence with flux quanta, the periods of a 2-form density H can be put into correspondence with charge quanta, and the periods of the 3-form density S can be put into correspondence with spin or action quanta. It is to be emphasized that by use of forms and form densities the previous equations, and those which follow, have an intrinsic significance. For any coordinate sequence abstractly labeled as (1, 2, 3, 4) and with corresponding 2-form [(N-2)-form density] coefficients labeled as $\mathbf{E}, \mathbf{B}$ (or $\mathbf{D}, \mathbf{H}$), the intrinsic equations are invariant in form. For example, in any coordinate system, if $\mathbf{B}$ is the sequence of coefficients associated with the (2,3), (3,1), and (1,2) coordinate pairs in the 2-form $\mathcal{F}$, and if H is the sequence of coefficients as-sociated with the (1,4), (2,4), and (3,4) coordinate pairs of the (N- 2)-form density H, then the magnetic energy density is given exactly by the engineering format, $2\mathbf{B} \circ \mathbf{H}$, even though the frame is not Cartesian. The results to be obtained, therefore, are independent of the choice of a reference coordinate frame when the engineering format is used.

3. INVARIANCE OF PERIODS

3.1. General theory

A vector field representing a transformation of the space may or may not preserve the system topology. If the vector field is a homeomorphism, then all topological properties are invariants of the transformation. In the context of this paper the pertinent topological properties are determined by the periods of one-, two-, and three-dimensions, and it is of some interest to determine the constraints on dynamical transformations which are necessary if these periods are to be individually invariant. Physically, the question may be stated as: For what transformations do the quanta of flux, charge, and spin remain invariant? Cartan has demonstrated that an extremal vector field which reversibly leaves the one-dimensional periods invariant is represented by Hamilton's equations of motion. Cartan's concept has been formulatedt in terms of a Lie derivative by the statement that,

$$L_{(\gamma \mathbf{V})}\Phi = L_{(\gamma \mathbf{V})} \oint \mathcal{A} = 0. \quad all \quad \gamma; \tag{3.1}$$

that is, it is necessary and sufficient that a vector field which leaves invariant the one-dimensional closed integral, $\oint \mathcal{A}$, independent of parametrization γ, be a Hamiltonian vector field. This work has been extended to nonhomeomorphic vector fields which indicate that the lack of topological invariance of the l-form $\mathcal{A}$ is related to dissipation, for if $\mathcal{A}$ is mechanically represented in terms of the system momentum, then a nonzero value for (7) implies a nonzero value for the cyclic work, $\oint f_\mu dq^\mu$. The result (7) implies that Hamiltonian systems are necessarily adiabatic. In an electromagnetic sense, the result (7) implies that such systems conserve flux quanta.

It is also of some interest to-study the invariance of the two-dimensional periods. For a vector field $\gamma\mathbf{V}$, invariance of the charge quanta implies that

$$L_{(\gamma\mathbf{V})}Q = L_{(\gamma\mathbf{V})}\iint_{2\,cycle} H = \iint_{2\,cycle}\{i(\gamma\mathbf{V})dH + d(i(\gamma\mathbf{V})H)\} = \iint_{2\,cycle} i(\gamma\mathbf{V})J = 0, \quad all \quad \gamma. \tag{3.2}$$

Expanding the last integral in electromagnetic format yields

$$\iint_{2\,cycle}\gamma(\mathbf{J}-\rho\mathbf{V})\circ d\mathbf{S} + \iint_{2\,cycle}\gamma(\mathbf{J}\times\mathbf{V})dt\hat{\ }dl = 0 \tag{3.3}$$

. If this result is to be true for all periods for all γ, then the integrand must vanish (by deRham's theorem). Hence it is necessary for invariance of the two-dimension periods that the current density be proportional to the velocity field:

$$\mathbf{J} = \rho\mathbf{V}. \tag{3.4}$$

This requirement is generally *assumed* in most magneto-hydrodynamic treatments, a priori, which then implies that a flow which starts without two-dimensional periods, remains without two-dimensional periods. Conversely, when $\mathbf{J} \neq \rho\mathbf{V}.$, then two-dimensional periods can be created or destroyed. This concept is related to another form of dissipation, intrinsically different from the dissipation associated with the one-dimensional periods (in the electromagnetic problem it takes work to separate charge). In a hydrodynamic context it is not apparent that all solutions to the Navier Stokes equations yield flow vector fields for which the two-dimensional periods remain invariant. The seat of hydrodynamic instabilities will be related to the creation of one-, two-, and three-dimensional periods.

Similarly, the invariance of the three-dimensional periods may be examined by means of the action of the Lie derivative on Sp, and exhibited in electromagnetic format as

$$\begin{aligned} L_{(\gamma\mathbf{V})}\mathrm{Sp} &= L_{(\gamma\mathbf{V})}\iiint_{3\,cycle}\mathcal{A}\hat{\ }H = \iiint_{3\,cycle} d(\mathcal{A}\hat{\ }H) = \iiint_{3\,cycle}\{\mathcal{F}\hat{\ }H - \mathcal{A}\hat{\ }J\} \\ &\iiint_{3\,cycle}[(\mathbf{B}\circ\mathbf{H} - \mathbf{D}\circ\mathbf{E}) - (\mathbf{A}\circ\mathbf{J} - \rho\phi)]i(\gamma\mathbf{V})\Omega. \end{aligned} \tag{3.5}$$

If all three-dimensional periods of spin are to be flow invariants independent of γ, then the integrand vanishes, putting a constraint on the magnetic and electric energy density difference. For a system without interactions, then, the field

Lagrangian must vanish for invariance of the spin quanta. Electromagnetically such a result is approximated by traveling waves for which $\mathbf{E}$ and $\mathbf{B}$ are in phase. Note that for standing waves the invariance result is only valid in a time average sense. (This notion is reminiscent of Wheeler's geometrodynamic froth.) The invariance of the flux quanta (one-dimensional) periods and charge quanta (two-dimensional) periods does not (necessarily) guarantee the invariance of the angular momentum quanta (three-dimensional) periods .

Summing up the topological invariance properties in terms of three fundamental theorems, it is to be noted that:

1. Flux quanta reversible invariance requires Hamilton's equations of motion.
2. Charge quanta reversible invariance requires $\mathbf{V} = \mathbf{J}/\rho$
3. (Spin) Angular momentum reversible quanta invariance requires

$$[(\mathbf{B} \circ \mathbf{H} - \mathbf{D} \circ \mathbf{E}) - (\mathbf{A} \circ \mathbf{J} - \rho\phi)] = 0.$$

3.2. Parametrization and reversibility

The above discussion describes topological invariance theorems with respect to vector fields $\{\mathbf{V}, 1\}$ parametrized by γ, where γ is an arbitrary function on the domain. In a space of N dimensions, vector fields have N components, any one of which may be selected as a parametrization function, leaving, by division, a local flow field $\gamma\{\mathbf{V}, 1\}$. For γ completely arbitrary, it is to be noticed that the inversion $\mathbf{V} \Rightarrow -\mathbf{V}\,(\gamma = 1 \Rightarrow \gamma = -1)$ changes the direction of the flow field (relative to the parameter). Invariance w. r. t. all γ implies that the object is not only invariant with respect to propagation down the flow lines, but also the invariance concept is independent of the propagation direction. Not all systems admit such a reversibility invariance.

For example, suppose γ is chosen to satisfy the continuity requirement that the volume element be invariant, then

$$L_{(\gamma\mathbf{V})}\Omega = 0, \tag{3.6}$$

or

$$[d\gamma/dt + \gamma\, div\, \mathbf{V}]\mathbf{\Omega} = [\partial\gamma/\partial t + \mathbf{V} \circ grad\gamma + \gamma\, div\, \mathbf{V}]\Omega = 0. \tag{3.7}$$

Then, if γ is continuous, solutions to (13) indicate that γ is one sign, either positive or negative: Even though $div\,\mathbf{V}$ may vary between positive and negative values, γ never goes through zero. Note that (13) implies a dual interpretation for γ. In addition to its interpretation as a parametrization function for the vector field $\mathbf{V}$, it may be interpreted also as a measure function for Ω.

For a particular choice of γ, the invariance of one-, two-, and-three-dimensional periods implies

$$L_{(\gamma \mathbf{V})} \oint_{1\,cycle} \mathcal{A} = 0, \quad \Rightarrow i(\mathbf{V})d\mathcal{A} = dP/\gamma \tag{3.8}$$

$$L_{(\gamma \mathbf{V})} \iint_{2\,cycle} H = 0, \quad \Rightarrow i(\mathbf{V})dH = d\theta/\gamma \tag{3.9}$$

$$L_{(\gamma \mathbf{V})} \iiint_{3\,cycle} S = 0, \quad \Rightarrow i(\mathbf{V})dS = d\alpha/\gamma, \quad fixed\ \gamma. \tag{3.10}$$

The forms P, θ,and α are respectively zero-, one-, and two-dimensional fields which are completely arbitrary at this level of analysis. By examing the requirement for invariance of the one-dimensional periods, the function P can be identified as the pressure addition to Hamiltonian field. For $A = p_\mu dq^\mu - H(p, q, t)dt$ on a state space of 2n + 1 dimensions, the modified Hamiltonian equations of motion become, for the vector field $\gamma[\mathbf{v}, \mathbf{f}, 1]$,

$$\mathbf{v} = +\partial H/\partial p_\mu + (1/\gamma)(\partial P/\partial p_\mu), \tag{3.11}$$

$$\mathbf{f} = -\partial H/\partial q^\mu - (1/\gamma)(\partial P/\partial q^\mu). \tag{3.12}$$

Such a field leaves the one-dimensional periods, $\oint_{1\,cycle} \mathcal{A}$, invariant for a given choice of parametrization, γ. Similar arguments may be constructed for the other pressurelike terms, θ and α.

The set of continuously parametrized vector fields which leave the periods invariant fall into two classes, those with positive parametrizations and those with negative parametrizations. The intersections of these two sets are those vector fields that satisfy the Hamilton equations of motions and yield absolute relative invariants which are independent of propagation direction. The compliment of the intersection forms two disjoint sets which produce relative integral invariants that are sensitive to the direction of propagation. It appears that a positive choice of γ (mechanically interpreted in its measure sense as a mass density) selects one of these disjoint sets such that in dissipative (irreversible) systems at least one of the periods is not preserved as an invariant. The concept of an entropy increase may be put into correspondence with this lack of period invariance in the forward (predictive) direction. It is interesting to observe that a statistical concept like pressure is related to the lack of reversibility of a topological period. The relationship between specific choices of the parametrization function and the concept of orientation is described in Appendix D.

The topological ideas presented above do not yield information about the magnitude of the quanta (why is Q=1.6 $\times 10^{-19}$ Coulombs, why is $h = 6.67 \times 10^{-34} joule - \mathrm{sec}$?), but they do indicate that the magnitudes of the Φ, Q, and S quanta can be interrelated, especially for those cases where separability permits,

$$\iiint_{3\,cycle} S = \iiint_{3\,cycle} \mathcal{A}\hat{}H = \oint_{1\,cycle} \mathcal{A} \iint_{2\,cycle} H \tag{3.13}$$

Then the flux quantum, charge quantum, and angular momentum quantum are not independent; the separability concept in effect imposes a topological constraint on the one-, two-, and three-dimensional gauges, in the sense of a gauge theory. Indeed, the Lagrange action potential S, built from the product $S = \mathcal{A}\hat{}H$, is yet another gauge theory [6] relating periods of $\mathcal{A}, H$, and S. Several classical gauge theories which impose certain constraints on the cohomology of spacetime are described in the next section. These are gauge theories of the second kind and are associated with restricted nonhomogeneous contact transformations. Gauge theories of the first kind based upon inhomogeneous contact transformations will be ignored in this article, except for the fact that they may be generalized to include the effects of dissipation. In certain cases gauge theories of the first kind are equivalent to gauge theories of the second kind. For example, it may be true that $\mathcal{A}\exp(i\theta) \equiv \mathcal{A} - d\psi$. However, such an equivalence requires that $\mathcal{A}$ be integrable; i.e., $\mathcal{A}\hat{}d\mathcal{A} = 0$, which is certainly not the case for the most general physical system. The concept of integrability will be shown below to be related to the notion of when the Lagrange and Hamiltonian formalisms are equivalent.

4. GAUGE THEORIES

4.1. Classical development

The recognition that the vector spaces of the fundamental forms, $F\,or\,H$, and $\mathcal{A}, H, or\ S$ are of the same dimensions leads to the notion that a map between the spaces of similar dimensionality effectively makes a statement about a choice of one-, two-, and three-dimensional gauges, and therefore influences a choice of quantization. It is the purpose of this section to demonstrate the topological basis for several historical gauge theories. One of the earliest gauge theories was due to Maxwell who utilized the idea of a constitutive map,

$$\chi : \, F \to H \tag{4.1}$$

to describe the electromagnetic properties of matter. In view of the fact that F is exact, but H need not be exact, the Maxwell constitutive map may be considered to be a gauge theory in that it specifies something about the two-dimensional periods of H, or the quantized charge. Historically this observation was not made.

Somewhat later Lorentz utilized the idea that charge-current meant particles in motion, and so developed a theory that maps the current, J, into the l-form $\mathcal{A}$,

$$L : \, J \to \mathcal{A} \tag{4.2}$$

The Lorentz map effectively makes statements about the gauge or one-dimensional periods of $\mathcal{A}$. To update the theory, consider a metrically based constitutive map, $X/Y_0 = \sqrt{g}(g^{\mu\alpha}g^{\nu\beta} - g^{\nu\alpha}g^{\mu\beta})$, an idea which is an extension of that introduced by some of the early proponents of the use of tensor analysis in physics. The factor Y_0 represents the functional admittance of spacetime. Its use permits the constitutive map, X, to be equated with the Hodge dualization operator, *, such thatf $H = Y_0 * F$. This idea gives Lorentz's classic wave-equation-with-source for the currents and potentials, when the generalized differential form identity,

$$\Box^2 \mathcal{A} = (d\delta + \delta d)\mathcal{A} = *J/Y_0 + d\delta\mathcal{A} - *H \, dY_0/Y_0^2, \tag{4.3}$$

is constrained by the Lorentz gauge condition, now explicitly stated as a topological requirement,

$$\delta\mathcal{A} = 0, \tag{4.4}$$

and the additional requirement that the admittance, Y_0, be constant over the domain. Note that the Lorentz theory makes a statement about the one-dimensional periods and the flux quanta. On a compact orientable manifold (23) requires by the HodgedeRham decom-position theorem that A contain no exact part. In hydrodynamics, the Lorentz gauge is related to the isochoric condition, commonly called (incorrectly) incompressibility .

4.2. Closed, but nonexact gauges

Most of the early gauge theories were not concerned with quantization, and ignored the latent possibilities associated with the periods of the gauge. Such a state-ment is not applicable to the theory of London [8] which effectively analyzed the theory of super conductivity and the flux quantum in terms of one-dimensional periods and introduced the gauge condition,

$$f : J \rightarrow \mathcal{A}$$

where explicitly

$$f : \mathcal{A} = - * (J/\rho). \tag{4.5}$$

For ρ a constant, this gauge implies that

$$F = d\mathcal{A} = -d(*J/\rho), \tag{4.6}$$

or for ρ equal to the London parameter, λ^{-1},

$$\mathbf{B} = curl\,(\lambda\mathbf{J}) \tag{4.7}$$

and

$$\mathbf{E} = +\partial(\lambda\mathbf{J})/\partial t. \tag{4.8}$$

A related and more modern development states that $\mathcal{A}$ and $(*J/\rho)$ are cohomologous; i . e ., let

$$f : \mathcal{A} = - * (J/\rho) + d\theta. \tag{4.9}$$

As J is exact; it satisfies the equation of continuity,

$$dJ = 0, \tag{4.10}$$

with its usual local representation

$$\partial\rho/\partial t + div\,\mathbf{J} = 0. \tag{4.11}$$

By defining $\Psi = \sqrt{\rho}\exp(i\theta/\hbar)$, the gauge condition and the equation of continuity imply a complex root, [9]

$$i\hbar\partial\Psi/\partial t = (1/2m)(-i\hbar\nabla - q\mathbf{A})^2\Psi + q\phi\Psi = 0, \tag{4.12}$$

which is Schrodinger's equation for a charged particle moving in an electromagnetic field given by the potential $(\mathbf{A}, \phi)$. Remarkably, the phase function θ of quantum theory appears to be related to the concept of the co-homology function between the 1-form A and the Hodge dual, $*(J/\rho)$.

4.3. Specific parametrizations and gauge theories

A remarkable feature of the invariance of two-dimensional periods is noted if a specific choice of parametrization of V is made to insure continuity. That is, choose a *mass* density γ such that $\gamma\mathbf{V}$ has zero divergence on the space. Then, with respect to this parametrized curve, the two-dimensional periods will be invariant if

$$\gamma\, i(\mathbf{V})J = d(Z), \tag{4.13}$$

where Z is some (N- 3) form density. In Cartesian coordinates

$$\gamma(\mathbf{J} - \rho\mathbf{V}) = curl\ \mathbf{Z} \tag{4.14}$$

and

$$\gamma(\mathbf{J} \times \mathbf{V}) = grad\,\phi + \partial\mathbf{Z}/\partial t \tag{4.15}$$

By letting $\mathbf{Z} \rightarrow \mathbf{0}, \phi \rightarrow 0$, the London gauge is retrieved, and by letting $curl\ \mathbf{Z} = grad\,\phi$, the Feynman gauge is obtained. It may be concluded that these two gauge theories (and hence the Schrodinger quantum theory) are gauge theories that leave invariant the two-dimensional periods of charge quanta, but not necessarily in a reversible manner.

If in four dimensions the gauge field $\mathbf{Z}$ is taken to be proportional to the 1-form $-\lambda(\mathcal{A})$, then

$$\mathbf{J} = \rho\mathbf{V} - (\lambda/\gamma)(\partial\mathbf{A}/\partial t + grad\,\phi) \tag{4.16}$$

which is both of the Ohmic format

$$\mathbf{J} = \rho\mathbf{V} + \sigma\mathbf{E}, \tag{4.17}$$

and related to the requirement that $\{\mathbf{V}, 1\}$ be a characteristic vector field.

Hence it is proved that Ohm's law for the current preserves two-dimension periods, but not reversibly. Ohmic dissipation does not involve topological variations of two-dimensional periods (but it is a dissipative mechanism that does involve topological variation of one- and three-dimensional periods). It is to be noted that only in four dimensions it is possible to use $Z = -\lambda\mathcal{A}$, for only in four dimensions does Z, an (N- 3)-form density, have the vector space qualities (the same dimension) of a 1-form, $\mathcal{A}$.

It is of physical interest to ask when two-dimensional periods are not topological invariants. In electromagnetism it is observed that friction (somehow) produces two-dimensional periods, and bombardment of matter with γ rays of more than 1. 02 MeV produces two-dimensional periods. In hydrodynamics, the notion of two-dimensional periods has yet to be experimentally identified, but it is suspected that they are related to (non-steady) dissipative mechanisms. The dissipative effects of radiation are associated with change of three-dimensional periods (spin). The dissipative effects of non-adiabatic processes are related to the change of one-dimensional periods (flux). These dissipative mechanisms intrinsically are distinct from those changes of two-dimensional periods - changes which must represent other dissipative mechanisms (See Appendix E). The relationship between forms, topology, and thermodynamics will appear elsewhere. Herein the pertinent theme to be remembered is that dissipation implies noninvariant periods, which in turn implies changing topology.

4.4. A HYDRODYNAMIC APPLICATION

The initial development of this theory of periods was in the language of electromagnetism for which it is well known that the desired 1-form is action per unit eharge. It would seem natural for hydrodynamic systems to develop the theory in terms of a fluid action per unit mass. Hence from the l-form,

$$A = udx + vdy + wdz - hdt, \tag{4.18}$$

it is possible to develop over spacetime the fundamental forms $F, H, J, A\hat{\ }H$, etc., and study the topological properties of such systems. Indeed, for a spatially Euclidean metric it is possible to demonstrate [10] how the Navier-Stokes equations take the form of a constraint relation:

$$L_{(\gamma\mathbf{V})}A = -dP/\rho + \nu\delta dA + d\theta. \tag{4.19}$$

By action of the exterior derivative on (38), the basic Helmholtz equation for vorticity becomes

$$L_{(\gamma \mathbf{V})} F = +d\rho\hat{}dP/\rho^2 + \nu d\delta F, \tag{4.20}$$

and demonstrates the effect of viscosity and equation of state on the generation of vorticity

. Every l-form may be adjusted by a gauge transformation (of the second kind) to its cohomological equivalent, $A - d\theta$. Furthermore, the dual of $(A - d\theta)$ is an (N-1)-form which locally satisfies the conditions of integrability, hence admits an integrating factor β such that the "current, "

$$J_m = \beta(*(A - d\theta)), \tag{4.21}$$

is closed. The following questions arise: Is there a global integrating factor? Can the integrability of A be accomplished by the gauge transformation, $d\theta$, alone implying that β is a constant, or $d\beta$ is orthogonal to $*(A - d\theta)$? If J_m is closed, is it exact? Is J_m a topological constraint such that, $J_m = J = d * dA$?

As a first example, consider the two-dimensional fluid ($w = 0$) on spacetime endowed with a metric $[1, 1, 1, -(c/n)^2]$. For simplicity, define $Y_0 n/c = \epsilon = (n^2/\mu c^2)$. Assume that the specific Hamiltonian h is given by the formula

$$h = [c^2 - \nu\partial(\ln\zeta)/\partial t], \tag{4.22}$$

where the $\{x, y, z\}$ components oof the vorticity are given by the usual notation, $curl\,\mathbf{V} = \{\xi, \eta, \zeta\}$. Now make the gauge transformation, based on the phase function,

$$\theta = \nu \ln\zeta, \tag{4.23}$$

and use the z component of vorticity, ζ, as the integrating factor β to yield

$$J_m = \zeta\{*[(u - \partial\theta/\partial x)dx + (v - \partial\theta/\partial y)dy]\}. \tag{4.24}$$

By assuming that n and c are constants, the closure of J_m, $dJ_m = 0$,yields

$$\partial\zeta/\partial t + u\partial\zeta/\partial x + v\partial\zeta/\partial y + \zeta(\partial u/\partial x + \partial v/\partial y) = \nu(\partial^2\zeta/\partial x^2 + \partial^2\zeta/\partial y^2), \tag{4.25}$$

which is precisely the Navier-Stokes-Helmholtz equation for the vorticity of a two-dimensional, compressible, viscous (barotropic) fluid, and demonstrates how the topologically based gauge theory is applicable to hydrodynamics.

It is to be noted that even for the three-dimensional case, if the phase function θ is chosen to be such that

$$(n/c)^2(h + \partial\theta/\partial t) = 1, \tag{4.26}$$

then the closure of J_m is always an equation of continuity, such that the volume measure $\Omega = \widetilde{\beta} dx\hat{}dy\hat{}dz\hat{}dt$ is an invariant of the flow field given by

$$\overline{V} = [u - \partial\theta/\partial x, v - \partial\theta/\partial y, w - \partial\theta/\partial z, 1]. \tag{4.27}$$

The density $\widetilde{\beta}$ is defined as $\beta\sqrt{g} = \beta(ic/n)$. If the gauge is chosen such that V is isochoric, then

$$\nabla^2\theta = div\mathbf{V} \tag{4.28}$$

but such a choice does not necessarily lead to the concept of incompressibility, for the closure of J_m and the constraint given by (45) yield the result that

$$d\beta/dt = \partial\{1 - (n/c)^2(h + \partial\theta/\partial t)\}(\beta c/n)/\partial t. \tag{4.29}$$

The density "β" is incompressible only if $d\beta/dt$ vanishes, which is not necessarily true, even though the fluid is isochoric.

Again focus attention on a two-dimensional fluid, and ask the question: Is J_m integrable? The answer, subject to (45), is always yes in two dimensions, which admits the concept of a stream function, ψ. Consider the 2-form density

$$-G/(ic/n) = (1/2)(\{xdy\hat{}dz - ydx\hat{}dz\} - \psi dt\hat{}dz). \tag{4.30}$$

The exterior derivative of G yields the 3-form density (for constant c/n)

$$-dG = (ic/n)(dx\hat{}dy\hat{}dz + \partial\psi/\partial x dx\hat{}dz\hat{}dt + \partial\psi/\partial y dy\hat{}dz\hat{}dt). \tag{4.31}$$

Direct comparison to $*(A - d\theta)$ yields the standard representations,

$$\begin{aligned} u &= +\partial\theta/\partial x + \partial\psi/\partial y \\ v &= +\partial\theta/\partial y - \partial\psi/\partial x \end{aligned} \tag{4.32}$$

for the velocity components in terms of the "potential" function θ and the "stream" function ψ. Note that

$$div\mathbf{V} = \nabla^2\theta, \tag{4.33}$$

and

$$\zeta = (curl\mathbf{V})_z = -\nabla^2\psi. \tag{4.34}$$

The remarkable feature of two dimensions is that the viscosity of the fluid and its associated dissipative features may be transformed away by means of a dilitation [compare (42) and (52b)]. This effect has been observed in other (mechanical) dissipative systems.[11] (Consider the simple example as given by the damped harmonic oscillator, followed by a transformation to a space whose coordinates are exponentially shrinking with time. In the shrinking coordinate system, the equation of motion is that of an undamped oscillator.)

The two-dimensional stream function permits the evaluation F:

$$F = d\overline{A} = d(A - d\theta) = -\nabla^2\psi(dx\hat{}dy) + (\partial^2\psi/\partial y\partial t)dt\hat{}dx + (-\partial^2\psi/\partial x\partial t)dt\hat{}dy. \tag{4.35}$$

which allows the z component of vorticity, $\zeta = -\nabla^2\psi$, to be compared with the z component of the electromagnetic $\mathbf{B}$ field. The dual of F is the 2-form density, H :

$$H = Y_0 * F = i(1/\mu)(-\nabla^2\psi)dz\hat{}dt - \epsilon(\partial^2\psi/\partial y\partial t)dy\hat{}dz - \epsilon(\partial^2\psi/\partial x\partial t)dz\hat{}dx \tag{4.36}$$

from which the electromagnetic" current density 3-form J, can be derived (for constant velocity, ϵ, and μ) in terms of a Hamiltonian format built on a function, χ, equal to the result of the application of the wave operator on ψ.

That is, define

$$\chi = \nabla^2\psi - (n/c)^2\partial^2\psi/\partial t^2 \tag{4.37}$$

then

$$J = -(i/\mu)(\partial\chi/\partial y dy\hat{}dz\hat{}dt + \partial\psi/\partial x dx\hat{}dz\hat{}dt) = -(i/\mu)d\chi\hat{}dz\hat{}dt. \tag{4.38}$$

Note that the free charge density ρ vanishes identically.

If χ is proportional to ψ such that (55) becomes the KleinGordon equation, then (56) indicates that the electromagnetic current is proportional to the "rotational" components of the velocity. Therefore, it is to be observed that the topological constraint $J_m = J$ is related to the constraint that the stream function, ψ, satisfies the Klein-Gordon equation.

For all cases, the form $F\hat{}H$ equals

$$F\hat{}H = i[(1/\mu)(\nabla^2\psi)^2 - \epsilon\{(\partial^2\psi/\partial y\partial t)^2 + (\partial^2\psi/\partial x\partial t)^2\}]dx\hat{}dy\hat{}dz\hat{}dt. \quad (4.39)$$

Except for very rapid variations of the stream function, the energy density of the fluid tends to be magneticlike. That is, the fluid field energy density is dominated by the vorticity and not by the fluid accelerations. Therefore, there exist Lorentz transformations which can eliminate the accelerations (E fields) entirely. A critical point in the fluid flow is reached when the energy density becomes electriclike.

The topological periods of one, two and three dimensions are determined by the forms $A, H, and\, A\hat{}H$. The periods of A are determined entirely by the stream function, ψ, as are the periods of H. However, the periods of $A\hat{}H$ depend upon the choice of the gauge function θ as well.

It is readily observed that if the fluid is steady there do not exist any timelike (dt =0) two-dimensional periods of H or three-dimensional periods of $A\hat{}H$. The one-dimensional periods are related to nonzero values of the circulation integral which can exist for steady (streamline) flows. Timelike two- and three-dimensional periods require that the flow be nonsteady.

Note, however, that for a fluid governed by the NavierStokes equations [set $\theta = L + \nu\delta A$ in (38)], the choice of parametrization $\gamma = \rho$ reduces the intrinsic equations of motion to the statement,

$$L_{(\rho\mathbf{V})}A = -d(\rho L - P) + \nu(d\delta + \delta d)A. \quad (4.40)$$

For the inviscid fluid this (common) choice of parametrization implies that the one-dimensional periods are invariant w. r. t. the flow $\rho\mathbf{V}$, for all $\mathbf{V}$. In the atmosphere where the dominant mechanism for vorticity creation is due to a nonbarotropic equation of state (and where viscosity is often neglected) this result implies that the one-dimensional periods are invariant along the flow lines of the momentum flux $\rho\mathbf{V}$ (which, of course, are different from the flow lines of $\mathbf{V}$). Only viscosity effects can change the vorticity, or the one-dimensional periods, of the fluid along the flow lines of $\rho\mathbf{V}$, the momentum flux.

5. SUMMARY

This article demonstrates that topological concepts of periods on manifolds may be put into correspondence (55) with the quantized concepts of flux, charge, and spin, each in an independent manner and without direct utilization of a quantum hypothesis. Physically important information is contained in the potentials in that the specification of a gauge selects an equivalence class in the topological sense. Physical theories of matter can be put into correspondence to various gauge theories relating or defining the one-, two-, and three-dimensional periods supported by spacetime. Processes without dissipation leave the topological periods invariant. Schrodinger's quantum theory can be interpreted as a gauge theory that leaves the two-dimensional periods invariant. Hamiltonian theories leave the one- dimensional periods invariant. On the other hand, there must exist three (independent) modes of dissipation depending upon whether or not the process individually modifies the one-, two-, or three-dimensional periods of the system.

6. ACKNOWLEDGMENT

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7. APPENDIX A:

7.1. THE BROUWER DEGREE OF A MAP THEOREM

Several proofs of the Brouwer theorem may be found in texts [2] on topology. The purpose of this appendix is to demonstrate its use in spacetime situations. Consider an initial space M of coordinates x^A and a final space N of coordinates y^μ, with a transitional map ϕ from M to N that need not be a homemorphism. Suppose $A = A_\mu dy^\mu$ is a l-form on N which may be retrodicted to M by the pullback, $\phi^* A$. Now suppose a closed curve $\overline{C}$ (one-dimensional manifold) may be immersed into M by $\overline{\psi}$ and another closed curve C may be immersed into N by ψ. Then a map f between $\overline{C}$ and C will have a degree such that one-dimensional integrals are related by Brouwer's theorem

$$\int_{\overline{C}} \overline{\psi}^* (\phi^* A) = \deg\, f \int_C \psi^* A \tag{7.1}$$

and deg f is an integer. If the image of $\overline{C}$ in M encircles a number of holes in M different from the number of holes encircles by C in N, then that information will appear in the degree of the map, f. The two curves $\overline{C}$ and C are not homotopic in a space of the same dimension, but may be homotopic in spaces of higher dimension . Now it is not necessary to think of the topological holes as literal holes in the space. A physical example of the holes may appear as regions of closed and open sets in the field built over spacetime. Consider a fluid stream in which there exist zones which are curl free and zones which are not. It is possible to visualize the zones which are rotational as "holes" in the field. As the fluid flows from an initial to a final configuration, the number of rotational zones can change. The first Betti number of field space then must not be an invariant. The flow cannot (by Cartan's theorem) be described by Hamilton's equations of motion. The changing topology will be exhibited by a nondiabatic dissipation. Figure 1 portrays the situation which is described by Brouwer's theorem. It is most remarkable that dissipative, radiative transitions from initial to final states take place in discrete increments in this topological analysis. These ideas are in agreement with a quantum theory point of view, but did not invoke a quantum hypothesis.

7.2. APPENDIX B: KURATOWSKI CLOSURE AND DIFFERENTIAL FORMS

The theory of exterior differential forms should not be considered as just another formalism of fancy. It should be considered as a theory that contains both geometrical information (as does classical tensor analysis) and topological information. Consider a system $\sum$ of forms and their exterior derivatives specified on some space. Then note that the operator

$$K = \mathbb{I} \oplus d \tag{7.2}$$

where $\mathbb{I}$ is the identity map and d is the exterior derivative, when operating on subsets of the system of forms, obeys the following rules:

1. $K\emptyset = \emptyset$,

2. $KA \supset A$

3. $K^n A = KA$,

4. $K(A \cup B) = KA \cup KB$

These rules permit K to be identified as a Kuratowski closure operation.[12] Hence the specification of the system $\sum$ defines a topology.

The most amazing property of differential forms is that with respect to maps which need not be invertible, and which may not have invertible Jacobians, the differential form is always retrodictable (but not necessarily predictable). The physicist should adopt the viewpoint that the most primitive field problem is: Given the final state, what was the initial state from which it came? This retrodictive question is more fundamental [13] than the usual predictive version: Given the initial state, what is the final state? With respect to nonhomeomorphic maps the retrodictive question is often deterministic, while the predictive question is not.

7.3. APPENDIX C: AN EXAMPLE OF A NONEXACT GAUGE FIELD

Consider the vector field,

$$\mathbf{A} = (\mathbf{i}y - \mathbf{j}x)/(x^2 + y^2), \tag{7.3}$$

on the space $E3 - \{0\}$, which has a hole. Now $curl\mathbf{A} = 0$, everywhere; therefore, the field $\mathbf{A}$ is a gauge field. The closed integral of $\mathbf{A}\circ d\mathbf{l}$ about the hole may be evaluated by means of the coordinate transformation $x = r\cos\theta, y = r\sin\theta$. The circulation integral about a loop encircling the origin is given by the expression

$$\oint A = \oint \mathbf{A}\circ d\mathbf{l} = \oint \{ydx - xdy\}/(x^2 + y^2). \tag{7.4}$$

Substitution of dx and dy in terms of r and θ yields

$$\oint A = \int_0^{2\pi} d\theta = 2\pi \tag{7.5}$$

As the circulation is not zero, $\mathbf{A}$ cannot be globally represented by the gradient of a single potential function and represents a gauge field. Under the transformation the closed loop in (x,y) space becomes an open curve in (r,θ) space.

7.4. APPENDIX D: PARAMETRIZATION, ORIENTATION AND OTHER TOPOLOGICAL IDEAS

It appears that the continuity of the parametrization function γ is related to the topological concept of orientability. When γ is continuous (and by convention chosen to be positive), then it is possible to mechanically identify γ with the mass density ρ and a choice of orientation. On the other hand, the identification of a γ which satisfies (12) with the charge density ρ is a notion associated with a nonorientable manifold that supports both positive and negative measures. An elementary geometric example of such phenomena is given by two separated spherical surfaces, one with an outward normal field and the other with an inward normal field. First, because the sets are disconnected they cannot be represented by a single immersion, for immersions have a unique range. The questions arises, can two spherical surfaces be represented by a submersion?, and the answer to the question is yes, if the spheres were oriented. The implicit equation

$$[(x-a)^2 + y^2 + z^2 - (a/2)^2] \times [(x+a)^2 + y^2 + z^2 - (a/2)^2] = 0$$

yields a closed set of two components representing two spheres of radii a/2 separated by 2a. The implicit representation, however, guarantees that the surfaces are closed and oriented. The normal (gradient) field is outbound on both spheres. If one of the factors is multiplied by minus one, then the implicit representation gives an inbound normal field on both spheres. The nonorientable two component spheres with one normal inbound and one normal out-bound cannot be given an implicit representation. Nature states this idea by demonstrating charge pairs with the field lines "existing" from the positive charge regions and "entering" the negative charge regions. Mass on the other hand is oriented, with field lines only existing from regions of (positive) mass.

A few remarks about the topological question of separability of the integration chains is of related physical significance. Some closed chains separate, others do not. For example the closed curves on a torus which yield cycles do not separate the space, but the shrinkable closed curves do. Sometimes a cut can be made that will separate and sometimes not. A cut along the closed "diametrical" curve on a Mobius band does not separate the space, but actually produces from a spin-2 representation a spin-2 representation, [14] which then may be folded into a double layered spin-2 representation again (fermions to gravitons to fermions??). On the other hand a cut along the nonshrinkable closed curve of a cylinder separates the space. In the physics of electromagnetic materials such phenomena often is observed. A conductor placed in an electric field is polarized. If the conductor is then cut in two, the charge is separated. On the other hand, a cut through a dielectric does not separate the charge.

Amperian magnetic needles when cut do not separate the flux quanta, which imply that such cuts do not separate and do not produce flux monopoles. The concept of a magnetic monopole has to be put in correspondence with the topological theory that there exists a one-dimensional cut which can separate the flux quanta, and yet $div\mathbf{B} = 0$ on the manifold. The existence of these cuts is an open question, but it is to be observed that this possible topological property of the theory does not require a source term to be appended to the 1st Maxwell set of equations ($dF = ddA = 0; not\ dF = J_m$). Flux quantization is permissable, with free flux quanta (monopoles) permissible, if separable cuts exist.

7.5. APPENDIX E: GAUGE INVARIANCE AND HAMILTONIAN THEORIES

The relationship of gauge and nonintegrability was introduced to the physics community by Weyl.[15] A most informative picture can be obtained by propagating a vector about some closed loop where infinitesimally the vector is kept "parallel" to itself. When the vector returns to its starting point, Weyl pointed out that there is no obvious reason why the length of the vector should be the same as it was when it started. The vector length happens to be length, or gauge, invariant with respect to parallel transport on Euclidean spaces, but it need not be invariant on other spaces.

A more general viewpoint notes that upon return to the starting point the final vector may be of different length, and also it may have a line of action not parallel to the initial vector. The decrement vector betwen this initial and final state may have a component in the "plane" of the curves and the initial vector, and perhaps a component out of the initial plane. " The decrement angle out of the plane is related to the torsion of the space, and the decrement angle in the plane is related to the curvature of the space. If the closed loop is defined by segments of vector fields, the propagated vector may or may not be algebraically closed with respect to these vectors. If the propagated vector, upon being transported around the loop, is not closed with respect to the vector generators of the loop, but in a sense points out of the plane of the loop, then the space has torsion. Immersed spaces do not have torsion; submersed spaces can have torsion. Now a great emphasis is placed on the gauge invariance of physical theories, some of it in an almost mystical sense. However, according to the ideas expressed in this text the notion of gauge invariance relates to a concept of topological invariance, and this concept must be extended also to two and three-dimen-sional gauges. It is to be noted that if the one-dimensional gauges (one-dimensional periods) are to be invariant with respect to a dynamical transformation generated by the field $\mathbf{V}$, for any parametrization γ, then

$$L_{(\gamma\mathbf{V})} \oint_{1\,cycle} \mathcal{A} = 0 \tag{7.6}$$

However, such a requirement leads to the necessary and sufficient conditions (Cartan's theorem) that the equations of motion are described by Hamilton's partial differential equations.[12]

A necessary and sufficient condition for a theory to be one-dimensionally gauge invariant (for any γ) is then that the theory be a Hamiltonian theory. It would appear that the predilection for gauge invariance possibly can be related to the demand for a quantum theory, which requires a Hamiltonian basis for quantization in the Dirac sense.

It should be recognized that a gauge invariant theory *cannot* explain the details of nonadiabatic phenomena, for dissipation implies that gauge is not invariant. As was mentioned in the text, the invariance of the one-dimensional gauges does not guarantee the invariance of the two- and three-dimensional periods, unless further constraints are put on the system. Hence, there may be dissipative effects in a physical system if the two-dimensional and three-dimensional periods are not in-variant. A Hamiltonian theory may still be dissipative in the sense that the two- and three-dimensional periods may not be invariant, even though the one-dimensional periods are invariant. For those topological systems where the p and (N-p) dimensional periods are dualistically related (flux dual to angular momentum) then a Hamiltonian analysis implies both conservation of flux and angular momentum, but still does not guarantee conservation of two-dimensional periods (charge pairs can be created in an independent dissipative manner).

For hydrodynamic systems where the unit source is mass (not charge) the integer relations of the one-dimensional periods have been experimentally observed, and the quantum of vortices in superfluids is h/m (not h/e). Physically, what the two-dimensional periods are for systems with mass as the source (gravity) is not known. Neither is there experimental evidence available at this time for three-dimensional periods in a hydrodynamic system. The relationship between creation of nonzero periods and instability concepts leads to the conjecture that these effects are to be observed in the turbulent state. (However see http://www.uh.edu/˜rkiehn 1997)

7.6. APPENDIX F: DUALITY AND PERIODS

It is known that topological periods are related to Betti numbers and that for certain spaces the Betti numbers obey the PoincareAlexander duality principle; that $B_p = B_{N-p}$, on compact oriented spaces.[16] The idea of duality can be used to explain the apparent paradox between the Bohr Sommerfeld quantization rule,

$$\oint p_\mu dq^\mu = n\hbar \tag{7.7}$$

which implies that mechanical action is related to a one-dimensional period, and the electromagnetic concept that action is related to a three-dimensional period [see Eq. (6)]. To resolve the paradox, consider those systems where the factorization of Eq. (19) is admissible; then the three-dimensional period of action becomes proportional to the one-dimensional period of flux mod the two-dimensional period of charge. Classically this result is expressed by the statement that the electromagnetic momentum is Q times the vector potential. Topologically, the concept is a duality constraint on the periods of dimension $p = 1$ and $N - p = 3$ for fixed values of Q, and states that the flux periods are proportional to the action periods. Under this topological constraint the three-dimensional period, or integral, of action may be reduced to a one-dimensional period, or integral, in the Bohr-Sommerfeld case.

In mechanics, a similar paradox arises when the Lagrangian approach is compared to the Hamiltonian action theory. The question asked is: When is the Lagrange integrand, Ldt, equivalent to the Hamiltonian action, $A = p_\mu dq^\mu - Hdt$? If the equivalence is to be globally valid, then it follows that $A\hat{}dA = 0.$; i.e., the Hamiltonian action must be integrable. Such special cases imply that the momentum vector either has zero curl, or that the curl of **p** be perpendicular to **p**. Many physical systems admit such descriptions, but others do not. A fluid with vorticity is a classic counter example to the exact case, and a fluid with a twisted secondary flow yields an example for the nonintegrable case.

The resolution to the paradox of LaGrange-Hamiltonian action equivalence is not only related to the duality principle, but also sheds some light on the ergodic problem of when phase averages are equivalent to time averages. In the duality sense, the Lagrange function is interpreted not as the coefficient of a 1-form, but as the decomposed coefficient of a 2N density in 2N + 1 space:

$$\mathcal{L} = L(p,q,t)dq^1\hat{}...dp \tag{7.8}$$

All 2N forms are decomposable, but they do not necessarily represent 2N volumes. The above format suggests that the Lagrange density approach appears to be associated with phase averages. For each 2N form in 2N + 1 space there exists a metric dual which is a 1-form, A. That is,

$$*\mathcal{L} = \mathcal{A}, \tag{7.9}$$

by means of the Hodge dualization operator *. The question may now be asked: Does the dual of $\mathcal{L}$ admit a single parameter (time) representation? If true, then there exists an ergodic equivalence between $\int \cdots \int \mathcal{L}$ as a phase average and $\int \mathcal{A}$ as a time average. The equivalence of the two methods requires again that the Hamiltonian action, A, be integrable:

$$A\hat{}dA = 0 = (*\mathcal{L})\hat{}d(*\mathcal{L}) \tag{7.10}$$

The Hamiltonian-Lagrange equivalence subsumes the ergodic principle.

The reverse question is somewhat different: Suppose there exists an integrable 1-form, $"L"d\tau = A$. Is the metrically related dual $*A = \sqrt{g}g^{0\nu}Ldq^1\hat{}..\widehat{dq^\nu}\hat{}...\hat{}dp_N$ divergence free? (In order words is the "phase volume" closed?) The answer is yes if $\sqrt{g}L$ is an integrating factor for the time like component of the metric $g^{0\nu}$.

A somewhat more general view considers the duality constraints between Sp and $\mathcal{A}$ in terms of a cohomology theory, and compares the closed one-dimensional integrals and the closed 2N+1 dimensional integrals on a state space of 2N + 1 variables p, q, t. In a preliminary way, assume that the single-dimensional integral implies the existence of a l-form $\mathcal{A} = p_\mu dq^\mu - Hdt$ on state space, and a one-parameter map $\phi : \tau \rightarrow p, q, t$, such that the pullback of the integral of the 1-form exists as a function on τ; i. e.,

$$\phi^* \int \mathcal{A} = \int L d\tau. \tag{7.11}$$

In the second situation, consider a map $\psi : \overline{q}, \overline{p} \rightarrow q, p, t$ such that the 2N dimensional form $\mathcal{S} = \rho dq^1\hat{}...dq^N\hat{}dp_1\hat{}...dp_N - J^1 dt\hat{}\widehat{dq^1}\hat{}...dq^N\hat{}dp_1\hat{}...dp_N.....$ has a pullback image,

$$\psi * \int \cdots \int \mathcal{S} \Rightarrow \int \cdots \int S(\overline{q}, \overline{p})d\overline{q}^1\hat{}...d\overline{p}_N \tag{7.12}$$

Time averages are equal to phase averages when it is possible to equate (dually) the forms of action L to action density $\mathcal{S}$. As the ergodic hypothesis asserts that the open integrals are equivalent, it follows that a similar question may be posed about the closed integrals; but then the question becomes when are the one-dimensional periods equivalent to the (N- 1)-dimensional periods. That is, assert $d\widehat{\mathcal{A}} = 0$ and $d\widehat{\mathcal{L}} = 0$ so that the question becomes

$$\oint \widehat{\mathcal{A}} = \underset{2n\,cycle}{\int \cdots \int} \widehat{\mathcal{S}} \tag{7.13}$$

but this question is similar to the notion that the flux quanta are equivalently related to the action quanta. The basic idea of importance here is the notion that the periods of $\widehat{\mathcal{A}}$ and the periods of $\widehat{\mathcal{S}}$ are dually related in ergodic systems. If the one-dimensional periods of $\widehat{\mathcal{A}}$ are quantized flux and if the periods of $\widehat{\mathcal{S}}$ are quantized angular momentum, then they are individually in the ratio of the integers by Brouwer's Theorem and directly related to one another (that is, dually equivalent) if $B_n = B_{n-p}$, e. g., if the space is compact and orientable. As not all spaces have $B_n = B_{n-p}$, the flux and action quanta are not always dually related, and not all systems are ergodic.

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8.1. CAPTIONS

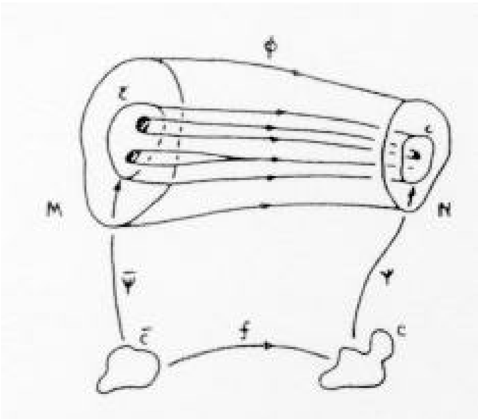

Figure 1 Brower's degree of a map theorem applied to hydrodynamic dissipative systems.